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Functional Analysis

A Note on weak amenability for free products of discrete quantum groups

Une Note sur les produits libres de groupes quantiques discrets faiblement moyennables

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ABSTRACT

We prove that the Cowling–Haagerup constant of a reduced free product of weakly amenable discrete quantum groups with Cowling–Haagerup constant equal to 1 is again equal to 1.

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RÉSUMÉ

Nous prouvons que la constante de Cowling–Haagerup d'un produit libre réduit de groupes quantiques discrets faiblement moyennables de constante de Cowling–Haagerup égale à 1 est encore égale à 1.

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1. Introduction

In geometric group theory, weak amenability is an approximation property which is satisfied by a large class of groups (for example free or even hyperbolic groups [6]) but is strong enough to give interesting properties for the von Neumann algebras associated to these groups (for example related to deformation/rigidity techniques [7,8]). The stability of this property under free products is still an open problem. However, using a very general version of the Khintchine inequality, E. Ricard and Q. Xu were able to prove in [9] that if (G_i) is a family of weakly amenable discrete groups *with Cowling–Haagerup constant equal to* 1, then their free product is again weakly amenable, and its Cowling–Haagerup constant is also equal to 1. The proof uses a classical characterization of the bounded functions on a group giving rise to completely bounded multipliers. This characterization having recently been generalized to arbitrary locally compact quantum groups by M. Daws in [3], we are able to prove an analogue of Ricard and Xu's result in the setting of discrete quantum groups. An extended version of this paper [4] is available at arxiv.org.

2. Definitions and notations

In this section, we introduce the notations and results about discrete quantum groups which will be used in this paper. For two Hilbert spaces H and K, $\mathcal{B}(H,K)$ will denote the set of bounded linear maps from H to K and $\mathcal{B}(H)=\mathcal{B}(H,H)$. In the same way, we use the notations $\mathcal{K}(H,K)$ and $\mathcal{K}(H)$ for compact linear maps. We will denote by $\mathcal{B}(H)_*$ the predual of $\mathcal{B}(H)$, i.e. the Banach space of all normal linear forms on $\mathcal{B}(H)$. On any tensor product $A\otimes B$, we define the flip operator $\Sigma : x\otimes y\mapsto y\otimes x$. We will use the usual leg-numbering notations: for an operator X acting on a tensor product we set $X_{12}:=X\otimes 1$, $X_{23}:=1\otimes X$ and $X_{13}:=(\Sigma\otimes 1)(1\otimes X)(\Sigma\otimes 1)$. For a subset B of a topological vector space C, $\overline{\operatorname{span}}B$ will

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denote the *closed linear span* of B in C. The symbol $\otimes$ will denote the *minimal* (or spatial) tensor product of C*-algebras or the topological tensor product of Hilbert spaces. Let A be a C*-algebra together with a distinguished state φ. Taking the completion of A with respect to the scalar product $\langle a, b\rangle = \varphi(a^*b)$ yields a Hilbert space which will be denoted $L^2(A, \varphi)$. If the scalar product is $\langle a, b\rangle^{op} = \varphi(ab^*)$, then the completion will be denoted $L^2(A, \varphi)^{op}$.

Let $\mathbb{G} = (C(\mathbb{G}), \Delta)$ be a compact quantum group in the sense of [12] and let $\widehat{\mathbb{G}} = (C_0(\widehat{\mathbb{G}}), \widehat{\Delta})$ be its dual discrete quantum group. The Haar state of $\mathbb{G}$ will be denoted h. We will always assume it to be faithful and identify $C(\mathbb{G})$ with its image under the GNS map. We will denote by $L^2(\mathbb{G})$ the Hilbert space of the GNS construction, by ξ_h the cyclic separating vector and by W the unitary operator on $L^2(\mathbb{G}) \otimes L^2(\mathbb{G})$ defined by $W^*(\xi \otimes a\xi_h) = \Delta(a)(\xi \otimes \xi_h)$ for $\xi \in L^2(\mathbb{G})$ and $a \in C(\mathbb{G})$. Then, W is a *multiplicative unitary* in the sense of [1], i.e. $W_{12}W_{13}W_{23} = W_{23}W_{12}$. We have the following equalities:

