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# Controllability of cascade coupled systems of multi-dimensional evolution PDEs by a reduced number of controls

## *Contrôlabilité de systèmes multi-dimensionnels couplés en cascade par un nombre réduit de contrôles*

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Presented by Jean-Michel Bony

This Note is dedicated to the memory of my father Abdallah Boussouira

Cette Note est dédicacée en mémoire de mon père Abdallah Boussouira

ABSTRACT

We prove controllability results for abstract systems of weakly coupled $N$ evolution equations in cascade by a reduced number of boundary or locally distributed controls ranging from a single up to $N-1$ controls. We give applications to cascade coupled systems of $N$ multi-dimensional hyperbolic, parabolic and diffusive equations.



RÉSUMÉ

Nous démontrons qu'il est possible de contrôler des systèmes de $N$ équations d'évolution faiblement couplées en cascade par un nombre réduit de contrôles frontière ou localement distribués, le nombre de contrôle pouvant varier de 1 à $N-1$. Nous donnons des applications aux systèmes couplés multi-dimensionnels en cascade hyperboliques, paraboliques et de Schrödinger.



### Version française abrégée

La contrôlabilité à zéro de systèmes couplés d'équations paraboliques ou diffusives par un nombre réduit de contrôles est une question ardue qui suscite beaucoup d'intérêt depuis plus d'une dizaine d'années, tout particulièrement dans les cas où les zones de couplage et de contrôle ne s'intersectent pas. Ces systèmes prennent la forme (1) avec $\theta = 0$ (resp. $\theta = \pi/2$) dans le cas parabolique (resp. dans le cas de Schrödinger), et où $\Omega$ est un ouvert à frontière suffisamment régulière dans $\mathbb{R}^d$, $Y = (y_1, \ldots, y_N)$ est l'état à contrôler, $\mathscr{C}$ est l'opérateur (borné) de couplage et $B$ celui du contrôle (borné ou non borné) et $v = (v_1, \ldots, v_m)$ le contrôle.

Ces systèmes ont été particulièrement étudiés dans le cas de systèmes paraboliques d'ordre 2 couplés en cascade, c'est-à-dire pour lesquels $N = 2$, $m = 1$ avec $Bv = (0, v\mathbf{1}_\omega)^t$ et $\mathscr{C}$ est donné par (2). Ces systèmes apparaissent naturellement dans l'étude de l'existence de contrôles insensibilisants pour l'équation de la chaleur scalaire [24,10,29,11,12,30]. Des résultats positifs de contrôlabilité à zéro [29,6,7,17,18,21] ont été obtenus dans les cas où $O \cap \omega \neq \emptyset$. Kavian et de Teresa [20] ont montré un résultat de continuation unique pour des systèmes paraboliques couplés en cascade d'ordre 2 dans les cas $O \cap \omega = \emptyset$.

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Indépendamment, la question de la contrôlabilité exacte indirecte de systèmes hyperboliques symétriques d'ordre 2 de la forme (3) a été étudiée par l'auteur dans [1,2] en introduisant une méthode d'énergie à deux niveaux (cas de couplages coercifs). Ces résultats ont été récemment étendus par l'auteur et Léautaud [4,5] aux cas de couplages partiellement coercifs et ont permis de déduire des résultats de contrôlabilité à zéro de systèmes couplés paraboliques symétriques dans des cas où $\omega \cap O = \emptyset$ (ou $\omega \cap \Gamma_1 = \emptyset$ dans le cas de contrôle frontière). Dans un travail récent Rosier et de Teresa [27], ont obtenu des résultats positifs de contrôlabilité de systèmes couplés en cascade d'ordre 2 hyperboliques sous une hypothèse forte de périodicité du semi-groupe associé à une seule équation libre (sans couplage) avec applications au cas de systèmes couplés en cascade d'ordre 2 paraboliques ou de Schrödinger en dimension 1 d'espace (cas parabolique) ou dans des carrés (cas Schrödinger avec condition de Neumann) dans des cas où $\omega \cap O = \emptyset$. Dehman, Léautaud et Le Rousseau [22, chapter 6, p. 231] ont montré un résultat de contrôlabilité pour des systèmes en cascade d'ordre 2 avec temps minimal de contrôle dans une variété riemannienne sans bord par des techniques d'analyse micro-locale et en utilisant le principe de travailler dans des espaces d'énergie affaiblie pour la composante non observée (cf. [1,2]).

