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Ordinary Differential Equations

Gibbs measure evolution in radial nonlinear wave and Schrödinger equations on the ball ☆

Mesures de Gibbs et équations non-linéaires des ondes et de Schrödinger sur la boule

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A R T I C L E I N F O

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A B S T R A C T

We establish new results for the radial nonlinear wave and Schrödinger equations on the ball in $\mathbb{R}^2$ and $\mathbb{R}^3$, for random initial data. More precisely, a well-defined and unique dynamics is obtained on the support of the corresponding Gibbs measure. This complements results from Burq and Tzvetkov (2008) [8,9] and Tzvetkov (2006, 2008) [10, 11].

R É S U M É

On démontre des résultats nouveaux sur les solutions radiales de l'équation des ondes et l'équation de Schrödinger sur la boule B dans $\mathbb{R}^2$ et $\mathbb{R}^3$ pour des conditions initiales aléatoires. Plus exactement, on établit une dynamique bien définie et unique sur le support de la mesure de Gibbs. Ceci complète des résultats de Burq et Tzvetkov (2008) [8,9] et Tzvetkov (2006, 2008) [10,11].

Version française abrégée

On considère les équations non linéaires (radiales et défocalisantes) des ondes (NLW) et Schrödinger (NLS) sur la boule B dans $\mathbb{R}^2$ et $\mathbb{R}^3$:

$$\left(\partial_t^2 - \Delta\right)w + |w|^\alpha w = 0 \tag{NLW}$$

$$(i\partial_t + \Delta)u - |u|^\alpha u = 0 \tag{NLS}$$

ainsi que leurs versions tronquées (en introduisant un projecteur P_N sur $[e_1, \dots, e_N]$, où les e_n sont les fonctions propres de Dirichlet sur B) et les mesures de Gibbs correspondantes. On établit des estimées espace-temps et une dynamique unique, quand $N \to \infty$, dans les modèles (NLW) en dimension 3 pour $\alpha < 4$ (le cas $\alpha < 3$ étant traité dans [8,9]), et (NLS) en dimension 2, α arbitraire (voir [11] pour le cas $\alpha < 4$) et en dimension 3 pour $\alpha = 2$.

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1. The equations and the Gibbs measure

Denote $B = B_d$ the unit ball in $\mathbb{R}^d$. We consider the defocusing nonlinear wave (NLW) and nonlinear Schrödinger (NLS) equation

$$\begin{cases} (\partial_t^2 - \Delta)w + |w|^\alpha w = 0 \\ (w, \partial_t w)|_{t=0} = (f_1, f_2) \end{cases} \tag{1}$$

$$\begin{cases} (i\partial_t + \Delta)u - |u|^\alpha u = 0 \\ u|_{t=0} = \phi \end{cases} \tag{2}$$

on the spatial domain B with Dirichlet boundary conditions and with radial initial data. Thus (f_1, f_2) is real valued and radial in (1), ϕ is a radial complex-valued function in (2). It is convenient to rewrite (1) as a first order equation in t, introducing the complex function $u = w + i(\sqrt{-\Delta})^{-1}\partial_t w$. Then (1) turns into the equation

$$\begin{cases} (i\partial_t - \sqrt{-\Delta})u + (\sqrt{-\Delta})^{-1}\big(|\operatorname{Re} u|^\alpha \operatorname{Re} u\big) = 0 \\ u|_{t=0} = \phi = f_1 + i(\sqrt{-\Delta})^{-1} f_2. \end{cases} \tag{3}$$

Both (2), (3) are Hamiltonian equations taking the respective forms $iu_t = \frac{\partial H}{\partial \bar u}$ and $iu_t = (\sqrt{-\Delta})^{-1}\frac{\partial H}{\partial \bar u}$ with Hamiltonians

$$H(\phi) = \int_B |\nabla\phi|^2 + \frac{2}{2+\alpha}\int_B |\phi|^{\alpha+2} \tag{4}$$

and

$$H(\phi) = \int_B |\nabla\phi|^2 + \frac{2}{2+\alpha}\int_B |\operatorname{Re}\phi|^{\alpha+2}. \tag{5}$$