$$C(\mathbb{G}) = \overline{\operatorname{span}}\big(id \otimes \mathcal{B}\big(L^2(\mathbb{G})\big)_*\big)(W),$$
$$\Delta(x) = W^*(1 \otimes x)W,$$
$$C_0(\widehat{\mathbb{G}}) = \overline{\operatorname{span}}\big(\mathcal{B}\big(L^2(\mathbb{G})\big)_* \otimes id\big), (W)$$
$$\widehat{\Delta}(x) = \Sigma W(x \otimes 1)W^*\Sigma.$$

These faithful representations on $L^2(\mathbb{G})$ allow us to define the von Neumann algebras $L^\infty(\mathbb{G}) = C(\mathbb{G})''$ and $\ell^\infty(\widehat{\mathbb{G}}) = C_0(\widehat{\mathbb{G}})''$, and one can check that $W \in L^\infty(\mathbb{G}) \otimes \ell^\infty(\widehat{\mathbb{G}})$. Finally, we set $\widehat{W} = \Sigma W^*\Sigma$.

3. Preliminaries

Let $\widehat{\mathbb{G}}$ be a discrete quantum group and $a \in \ell^\infty(\widehat{\mathbb{G}})$. The *left multiplier* associated to a is the map m_a on $(id \otimes \mathcal{B}(L^2(\mathbb{G}))_*)(W)$ defined by $(m_a \otimes id)(W) = (1 \otimes a)W$. A net (a_i) of elements of $\ell^\infty(\widehat{\mathbb{G}})$ is said to *converge pointwise* to $a \in \ell^\infty(\widehat{\mathbb{G}})$ if $a_i p \to ap$ in $\mathcal{B}(L^2(\mathbb{G}))$ for any minimal central projection $p \in \ell^\infty(\widehat{\mathbb{G}})$. An element $a \in \ell^\infty(\widehat{\mathbb{G}})$ is said to have *finite support* if ap vanishes for all but finitely many central projections $p \in \ell^\infty(\widehat{\mathbb{G}})$.

For a discrete group G, it is known that a bounded function $\varphi : G \to \mathbb{C}$ gives rise to a completely bounded multiplier if and only if there exist a Hilbert space K and two families $(\xi_s)_{s\in G}$ and $(\eta_t)_{t\in G}$ of vectors in K such that $\varphi(s) = \langle \eta_t, \xi_{st}\rangle$ (which is usually written $\varphi(st^{-1}) = \langle \eta_t, \xi_s\rangle$). Moreover, the completely bounded norm is then $\inf \|(\xi_s)\|_\infty \|(\eta_t)\|_\infty$ (see e.g. [2]). The following theorem [3, Prop. 4.1 and Thm. 4.2] gives the quantum analogue of this characterization:

Theorem 3.1. *(See Daws [3].) Let $\widehat{\mathbb{G}}$ be a discrete quantum group and $a \in \ell^\infty(\widehat{\mathbb{G}})$. Then m_a extends to a completely bounded multiplier on $\mathcal{B}(L^2(\mathbb{G}))$ if and only if there exist a Hilbert space K and two maps $\alpha, \beta \in \mathcal{B}(L^2(\mathbb{G}), L^2(\mathbb{G}) \otimes K)$ such that $(1 \otimes \beta)^*\widehat{W}_{12}^*(1 \otimes \alpha)\widehat{W} = a \otimes 1$. Moreover, we then have $m_a(x) = \beta^*(x \otimes 1)\alpha$ and we can choose α and β to have norm equal to $\sqrt{\|m_a\|_{cb}}$.*