Nous généralisons ces résultats aux cas de systèmes couplés en cascade hyperboliques, paraboliques ou de Schrödinger d'ordre 2 sans hypothèse de périodicité dans les Théorèmes 2.1, 2.2 et 2.3 donnés dans la partie anglaise. Nous indiquons par ailleurs que ces différents résultats se généralisent aux cas de systèmes couplés en cascade hyperboliques, paraboliques ou de Schrödinger d'ordre $N$ avec $N-p$ contrôles, avec $N \geqslant 2$ et $p$ variant de $N-1$ à 1 et des régions de couplage qui n'intersectent pas les zones de contrôle (frontière ou localement distribué). En particulier, nous montrons qu'il est possible de contrôler un système couplé multi-dimensionnel en cascade d'ordre $N$, hyperbolique parabolique ou de type diffusif, par un seul contrôle frontière ou localement distribué, la zone de contrôle n'intersectant aucune des zones de couplages localisés.

Ces résultats sont basés sur une généralisation de la méthode d'énergie à deux niveaux [2] et de son extension récente [5] introduite pour des systèmes couplés symétriques hyperboliques, à des systèmes couplés en cascade hyperboliques d'ordre $N$, $N \geqslant 2$.

## 1. Introduction

The question of null controllability results for coupled parabolic or diffuse equations is a challenging issue since more than a decade, especially in the cases of localized coupling and control regions with empty intersection and in case of boundary control and localized couplings as well. Such $N$-coupled parabolic or diffusive control systems are given as

$$\begin{cases} e^{i\theta} y_t - \Delta y + \mathscr{C} y = Bv, & \text{in } Q_T = \Omega \times (0,T), \\ y = 0, & \text{on } \Sigma_T = \partial\Omega \times (0,T), \\ y(0,.) = y_0(.), & \text{in } \Omega, \end{cases} \tag{1}$$

with $\theta = 0$ (resp. $\theta = \pi/2$) in the parabolic case (resp. for Schrödinger case) and where $\Omega$ is an open non-empty subset in $\mathscr{R}^d$ with a smooth boundary $\Gamma$, $Y = (y_1, \ldots, y_N)$ is the state to be controlled, $\mathscr{C}$ is a coupling bounded operator on $(L^2(\Omega))^N$, $B$ is either a bounded control operator from $(L^2(\Omega))^m$ to $(L^2(\Omega))^N$ or may act only on a part of the boundary of $\Omega$ for some components of the above system, and $v$ is the control.

The above systems have received a lot of attention in the case of cascade 2-coupled parabolic systems, that is when $N=2$ and $\mathscr{C}$ has the form

$$\mathscr{C} = \begin{pmatrix} 0 & \mathbf{1}_O \\ 0 & 0 \end{pmatrix} \tag{2}$$

where $m=1$ and $Bv = (0, v\mathbf{1}_\omega)^t$. Here $O$ and $\omega$ are open non-empty subsets of $\Omega$ standing respectively for the coupling and control regions and $\mathbf{1}_O$ stands for the characteristic function of the set $O$.

De Teresa [29] has studied null controllability results for 2-coupled cascade parabolic systems, motivated by the determination of insensitizing controls for the heat equation in the case $\omega \cap O \neq \emptyset$. We also refer to [29,6,7,17,18,21] for null controllability results on coupled parabolic systems by a single control force for either constant coupling operators and locally distributed control, or localized coupling operators and locally distributed control regions with a non-empty intersection between control and coupling regions. These results are based on Carleman estimates for the observability of the adjoint system. In the case $\omega \cap O = \emptyset$, Kavian and de Teresa [20] proved a unique continuation result for a 2-coupled cascade systems of parabolic equations. We refer the reader to the survey paper [8] which presents the state-of-the-art on coupled parabolic systems. Also, local null controllability results have been obtained for nonlinearly coupled 2-systems of parabolic equations in [15].