Denote $\{e_n\}_{n\geqslant 1}$ the radial Dirichlet eigenfunctions of B and

$$P_N\phi = \sum_{n=1}^N \phi_n e_n$$

the projection operator. The 'truncated' equations

$$(i\partial_t + \Delta)u - P_N\big(|u|^\alpha u\big) = 0 \tag{6}$$

$$(i\partial_t - \sqrt{-\Delta})u - P_N\big((\sqrt{-\Delta})^{-1}\big(|\operatorname{Re} u|^\alpha \operatorname{Re} u\big)\big) = 0 \tag{7}$$

where $u(t) = \sum_{n=1}^N u_n(t)e_n$ are globally well-posed in time and correspond to finite dimensional Hamiltonian models. The Gibbs measure

$$\mu_G^{(N)}(d\phi) = e^{-H(\phi)}\prod_1^N d^2\phi \tag{8}$$

is invariant under their respective flow.

2. Statement of the main results

Our results are the continuation of those obtained in [8,9] and [10,11], as we address various cases that were not treated in these papers.

We consider random initial data given by a Gaussian process

$$\phi_\omega = \sum_{n=1}^N \frac{g_n(\omega)}{n\pi} e_n \tag{9}$$

with $\{g_n\}_{n\geqslant 1}$ independent normalized complex Gaussian random variables. The free measure $\mu_F^{(N)}$ induced by the map $\omega \mapsto \phi_\omega$ allows to re-express the Gibbs measure as

$$\mu_G^{(N)} = e^{-\frac{1}{\alpha+2}\int|\phi|^{\alpha+2}}\mu_F^{(N)}. \tag{10}$$

Thus, the Gibbs measure is a weighted version of the free measure and has the advantage of being preserved under the flow. This fact is crucial in the papers cited above and also in the results discussed here. Note that $\phi_\omega \in H^{\frac{1}{2}-}(B)$ almost surely (a.s.). Fixing $\phi = \phi_\omega$ and considering the truncated solutions $u_\phi^N = u^N$, $u^N|_{t=0} = P_N\phi$ (which are well-defined globally in time), there are two natural issues. The first is to establish space–time regularity estimates on u^N that are uniform in N. The second is to prove that for $N \to \infty$, the sequence $\{u^N\}$ converges to a unique limit. Of course, these properties are only valid a.s. in ω.

Theorem 2.1 *(3D NLW). Let $\alpha < 4$. For almost all ω, the solutions u^N of (7), $u^N|_{t=0} = P_N(\phi_\omega)$ satisfy*

$$\sup_N \left\| u^N(t) - e^{it\sqrt{-\Delta}}(P_N\phi) \right\|_{H_x^s} < \infty \tag{11}$$

for all $s < \frac{5-\alpha}{2}$ and $t \in \mathbb{R}$.

Moreover, considering u^N as random variables in ω, the sequence $\{u^N\}$ converges in mean in the space $C_{t<T}H_x^s$ for $s < \frac{1}{2}$, $T < \infty$ arbitrary.

The case $\alpha < 3$ is covered by Theorem 1 in [9].

Theorem 2.2 *(2D NLS). Let $\alpha \in 2\mathbb{Z}_+$ be arbitrary and u^N the solutions of (6), $u^N|_{t=0} = P_N(\phi_\omega)$. Then the sequence $\{u^N\}$ converges in the mean in the space $C_{t<T}H_x^s$ for $s < \frac{1}{2}$, $T < \infty$.*

The assumption $\alpha \in 2\mathbb{Z}_+$ is not essential, and more general sufficiently smooth defocusing nonlinearities may be handled as well. The subquintic case was treated in [11].