Proof. We only give the construction of α and β since we will need their precise form later on. Assume m_a to be completely contractive. By Wittstock's factorization theorem, there are a representation $\pi : \mathcal{B}(L^2(\mathbb{G})) \to \mathcal{B}(K)$ and two isometries $P, Q \in \mathcal{B}(L^2(\mathbb{G}), K)$ such that for all $x \in \mathcal{B}(L^2(\mathbb{G}))$, $m_a(x) = Q^*\pi(x)P$. Then setting $\alpha = (id \otimes \pi)(\widehat{W})(1 \otimes P)\widehat{W}(id \otimes \xi_h)$ and $\beta = (id \otimes \pi)(\widehat{W})(1 \otimes Q)\widehat{W}(id \otimes \xi_h)$ yields the result. □

From this we easily deduce that a multiplier m_a on a discrete quantum group $\widehat{\mathbb{G}}$ has a completely bounded extension to $\mathcal{B}(L^2(\mathbb{G}))$ if and only if it has a completely bounded extension to $C(\mathbb{G})$ or to $L^\infty(\mathbb{G})$ and that the completely bounded norms of these extensions are all equal. This justifies the following definition of weak amenability for discrete quantum groups:

Definition 3.2. A discrete quantum group $\widehat{\mathbb{G}}$ is said to be *weakly amenable* if there exists a net (a_i) of elements of $\ell^\infty(\widehat{\mathbb{G}})$ such that a_i has finite support for all i, the net (a_i) converges pointwise to 1 and the maps m_{a_i} satisfy $\limsup \|m_{a_i}\|_{cb} < \infty$. The infimum of $\limsup \|m_{a_i}\|_{cb,\mathcal{B}(L^2(\mathbb{G}))}$ for all nets satisfying the above conditions is denoted $\Lambda_{cb}(\widehat{\mathbb{G}})$ and called the *Cowling–Haagerup constant* of $\widehat{\mathbb{G}}$. By convention, $\Lambda_{cb}(\widehat{\mathbb{G}}) = \infty$ if $\widehat{\mathbb{G}}$ is not weakly amenable.

4. Free products of weakly amenable discrete quantum groups

Recall that given two discrete quantum groups $\widehat{\mathbb{G}}$ and $\widehat{\mathbb{H}}$, there is a unique compact quantum group structure on the reduced free product $C(\mathbb{G}) * C(\mathbb{H})$ with respect to the Haar states which is compatible with the canonical inclusions (it is defined in [11]). By analogy with the classical case, the dual of this compact quantum group will be called the *reduced free product* of $\widehat{\mathbb{G}}$ and $\widehat{\mathbb{H}}$ and denoted $\widehat{\mathbb{G}} * \widehat{\mathbb{H}}$. We will now prove that a reduced free product of weakly amenable discrete quantum groups with Cowling–Haagerup constant equal to 1 has Cowling–Haagerup constant equal to 1. This result has been proved in the classical case by E. Ricard and Q. Xu [9, Thm. 4.3] using the following result [9, Prop. 4.11]:

Theorem 4.1. *(See Ricard and Xu [9].) Let $(B_i, \psi_i)_{i\in I}$ be unital C*-algebras with distinguished states (ψ_i) having faithful GNS constructions. Let $A_i \subset B_i$ be unital C*-subalgebras such that the states $\varphi_i = \psi_{i|A_i}$ also have faithful GNS construction. Assume that* *for each i, there is a net of finite rank maps $(V_{i,j})$ on A_i converging to the identity pointwise, preserving the state and such that* $\limsup_j \|V_{i,j}\|_{cb} = 1$. *Assume moreover that for each pair (i, j), there is a completely positive unital map $U_{i,j} : A_i \to B_i$ preserving the state and such that* $\|V_{i,j} - U_{i,j}\|_{cb} + \|V_{i,j} - U_{i,j}\|_{\mathcal{B}(L^2(A_i,\varphi_i),L^2(B_i,\psi_i))} + \|V_{i,j} - U_{i,j}\|_{\mathcal{B}(L^2(A_i,\varphi_i)^{op},L^2(B_i,\psi_i)^{op})} \to 0$. *Then, the reduced free product of the family (A_i, φ_i) has Cowling–Haagerup constant equal to* 1.