On the other hand and independently, the question of controllability of symmetric weakly 2-coupled hyperbolic systems by a single control has been first addressed by the author in [1,2] by means of a *two-level energy method*. These systems have the form

$$\begin{cases} y_{1,tt} - \Delta y_1 + C y_2 = Bv, & \text{in } Q_T = \Omega \times (0,T), \\ y_{2,tt} - \Delta y_2 + C^\star y_1 = 0, & \text{in } Q_T = \Omega \times (0,T), \\ y_i = 0 \; i = 1,2, & \text{on } \Sigma_T = \partial\Omega \times (0,T), \\ (y_i, y_{i,t})(0,.) = \big(y_i^0, y_i^1\big)(.) \; i = 1,2, & \text{in } \Omega. \end{cases} \tag{3}$$

This method has been introduced in [1,2] in a general abstract setting to prove positive controllability results for coercive bounded coupling operators $C$ (case of globally distributed couplings) and unbounded control operators (case of boundary control). These results have been recently extended by the author and Léautaud in [4,5] to the case of symmetric weakly 2-coupled hyperbolic systems with localized couplings and localized distributed control as well as boundary control. Moreover, using the transmutation method [25,26], applications to symmetric 2-coupled systems of parabolic and diffusive equations have also been deduced. These results are valid for multi-dimensional wave-like equations under the condition that both the coupling and control regions satisfy the Geometric Control Condition of Bardos, Lebeau and Rauch [9] (see also [13,14] for weaker smoothness assumptions on $\Omega$ and the coefficients of the elliptic operator $A$), in particular for cases $Cz = pz$ and $Bv = bv$ where $p$ and $b$ are nonnegative functions with supports containing respectively $\overline{O}$ and $\overline{\omega}$ with $\omega \cap O = \emptyset$. In a recent work, Rosier and de Teresa [27] considered a 2-coupled system of cascade hyperbolic equations under a strong hypothesis, that is a periodicity assumption of the semigroup associated to a single uncoupled equation. They give applications to 2-coupled systems of cascade one-dimensional heat equations and to 2-coupled systems of cascade Schrödinger equations in an $n$-dimensional interval with empty intersection between the control and coupling regions. The method is linked to Dáger's [16] approach and strongly relies on the periodicity assumption of the semigroup for the single free equation. There is also a recent result for 2-coupled cascade systems with localized control by Dehman, Léautaud and Le Rousseau [22, chapter 6, p. 231] in a $\mathscr{C}^\infty$ compact connected riemannian manifold without boundary with characterization of the minimal control time using micro-local analysis and the idea to work in weakened energy spaces for the unobserved component (see [1,2]).

This Note concerns the exact controllability of coupled $N$ systems of second order hyperbolic abstract equations in cascade by a reduced number of either boundary or locally distributed controls. We give sufficient conditions on the control and coupling operators for exact controllability to hold in case of $N - p$ controls, $p$ varying from 1 to $N - 1$. We then give applications to cascade systems of wave equations, parabolic equations and Schrödinger equations. We first introduce the abstract setting. Let $H$ and $G$ denote Hilbert spaces with respective norm $|\cdot|$, $|\cdot|_G$ and scalar product $\langle\, ,\rangle$, $\langle\, ,\rangle_G$. We consider the following control cascade system:

$$\begin{cases} y_1'' + Ay_1 + C^\star y_2 = 0, \\ y_2'' + Ay_2 = Bv, \\ \big(y_i, y_i'\big)(0) = \big(y_i^0, y_i^1\big) \quad \text{for } i = 1, 2, \end{cases} \tag{4}$$

where $A$ satisfies

$$\text{(A1)} \quad \begin{cases} A : D(A) \subset H \mapsto H, A^\star = A, \\ \exists\, \omega > 0, |Au| \geqslant \omega |u| \quad \forall u \in D(A), \end{cases} \tag{5}$$

and where $C$ is a bounded operator in $H$, $B \in \mathscr{L}(G, H)$ (resp. $B \in \mathscr{L}(G, (D(A))'))$ is the control operator in the case of bounded (resp. unbounded) control, and $v$ is the control. We set $\mathbf{B}^*(w, w') = Bw'$ (resp. $\mathbf{B}^*(w, w') = Bw$) when $B \in \mathscr{L}(G, H)$ (resp. $B \in \mathscr{L}(G, (D(A))')$). We also set $H_k = D(A^{k/2})$ for $k \in \mathbb{N}$, with the convention $H_0 = H$. The set $H_k$ is equipped with the norm $|\cdot|_k$ defined by $|A^{k/2}\cdot|$ and the associated scalar product. It is a Hilbert space. We denote by $H_{-k}$ the dual space of $H_k$ with the pivot space $H$. We equip $H_{-k}$ with the norm $|\cdot|_{-k} = |A^{-k/2}\cdot|$. We also define the *local* natural energy as

$$e_1\big(W(t)\big) = \frac{1}{2}\big(\big|A^{1/2}w\big|^2 + |w'|^2\big), \tag{6}$$

where $W = (w, w')$.

