



Functional Analysis

Classification of noncommutative domain algebras

Classification des algèbres de domaines noncommutatifs

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ABSTRACT

Noncommutative domain algebras are noncommutative analogues of the algebras of holomorphic functions on domains of \mathbb{C}^n defined by holomorphic polynomials, and they generalize the noncommutative Hardy algebras. We present here a complete classification of these algebras based upon techniques inspired by multivariate complex analysis, and more specifically the classification of domains in hermitian spaces up to biholomorphic equivalence.

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R É S U M É

Les algèbres de domaines noncommutatifs sont des analogues des algèbres de fonctions holomorphes sur un domaine défini par des inéquations polynômiales dans \mathbb{C}^n . Elles généralisent également les algèbres de Hardy. Nous présentons dans cette Note la classification complète de ces algèbres, obtenue par des méthodes inspirées de l'étude des fonctions biholomorphes entre domaines d'espaces hermitiens.

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1. Introduction

Noncommutative domain algebras, introduced by Popescu in [11] are universal objects in the category of operator algebras for certain polynomial relations and are noncommutative analogues of algebras of holomorphic functions on domains in hermitian spaces, as well as generalizations of Hardy algebras. This Note presents a classification of these algebras up to completely isometric isomorphism, based upon their defining symbols, and thus completes the work initiated by the authors in [2] and [3]. Our methods are based on the deep interplay between analysis of this type of operator algebras and analysis of multivariate holomorphic functions. This Note also exploits an observation made in [9] which helps us conclude Theorem 2.2.

Algebras of weighted shifts on Fock spaces [10,5,7,6] find their origin in the study of row contractions, dilations, and commutant lifting theorems. Their applications became important in interpolation problems [8,4] and the study of E_0 -semigroups [9] or noncommutative complex analysis [2,1,3].

In [2] and [3], the authors of this Note studied the geometry of the spectra of noncommutative domains. They used Thullen's [13] and Sunada's [12] classification of Reinhardt domains and combinatorial arguments to fully classify a large

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class of domain algebras. However, one important case remained unsolved, and this Note presents the solution for this last unresolved class, thus completing the classification of all noncommutative domain algebras with polynomial symbols.

Noncommutative domain algebras are naturally – and best – represented as norm closed algebras of operators on full Fock spaces. For any (complex) Hilbert space \mathcal{H} , the associated full Fock space, denoted by $\mathcal{F}(\mathcal{H})$, is the Hilbert sum $\bigoplus_{n \in \mathbb{N}} \mathcal{H}^{\otimes n}$ with the convention that $\mathcal{H}^0 = \mathbb{C}$. It is useful for the definition of noncommutative domain algebras to introduce the following alternative description of $\mathcal{F}(\mathcal{H})$. Let \mathbb{F}_+^n be the free semigroup over $\{1, \dots, n\}$ whose neutral element will be denoted by \emptyset by abuse of notation. For any elements t_1, \dots, t_n of any unital associative algebra with unit $\mathbb{1}$, and for any nonempty word $\alpha = i_1 \dots i_k$ in \mathbb{F}_+^n , with $i_1, \dots, i_k \in \{1, \dots, n\}$, we denote by t_α the product $\prod_{j=1}^k t_{i_j}$, while t_\emptyset will be set to $\mathbb{1}$ by convention. In particular, $\mathcal{F}(\mathbb{C}^n)$ is a unital associative algebra for the tensor product. We fix henceforth the canonical basis $\{e_1, \dots, e_n\}$ of \mathbb{C}^n and observe that $(e_\alpha)_{\alpha \in \mathbb{F}_+^n}$ is a Hilbert basis of $\mathcal{F}(\mathbb{C}^n)$, which allows us to identify $\mathcal{F}(\mathbb{C}^n)$ with $\ell^2(\mathbb{F}_+^n)$.

Fix $n \in \mathbb{N}, n > 0$. Let $\mathcal{P}_n = \mathbb{C}[X_1, \dots, X_n]$ be the algebra of polynomials in n noncommuting indeterminates X_1, \dots, X_n – which is the free associative algebra generated by \mathbb{F}_+^n , where each generator $i \in \{1, \dots, n\}$ of \mathbb{F}_+^n is identified with X_i . Thus any element of \mathcal{P}_n is of the form $\sum_{\alpha \in \mathbb{F}_+^n} a_\alpha X_\alpha$ for an almost zero family $(a_\alpha)_{\alpha \in \mathbb{F}_+^n}$ of complex numbers. For any $f = \sum_{\alpha \in \mathbb{F}_+^n} a_\alpha X_\alpha \in \mathcal{P}_n$ and any Hilbert space \mathcal{H} whose C^* -algebra of bounded linear operators will be denoted by $\mathcal{B}(\mathcal{H})$, the noncommutative domain $\mathcal{D}_f(\mathcal{H})$ is defined by:

$$\mathcal{D}_f(\mathcal{H}) = \left\{ (T_1, \dots, T_n) \in \mathcal{B}(\mathcal{H})^n : \sum_{\alpha \in \mathbb{F}_+^n} a_\alpha T_\alpha T_\alpha^* \leq \mathbb{1}_{\mathcal{H}} \right\},$$

where we used the notation T_α^* for $(T_\alpha)^*$.