Theorem 2.3 *(3D NLS). Let $d = 3$ and consider Eq. (6) with $\alpha = 2$. The solutions u^N, $u^N|_{t=0} = P_N(\phi_\omega)$ converge in the mean in the space $C_{t<T}H_x^s$, $s < \frac{1}{2}$, $T < \infty$.*

3. Comments on the proofs

As in the many earlier works, the arguments are a combination of probabilistic and harmonic analysis techniques; see for instance the classical works [3–5] on this topic and the survey of these results in [6]. We only comment on the proof of Theorem 2.3, which is by far the most delicate.

The starting point is Duhamel's formula on a fixed time interval $[0, T]$

$$u^N(t) = u(t) = e^{it\Delta}(P_N\phi) + i\int_0^t e^{i(t-\tau)\Delta} P_N\big(u|u|^2\big)(\tau)\,\mathrm{d}\tau. \tag{12}$$

The spaces $X_{s,b} = X_{s,b}([0, T])$ are defined in the usual way, see also [1,2] where they were first introduced. Let $s \geqslant 0$ and $b \geqslant 0$. For functions f on $B_3 \times [0, T]$, admitting a representation of the form

$$f(x,t) = \sum_{n=1}^{\infty}\left[\int_{-\infty}^{\infty} f_{n,\lambda}\, e^{2\pi i\lambda t}\,\mathrm{d}\lambda\right] e_n(x) \quad \text{for } x \in B_3,\ 0 \leqslant t \leqslant T \tag{13}$$

where

$$\left(\sum_n \int n^{2s}\big(1 + \big|n^2 - \lambda\big|^{2b}\big)|f_{n,\lambda}|^2\,\mathrm{d}\lambda\right)^{\frac{1}{2}} < \infty \tag{14}$$

we define $\|f\|_{s,b}$ as the inf (14) over all representations (13).

With these notations, it follows that for $\frac{1}{2} < b < 1$

$$\left\|\int_0^t e^{i(t-\tau)\Delta} f(\tau)\,\mathrm{d}\tau\right\|_{s,b} \leqslant C\left(\sum_n \int \frac{n^{2s}|f_{n,\lambda}|^2}{(1 + |n^2 - \lambda|^2)^{1-b}}\,\mathrm{d}\lambda\right)^{\frac{1}{2}}. \tag{15}$$

The inclusions $X_{0,\frac{1}{4}+} \subset L_x^{3-}L_t^2$ and $X_{0+,\frac{1}{2}+} \subset L_x^3 L_t^4$ imply that $X_{0+,b_1} \subset L_x^3 L_t^{\frac{4}{3-4b_1}}$ for $\frac{1}{4} < b_1 < \frac{1}{2}$. It follows by duality that for $\frac{1}{2} < b < \frac{3}{4}$

$$(15) \leqslant \left\|(\sqrt{-\Delta})^{s+} f\right\|_{L_x^{\frac{3}{2}} L_t^{\frac{4}{5-4b}}}. \tag{16}$$

From the Gibbs measure conservation under the flow, one derives the a priori inequality (on finite time intervals)

$$\big\|(\sqrt{-\Delta})^s u\big\|_{L^p_x L^q_t} < C \quad \text{for } p < \frac{6}{1+2s},\ q < \infty. \tag{17}$$

Using (12), (16), (17), it follows that

$$\|u\|_{s,b} < C \quad \text{for } s < \frac{1}{2},\ b < \frac{3}{4}. \tag{18}$$

Recall that $u = u_\phi$, $u|_{t=0} = \phi$ and statements such as (17), (18) require exclusion of small-measure ϕ-sets. We do not elaborate on the quantitative aspects of these matters here.

Next, in order to establish convergence properties for $N \to \infty$, let $N \geqslant N_0$ and estimate using (12) and the preceding

$$\big\|u^N - u^{N_0}\big\|_{0,b} \leqslant \left\| \int_0^t e^{i(t-\tau)\Delta}\big(P_{N_0}u^N\big|P_{N_0}u^N\big|^2 - u^{N_0}\big|u^{N_0}\big|^2\big)(\tau)\,\mathrm{d}\tau \right\|_{0,b} + N_0^{-\frac{1}{4}}. \tag{19}$$