We are interested in the *indirect exact controllability* by $L^2$ controls for the above system. That is, we are concerned with the issue whether for a sufficiently large time $T$, for all initial data $(y_1^0, y_2^0, y_1^1, y_2^1)$ in a suitable space, it is possible to find a control $v \in L^2((0, T); G)$ such that the solution $Y = (y_1, y_2, y_1', y_2')$ of (4) satisfies $Y(T) = 0$. Here the control appears only in the equation for the second component, thus if exact controllability holds it means that the first component is *indirectly* controlled, indeed through the coupling with a *directly* controlled equation.

We shall assume that, the adjoint of $B$ is an admissible observation for one equation, that is

$$\text{(A2)} \quad \begin{cases} \forall T > 0\ \exists C > 0, \text{ such that all the solutions } w \text{ of } w'' + Aw = f \text{ satisfy} \\ \displaystyle\int_0^T \big\|\mathbf{B}^*(w, w')\big\|_G^2\, dt \leqslant C\left(e_1\big(W(0)\big) + e_1\big(W(T)\big) + \int_0^T e_1\big(W(t)\big)\, dt + \int_0^T \|f\|_H^2\, dt\right), \end{cases} \tag{7}$$

where $W = (w, w')$. Thanks to this hypothesis, the solution of (4) can be defined by the method of transposition [23]. More precisely, for any $Y_0 = (y_1^0, y_2^0, y_1^1, y_2^1) \in H_2 \times H_1 \times H_1 \times H_0$ (resp. any $Y_0 \in H_1 \times H_0 \times H_0 \times H_{-1}$) and any $v \in L^2((0, T); G)$, (4) admits a unique solution $Y \in \mathscr{C}([0, T]; H_2 \times H_1 \times H_1 \times H_0)$ (resp. $\mathscr{C}([0, T]; H_1 \times H_0 \times H_0 \times H_{-1})$) when $B \in \mathscr{L}(G, H)$ (resp. $B \in \mathscr{L}(G, (D(A))')$). We refer to [2,5] for more details.

## 2. Main results for 2-coupled cascade systems

We assume the following observability inequalities for a single equation:

$$\text{(A3)} \quad \begin{cases} \exists T_1 > 0, T_2 > 0, \text{ such that all the solutions } w \text{ of } w'' + Aw = 0 \text{ satisfy} \\ \displaystyle\int_0^T \left\| \mathbf{B}^*(w, w') \right\|_G^2 \, dt \geqslant C_1(T) e_1\big(W(0)\big), \quad \forall T > T_1, \\ \displaystyle\int_0^T \left\| \Pi_p w' \right\|_H^2 \, dt \geqslant C_2(T) e_1\big(W(0)\big), \quad \forall T > T_2, \end{cases} \tag{8}$$

and that $C$ satisfies

$$\text{(A4)} \quad \begin{cases} C \in \mathscr{L}(H_k) \text{ for } k \in \{0, 1, 2\}, \ \|C\| = \beta, \ |Cw|^2 \leqslant \beta \langle Cw, w \rangle \ \forall w \in H, \\ \exists \alpha > 0 \text{ and } \Pi_p \in \mathscr{L}(H) \text{ such that, } \alpha |\Pi_p w|^2 \leqslant \langle Cw, w \rangle \ \forall w \in H. \end{cases} \tag{9}$$

The main results of this Note are the following:

**Theorem 2.1.** *Assume that the hypotheses* (A1)–(A4) *are satisfied.*

(i) *Let $\mathbf{B}^*(w, w') = Bw'$ with $B \in \mathscr{L}(G, H)$. Then, there exists a time $T^\star \geqslant \max(T_1, T_2)$ such that for all $T > T^\star$, and all $Y_0 \in H_2 \times H_1 \times H_1 \times H_0$, there exists a control function $v \in L^2((0, T); G)$ such that the solution $Y = (y_1, y_2, y_1', y_2')$ of* (4) *satisfies $Y(T) = 0$.*