If f is an element of the subset $\mathcal{S}_n \subseteq \mathcal{P}_n$ defined by:

$$\mathcal{S}_n = \left\{ f = \sum_{\alpha \in \mathbb{F}_+^n} a_\alpha X_\alpha \mid a_\emptyset = 0, \forall \alpha \in \mathbb{F}_+^n a_\alpha \geq 0 \wedge (|\alpha| = 1 \Rightarrow a_\alpha > 0) \right\},$$

where $|\alpha|$ is the word-length of $\alpha \in \mathbb{F}_+^n$, one can construct an explicit, universal n -tuples of operators in $\mathcal{D}_f(\mathcal{F}(\mathbb{C}^n))$. We shall refer to elements of \mathcal{S}_n as n -symbols, while Popescu refers to them as *positive regular n -free formal polynomials*. Let us fix an n -symbol $f = \sum_{\alpha \in \mathbb{F}_+^n} a_\alpha X_\alpha$. We define the weighted shifts W_1^f, \dots, W_n^f on $\mathcal{F}(\mathbb{C}^n)$ by extending:

$$W_j^f(e_\alpha) = \sqrt{\frac{b_\alpha}{b_{j \cdot \alpha}}} (e_j \otimes e_\alpha) \quad \text{where } b_\alpha = \sum_{k=1}^{|\alpha|} \sum_{\substack{\gamma_1 \dots \gamma_k = \alpha \\ |\gamma_1| \geq 1, \dots, |\gamma_k| \geq 1}} a_{\gamma_1} \dots a_{\gamma_k} \quad \text{for all } \alpha \in \mathbb{F}_+^n. \tag{1.1}$$

We then have the following fundamental universal property:

Theorem 1.1. (See Popescu [11].) *Let $n \in \mathbb{N}, n > 0$, and let $f \in \mathcal{S}_n$. Let \mathcal{A}_f be the norm closure of the associative algebra generated by the set of operators $\{W_1^f, \dots, W_n^f\}$ defined by (1.1) on $\mathcal{F}(\mathbb{C}^n)$. We have $(W_1^f, \dots, W_n^f) \in \mathcal{D}_f(\mathcal{F}(\mathbb{C}^n))$. Moreover, for any Hilbert space \mathcal{H} and any $(T_1, \dots, T_n) \in \mathcal{D}_f(\mathcal{H})$, there exists a unique completely contractive algebra morphism φ from \mathcal{A}_f onto the norm closure of the algebra generated by T_1, \dots, T_n such that $\varphi(W_j^f) = T_j$ for $j = 1, \dots, n$.*

The purpose of this Note is to completely classify the noncommutative domain algebras \mathcal{A}_f defined for $f \in \mathcal{S}_n$ in Theorem 1.1 in term of their symbol f .

2. Main result

We define the following equivalence relation on the set \mathcal{S}_n of symbols [3, Definition 2.1]:

Definition 2.1. Let $n \in \mathbb{N}, n > 0$. Two elements $f, g \in \mathcal{S}_n$ are scale-permutation equivalent when there exist a permutation σ of $\{1, \dots, n\}$ and scalars $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ such that $f(X_1, \dots, X_n) = g(\lambda_1 X_{\sigma(1)}, \dots, \lambda_n X_{\sigma(n)})$.

Our main theorem is the following complete classification result:

Theorem 2.2. *Let $f \in \mathcal{S}_n$ and $g \in \mathcal{S}_m$ be two symbols. The noncommutative domain algebras \mathcal{A}_f and \mathcal{A}_g are completely isometrically isomorphic if and only if $n = m$ and f and g are scale-permutation equivalent.*

Before we provide the proof of this result, let us recall from [2] the following fundamental duality construction upon which our work relies. Let $f \in \mathcal{S}_n$ for some $n \in \mathbb{N}, n > 0$. Let $k \in \mathbb{N}, k > 0$. Let $T = (T_1, \dots, T_n) \in \mathcal{D}_f(\mathbb{C}^k)$. By universality

of \mathcal{A}_f , there exists a completely contractive morphism $\langle T, \cdot \rangle_k : \mathcal{A}_f \rightarrow \mathcal{B}(\mathbb{C}^k)$ such that $\langle T, W_j^f \rangle_k = T_j$ for $j = 1, \dots, n$. Moreover, the function $\langle \cdot, a \rangle_k$ is holomorphic on the interior of $\mathcal{D}_f(\mathbb{C}^k)$ and extends by continuity to $\mathcal{D}_f(\mathbb{C}^k)$. We shall abuse the terminology and call such a function holomorphic on the compact $\mathcal{D}_f(\mathbb{C}^k)$.