Theorem 4.2. *Let $(\widehat{\mathbb{G}}_i)_{i\in I}$ be a family of discrete quantum groups with Cowling–Haagerup constant equal to* 1. *Then* $\Lambda_{cb}(*_i\widehat{\mathbb{G}}_i) = 1$.

Proof. Let $\widehat{\mathbb{G}}$ be a discrete quantum group, let $0 < \eta < 1$ and let $a \in \ell^\infty(\widehat{\mathbb{G}})$ with $\|m_a\|_{cb} \leqslant 1 + \eta$.

Lemma 4.3. *Assume that $a = \widehat{S}(a)^*$ and that $m_a(1) = 1$, then there exists a u.c.p. map on $\mathcal{B}(L^2(\mathbb{G}))$ approximating m_a up to 6η in completely bounded norm.*

Proof. Let α, β denote the two maps given by Theorem 3.1 with $\|\alpha\|, \|\beta\| \leqslant \sqrt{1+\eta}$ and set $\gamma = (\alpha + \beta)/2$ and $\delta = (\alpha - \beta)/2$. We know from [5, Prop. 2.6] that for any $x \in C(\mathbb{G})$, $\alpha^*(x\otimes 1)\beta = m_a(x^*)^* = m_{\widehat{S}(a)^*}(x)$, thus $m_a(x) = M_\gamma(x) - M_\delta(x)$, where $M_\gamma(x) = \gamma^*(x\otimes 1)\gamma$ and $M_\delta(x) = \delta^*(x\otimes 1)\delta$. The maps M_γ and M_δ are completely positive and evaluating at 1 yields $\|M_\gamma\|_{cb} \leqslant 1 + \eta$ and $\|M_\delta\|_{cb} \leqslant \eta$. We now want to perturb M_γ into a *unital* completely positive map. To do this, it suffices to define $\tilde{\gamma} = \gamma(\gamma^*\gamma)^{-1/2}$ ($\gamma^*\gamma$ is invertible because $\|1 - \gamma^*\gamma\| < 1$). Then, $M_{\tilde{\gamma}}$ is a u.c.p. map approximating m_a up to 6η in completely bounded norm. □

Set $D = \mathcal{B}(L^2(\mathbb{G}))$. We now have to prove that the previous approximation also works when the maps are seen as operators on $L^2(D, \tau)$ and $L^2(D, \tau)^{op}$, where $\tau(x) = \langle x.\xi_h, \xi_h\rangle$. Let us start with a technical lemma.

Lemma 4.4. *Let T be any bounded linear operator from $L^2(\mathbb{G})$ to K and set*

$$A(T) = (id \otimes \pi)(\widehat{W})^*(1 \otimes T)\widehat{W}(id \otimes \xi_h) \in \mathcal{B}\big(L^2(\mathbb{G}), L^2(\mathbb{G}) \otimes K\big)$$

and $M_{A(T)}(x) = A(T)^*(x \otimes 1)A(T)$. Then $\tau(M_{A(T)}(x^*x)) \leqslant \|T\|^2\tau(x^*x)$ and $M_{A(T)}$ is a bounded operator on $L^2(D, \tau)$ of norm less than $\|T\|^2$. If moreover $A(T)^*A(T)$ is invertible, then $M_{A(T)|A(T)|^{-1}}$ is τ-invariant.