(ii) *Let $\mathbf{B}^*(w, w') = Bw$ with $B \in \mathscr{L}(G, H_2')$. Then, there exists a time $T^\star \geqslant \max(T_1, T_2)$ such that for all $T > T^\star$, and all $Y_0 \in H_1 \times H_0 \times H_0 \times H_{-1}$, there exists a control function $v \in L^2((0, T); G)$ such that the solution $Y = (y_1, y_2, y_1', y_2')$ of* (4) *satisfies $Y(T) = 0$.*

Let us now give applications to 2-coupled cascade parabolic and Schrödinger systems. We consider the locally distributed control system

$$\begin{cases} e^{i\theta} y_{1,t} - \Delta y_1 + c y_2 = 0, \quad \text{in } Q_T = \Omega \times (0, T), \\ e^{i\theta} y_{2,t} - \Delta y_2 = bv, \quad \text{in } Q_T = \Omega \times (0, T), \\ y_1 = y_2 = 0, \quad \text{on } \Sigma_T = \partial\Omega \times (0, T), \\ y_i(0, .) = y_i^0(.), \quad \text{in } \Omega, i = 1, 2, \end{cases} \tag{10}$$

and the boundary control system

$$\begin{cases} e^{i\theta} y_{1,t} - \Delta y_1 + c y_2 = 0, \quad \text{in } Q_T = \Omega \times (0, T), \\ e^{i\theta} y_{2,t} - \Delta y_2 = 0, \quad \text{in } Q_T = \Omega \times (0, T), \\ y_1 = 0, y_2 = bv, \quad \text{on } \Sigma_T = \partial\Omega \times (0, T), \\ y_i(0, .) = y_i^0(.), \quad \text{in } \Omega, i = 1, 2, \end{cases} \tag{11}$$

where we assume for the sequel $\theta \in [-\pi/2, \pi/2]$, $c \geqslant 0$ on $\Omega$, $\{c > 0\} \supset \overline{O}$ and $b \geqslant 0$ on $\Omega$, $\{b > 0\} \supset \overline{\omega}$ (resp. $b \geqslant 0$ on $\Gamma$, $\{b > 0\} \supset \overline{\Gamma_1}$) in the case of system (10) (resp. (11)), where $O$ and $\omega$ are open subsets of $\Omega$, and where $\Gamma_1 \subset \Gamma$.

**Theorem 2.2.** *Assume that the subsets $O$ and $\omega$ (resp. $O$ and $\Gamma_1$) satisfy the Geometric Control Condition and that $\theta = 0$. Then, for all $T > 0$, for all initial data $(y_1^0, y_2^0) \in (L^2(\Omega))^2$ (resp. $(y_1^0, y_2^0) \in (H^{-1}(\Omega))^2$), there exists a control $v \in L^2((0, T) \times \Omega)$ (resp. $v \in L^2((0, T) \times \Gamma_1)$) such that the solution of* (10) (*resp.* (11)) *satisfies $(y_1, y_2)(T, .) = 0$ in $\Omega$.*

**Theorem 2.3.** *Assume that the subsets $O$ and $\omega$ (resp. $O$ and $\Gamma_1$) satisfy the Geometric Control Condition and that $\theta = \pi/2$. Then, for all $T > 0$, for all initial data $(y_1^0, y_2^0) \in H_0^1(\Omega) \times L^2(\Omega)$ (resp. $(y_1^0, y_2^0) \in L^2(\Omega) \times H^{-1}(\Omega)$), there exists a control $v \in L^2((0, T) \times \Omega)$ (resp. $v \in L^2((0, T) \times \Gamma_1)$) such that the solution of* (10) (*resp.* (11)) *satisfies $(y_1, y_2)(T, .) = 0$ in $\Omega$.*