Proof of Theorem 2.2. In this proof, isomorphisms are always meant for completely isometric isomorphisms of operator algebras.

By [2, Lemma 4.4], whenever two symbols are scale-permutation equivalent, their associated noncommutative domains are isomorphic. It is thus sufficient to prove the converse here.

Let us denote the set of all isomorphisms from \mathcal{A}_f to \mathcal{A}_g by $\mathcal{I}(\mathcal{A}_f, \mathcal{A}_g)$. By [2, Theorem 3.7], for any $\Psi \in \mathcal{I}(\mathcal{A}_f, \mathcal{A}_g)$, there exists a necessarily unique biholomorphic map $\widehat{\Psi} : \mathcal{D}_g(\mathbb{C}) \rightarrow \mathcal{D}_f(\mathbb{C})$ such that, for all $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathcal{D}_g(\mathbb{C})$ and $a \in \mathcal{A}_f$, we have $\langle \lambda, \Psi(a) \rangle_1 = \langle \widehat{\Psi}(\lambda), a \rangle_1$. Moreover, by [2, Theorem 3.18] and [3, Theorem 3.2], if $\widehat{\Psi}(0) = 0$ then f and g are scale-permutation equivalent.

Thus, let us assume that there exists an isomorphism $\Phi : \mathcal{A}_f \rightarrow \mathcal{A}_g$. In [3, Theorem 3.4], we were able to show that if either $\mathcal{D}_f(\mathbb{C})$ or $\mathcal{D}_g(\mathbb{C})$ is not biholomorphic to the unit ball $\mathbb{B}_n = \{(z_1, \dots, z_n) \in \mathbb{C}^n : \sum_{j=1}^n z_j \bar{z}_j \leq 1\}$ of \mathbb{C}^n , then $\widehat{\Psi}(0) = 0$, which implies that f and g are scale-permutation equivalent. Thus, it remains to consider the case where $\mathcal{D}_f(\mathbb{C})$ and $\mathcal{D}_g(\mathbb{C})$ are \mathbb{B}_n , up to replacing f and g by scale-permutation equivalent symbols.

Let $\omega = \widehat{\Phi}^{-1}(0)$. If $\omega = 0$ then by [3, Theorem 3.2], we can already conclude that f and g are scale-permutation equivalent. Henceforth, we shall assume $\omega \neq 0$. We adapt an argument in [9]. Let $D_g = \mathbb{C}\omega \cap \mathbb{B}_n$ and $D_f = \widehat{\Phi}(D_g)$. By construction, D_g is a disk in the plane $\mathbb{C}\omega$. Now, as a conformal self-map of the unit ball, $\widehat{\Phi} = \gamma \circ \varphi_\omega$ where φ_ω is the conformal map $\varphi_\omega = \frac{\psi_\omega}{|\psi_\omega|^2}$ where $\psi_\omega : z \in \mathbb{B}_n \mapsto \omega + (1 - |\omega|^2) \frac{\omega - x}{|\omega - x|^2}$, and γ is unitary. Indeed, φ_ω is easily seen to be a conformal map such that $\varphi_\omega \circ \varphi_\omega$ is the identity and $\varphi_\omega(0) = \omega$. Thus $\widehat{\Phi} \circ \varphi_\omega(0) = 0$ and hence, by Cartan's Lemma, $\widehat{\Phi} \circ \varphi_\omega$ is a unitary γ . Hence $\widehat{\Phi} = \widehat{\Phi} \circ \varphi_\omega \circ \varphi_\omega = \gamma \circ \varphi_\omega$. Now, φ_ω maps the plane $\mathbb{C}\omega$ to itself by construction, so $D_f = \widehat{\Phi}(D_g) = \gamma(D_g)$, and since γ is unitary, D_f is also a disk.