Denoting $u_1 = u^{N_0} - P_{N_0}u^N$ and u_2, u_3 factors u^{N_0}, $P_{N_0}u^N$, the integrand in (19) leads to trilinear expressions of the form

$$\sum_{n,n_1,n_2,n_3} \left[\int_0^t \widehat{u_1(\tau)}(n_1)\overline{\widehat{u_2(\tau)}(n_2)}\widehat{u_3(\tau)}(n_3)e\big(-n^2\tau\big)\,\mathrm{d}\tau\right]\left(\int e_n e_{n_1} e_{n_2} e_{n_3}\,\mathrm{d}x\right) e_n e\big(n^2 t\big) \tag{20}$$

where

$$\left|\int e_n e_{n_1} e_{n_2} e_{n_3}\right| \leqslant C \min(n, n_1, n_2, n_3). \tag{21}$$

Our analysis of (20) is based on arguments closely related to those in [4]. We first break up (20) in dyadic regions $n \sim N$, $n_i \sim N_i$ $(i = 1, 2, 3)$ and distinguish the contributions

$$\big|n^2 - n_1^2 + n_2^2 - n_3^2\big| \geqslant \min(N, N_1, N_2 + N_3)^{\frac{1}{100}} \tag{22}$$

and

$$\big|n^2 - n_1^2 + n_2^2 - n_3^2\big| < \min(N, N_1, N_2 + N_3)^{\frac{1}{100}}. \tag{23}$$

The contribution (22) is handled using $X_{s,b}$-spaces and inequalities of the type

$$\iint \bar{v} u_1 \bar{u}_2 u_3\,\mathrm{d}x\,\mathrm{d}t \lesssim \|v\|_{\sigma_1, 1-b-}\|u_1\|_{\sigma_2, \frac{1}{2}-}\|u_2\|_{\sigma_3, \frac{3}{4}-}\|u_3\|_{L^{6-}_x L^q_t} \tag{24}$$

with $\sigma_1, \sigma_2 \geqslant 0$, $\sigma_1 + \sigma_2 > 0$, $\sigma_3 = \frac{1}{2}-$ or $\sigma_1 = \sigma_2 = 0$, $\sigma_3 > \frac{1}{2}$ and $q > \frac{4}{3-4b}$.

Contributions from (23) are evaluated using further probabilistic considerations, in the spirit of [4] and exploiting the random nature of u_2, u_3.

The most significant terms are

$$\sum_n \left[\int_0^t \widehat{u_1(\tau)}(n)\left(\sum_m \big|\widehat{u(\tau)}(m)\big|^2\left(\int e_n^2 e_m^2\right)\right) e\big(-n^2\tau\big)\right] e_n e\big(n^2 t\big). \tag{25}$$

Replacing the inner sum u by the free solution $e^{it\Delta}\phi = \sum \frac{g_n(\omega)}{n} e_n e(n^2 t)$ leads to an expression of the form

$$\sum_{n<N_0} \log n \left[\int_0^t \widehat{u_1(\tau)}(n) e\big(-n^2\tau\big)\,\mathrm{d}\tau\right] e_n e\big(n^2 t\big). \tag{26}$$

In order to obtain a contractive estimate in u_1, the presence of the $\log n$ factors requires to restrict $t \in [0, T]$, with $T \sim \frac{1}{\log N_0}$.

Moreover, the norm $\|\cdot\|_{0,b}$ has to be slightly weakened to a norm $|||\cdot|||_{0,b}$ by allowing in addition to (13), (14) also expressions

$$f_1(x,t) = \psi(t)\left[\sum_n b_n e_n e\big(n^2 t\big)\right]$$

where

$$|||f_1|||_{0,b} = \left(\|\psi\|_\infty + \|\psi\|_{H^{1/2}}\right)\left(\sum |b_n|^2\right)^{\frac{1}{2}} < \infty.$$

Note that the logarithmic divergency above is barely compatible with the error term in (19).

Remark. An alternative approach of interest would be to apply the normal forms approach on finite time intervals (cf. [7]) in order to make reductions of the Hamiltonian by suitable symplectic transformations.

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