**Remarks.** The above geometric conditions on the coupling region $O$ and the control region $\omega$ (resp. $\Gamma_1$) hold for various examples of subsets $O$ and $\omega$ (resp. $O$ and $\Gamma_1$) such that $O \cap \omega = \emptyset$ (resp. $O \cap \Gamma_1 = \emptyset$) for one-dimensional as well as multi-dimensional sets $\Omega$. In particular it holds for arbitrary open non-empty subsets $O$ and $\omega$ in the one-dimensional case. In the multi-dimensional setting, (*GCC*) is a necessary condition for the internal and boundary observability and controllability of the scalar wave equation when $\partial\Omega$ has no contact of infinite order with its tangent (see [9]). Hence this condition is quasi-optimal for the wave equations. This condition is not natural for the internal (resp. boundary) observability and controllability of the scalar heat equation, which holds for arbitrary non-empty region of $\Omega$ (resp. $\Gamma$). Moreover, (*GCC*) is also not a necessary condition in some cases for the scalar Schrödinger equation, in particular for rectangles (see [19,28]). The above theorems strongly extends Rosier and de Teresa's results when the Geometric Control Condition is fulfilled. It is, to our knowledge, the first result showing that null controllability of 2-coupled cascade parabolic systems holds in a multi-dimensional setting with empty intersection between the coupling and control regions in the case of boundary control. It also extends the results for 2-coupled cascade Schrödinger systems in a multi-dimensional setting without any further periodicity assumption. The time $T^\star$ in Theorem 2.1 is not optimal contrarily to the results in [22]. On the other hand, the approach in this Note is based on direct arguments contrarily to [22], which is based on contradiction arguments. One can note also that the results of this Note hold without smallness conditions on the coupling operators.

## 3. Further generalizations to $N$-coupled cascade systems

We more generally consider $N$-coupled control hyperbolic systems driven by $N-p$ controls, where $p \in \{1, \ldots, N-1\}$ as follows

$$\begin{cases} y_1'' + Ay_1 + C_{21}^\star y_2 + \cdots C_{N1}^\star y_N = 0, \\ y_2'' + Ay_2 + C_{32}^\star y_3 + \cdots C_{N2}^\star y_N = 0, \\ \vdots \\ y_p'' + Ay_p + C_{p+1p}^\star y_p + \cdots C_{Np}^\star y_N = 0, \\ \vdots \\ y_{p+1}'' + Ay_{p+1} C_{p+2p+1}^\star y_{p+1} + \cdots C_{Np+1}^\star y_N = B_{p+1} v_{p+1}, \\ \vdots \\ y_{N-1}'' + Ay_{N-1} + C_{NN-1}^\star y_N = B_{N-1} v_{N-1}, \\ \vdots \\ y_N'' + Ay_N = B_N v_N, \\ (y_i, y_i')(0) = (y_i^0, y_i^1) \text{ for } i = 1, \ldots N, \end{cases} \tag{12}$$

where $A$ satisfies (A1), the coupling operators $C_{ij}$ are bounded in $H$ for all $i \in \{2, \ldots, N\}$ and all $j \in \{1, \ldots, i-1\}$. We recover system (12) when $N = 2$, setting $C_{21} = C$, $B_2 = B$ and $v_2 = v$. For each $k \in \{p+1, \ldots, N\}$, the control operators $B_k$ can either satisfy $B_k \in \mathscr{L}(G_k, H)$ (bounded case) or $B_k \in \mathscr{L}(G_k, (D(A))')$ (unbounded case) where $G_k$ are given Hilbert spaces. Moreover we consider the case of $L^2$ controls, that is, we assume that the controls $v_k \in L^2((0,T); G_k)$ for $k \in \{p+1, \ldots, N\}$.

In [3], we give sufficient conditions on the coupling operators $C_{ij}$ for $i \in \{2, \ldots, N\}$, $j \in \{1, \ldots, i-1\}$, on the control operators $B_k$ for $k \in \{p+1, \ldots, N\}$, so that for sufficiently large time $T$, there exist controls $v_k \in L^2((0,T); G_k)$ for $k \in \{p+1, \ldots, N\}$, so that the solution $Y = (y_1, \ldots, y_N)^t$ of (12) satisfies $Y(T) = 0$. Thanks to the transmutation method, we give further applications to the null controllability of $N$-coupled parabolic or diffusive control cascade systems by either 1, 2, up to $N-1$ controls, each of them possibly chosen either locally distributed or localized on a part of the boundary. Moreover these results hold for localized couplings and localized or boundary controls such that none of the coupling regions meet the control regions. In particular, we give nontrivial examples of $N$-coupled systems with $N$ an arbitrary integer greater than 2 which can be driven to equilibrium at time $T$ by a single either locally distributed or boundary control. The control spaces depend on the number of requested controls.

These results are based on a generalization of the two-level energy method [2] and its recent extension [5] for 2-coupled symmetric hyperbolic control systems under a smallness condition on the coupling operator, to $N$-coupled cascade hyperbolic control systems without smallness conditions. They also rely on the obtention of suitable observability estimates for the adjoint system.

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