We now set $G = \{z \in D_g : \exists \Psi \in \mathcal{I}(\mathcal{A}_f, \mathcal{A}_g), \widehat{\Psi}(z) = 0\}$ and $F = \{z \in D_f : \exists \Psi \in \mathcal{I}(\mathcal{A}_f, \mathcal{A}_f), \widehat{\Psi}(0) = z\}$. The first observation is that G and F are circular domains. Indeed, note first that for any $\lambda \in \mathbb{C}$, $|\lambda| = 1$, we can extend the function which, to each $j \in \{1, \dots, n\}$, maps W_j^g to $\bar{\lambda} W_j^g$, to an automorphism Λ of \mathcal{A}_g by [2, Lemma 4.4]. Now, for any $b \in G$ there exists $\Psi \in \mathcal{I}(\mathcal{A}_f, \mathcal{A}_g)$ such that $\widehat{\Psi}(b) = 0$. We have $\widehat{\Lambda \circ \Psi} = \widehat{\Psi} \circ \widehat{\Lambda}$ so $\widehat{\Lambda \circ \Psi}(\lambda b) = 0$ and $\Psi \circ \Lambda \in \mathcal{I}(\mathcal{A}_f, \mathcal{A}_g)$, so $\lambda b \in G$ since $\lambda b \in D_g$ as D_g is a disk, hence circular. The same argument of course applies to F .

Now, by construction, $\omega \in G$. Thus G contains the circle $\mathbb{T}_g = \{\lambda\omega : |\lambda| = 1\}$. Since $\widehat{\Phi}$ restricted to the disk D_g is a Möbius map whose poles lie outside of D_g , it preserves circles. Let $\mathbb{T}_f = \widehat{\Phi}(\mathbb{T}_g)$. On the other hand, if $b \in G$ then there exists $\Psi \in \mathcal{I}(\mathcal{A}_f, \mathcal{A}_g)$ such that $\widehat{\Psi}(b) = 0$, so $\widehat{\Phi}(b) = \widehat{\Phi} \circ \widehat{\Psi}^{-1}(0) = \widehat{\Psi}^{-1} \circ \widehat{\Phi}(0)$ and $\Psi \circ \Phi \in \mathcal{I}(\mathcal{A}_f, \mathcal{A}_f)$. Hence $\widehat{\Phi}(b) \in F$. Since $\omega \in \mathbb{T}_g$ we conclude that $0 \in \mathbb{T}_f$.

Hence, F contains the circle \mathbb{T}_f containing the origin. Since F is circular, one checks easily that F contains the interior of \mathbb{T}_f . Since $G = \widehat{\Phi}^{-1}(F)$ by a similar argument as above, we conclude that G contains the interior of $\mathbb{T}_g = \widehat{\Phi}^{-1}(\mathbb{T}_f)$, which in turn contains 0. Hence, there exists $\Psi \in \mathcal{I}(\mathcal{A}_f, \mathcal{A}_g)$ such that $\widehat{\Psi}(0) = 0$. By [3, Theorem 3.2], we conclude that f and g are scale-permutation equivalent. \square

References

- [1] A. Arias, F. Latrémolière, Ergodic actions of convergent Fuchsian groups on noncommutative Hardy algebras, Proc. Amer. Math. Soc. 139 (7) (2011) 2485–2496, arXiv:1010.5840.
- [2] A. Arias, F. Latrémolière, Isomorphisms of non-commutative domain algebras, J. Oper. Theory 66 (2) (2011) 425–450, arXiv:0902.0195.
- [3] A. Arias, F. Latrémolière, Isomorphisms of non-commutative domains II, J. Oper. Theory (2011), 15 pp., arXiv:1010.5838, in press.
- [4] A. Arias, G. Popescu, Noncommutative interpolation and Poisson transforms, Israel J. Math. 115 (2000) 205–234.
- [5] W. Arveson, Subalgebras of C^* -algebras. III. Multivariable operator theory, Acta Math. 181 (2) (1998) 159–228.
- [6] K. Davidson, E. Katsoulis, D. Pitts, The structure of free semigroup algebras, J. Reine Angew. Math. 533 (2001) 99–125.
- [7] K. Davidson, D. Pitts, The algebraic structure of non-commutative analytic Toeplitz algebras, Math. Ann. 311 (1998) 275–303.
- [8] K. Davidson, D. Pitts, Nevanlinna–Pick interpolation for non-commutative analytic Toeplitz algebras, Integral Equations Operator Theory 31 (3) (1998) 321–337.
- [9] K. Davidson, C. Ramsey, O. Shalit, The isomorphism problem for some universal operator algebras, Adv. Math. 228 (1) (2011) 167–218.
- [10] G. Popescu, Von Neumann inequality for $(B(\mathcal{H})^n)_1$, Math. Scand. 68 (2) (1991) 292–304.
- [11] G. Popescu, Operator theory on noncommutative domains, Mem. Amer. Math. Soc. 205 (954) (2010), vi+124 pp.
- [12] T. Sunada, Holomorphic equivalence problem for bounded Reinhardt domains, Math. Ann. 235 (1978) 111–128.
- [13] P. Thullen, Zu den Abbildungen durch analytische Funktionen mehrerer Veränderlichen, Math. Ann. 104 (1931) 244–259.