Proof. The inequality $\tau(M_{A(T)}(x^*x)) \leqslant \|T\|^2\tau(x^*x)$ and the τ-invariance follow from a straightforward calculation using the fact that $\widehat{W}^*(\xi_h \otimes \xi) = (\xi_h \otimes \xi)$ for any $\xi \in L^2(\mathbb{G})$ and that $(id \otimes \pi)(\widehat{W})(\xi_h \otimes \xi') = \xi_h \otimes \xi'$ for any $\xi' \in K$. Applying Kadison's inequality to the completely positive map $M_{A(T)}$ then yields the norm estimate. □

Applying this lemma to $M_\delta = A([P - Q]/2)$ proves that M_γ approximates m_a up to η in $\mathcal{B}(L^2(D, \tau))$. We still have to control $\|M_{\tilde{\gamma}} - M_\gamma\|_{\mathcal{B}(L^2(D,\tau))}$. Note that we already know from Lemma 4.4 that $M\tilde{\gamma}$ is τ-invariant.

Lemma 4.5. $\tau((M_\gamma(x) - M_{\tilde{\gamma}}(x))^*(M_\gamma(x) - M_{\tilde{\gamma}}(x)))^{1/2} \leqslant 5\eta\tau(x^*x)^{1/2}$.

Proof. Using Lemma 4.4 for $\gamma = A([P + Q]/2)$, we see that $\tau(M_{\tilde{\gamma}-\gamma}(x^*x)) \leqslant \eta^4\tau(x^*x)$ and from this we deduce that $M_{\tilde{\gamma}-\gamma}$ has norm less than η^2. Decomposing $M_{\tilde{\gamma}}(x)$ as $M_\gamma(x) + M_{\tilde{\gamma}-\gamma}(x) + (\tilde{\gamma} - \gamma)^*(x \otimes 1)\gamma + \gamma^*(x \otimes 1)(\tilde{\gamma} - \gamma)$ then yields the result. □

Lemma 4.6. *$M_{\tilde{\gamma}}$ is also an approximation in $\mathcal{B}(L^2(D, \tau)^{op})$.*

Proof. To estimate the opposite L^2-norm, one only needs to do all the previous computations exchanging P and Q. Since they play symmetric rôles, we get the same result. □

We are now ready to prove the theorem. For each i, set $A_i = C(\mathbb{G}_i)$ and $B_i = \mathcal{B}(L^2(\mathbb{G}_i))$. Consider a sequence $(a_{i,j})$ of finitely supported elements in $\ell^\infty(\widehat{\mathbb{G}}_i)$ converging pointwise to the identity and such that $\limsup_j \|m_{a_{i,j}}\|_{cb} = 1$. Note that since $m_{a_{i,j}}(1) = \hat{\varepsilon}(a_{i,j}).1 \to 1$, we can assume that $\hat{\varepsilon}(a_{i,j})$ is non-zero and divide by it so that $m_{a_{i,j}}$ becomes unital. For any $0 < \eta < 1$, there is a $j(\eta)$ such that $a_{i,j(\eta)}$ satisfies $\|m_{a_{i,j(\eta)}}\|_{cb} \leqslant 1 + \eta$ (and the same inequality holds for $\widehat{S}(a_{i,j(\eta)})$). Since $* \circ \widehat{S} \circ * \circ \widehat{S} = id$, we can replace $a_{i,j}$ by $(a_{i,j} + \widehat{S}(a_{i,j})^*)/2$ so that all the hypotheses of Lemma 4.3 are satisfied. The procedure described above then yields a unital completely bounded map approximating $m_{a_{i,j(\eta)}}$ up to 6η in completely bounded norm and in both L^2-norms. Applying Theorem 4.1 proves that $\Lambda_{cb}(*_iA_i) = 1$, which implies that $\Lambda_{cb}(*_i\widehat{\mathbb{G}}_i) = 1$ (the argument of [5, Thm. 5.14] for discrete Kac algebras is easily extended to the general case, see e.g. [4, Thm. 3.11]).

Example 1. Let (G_i) be any family of compact groups, then their duals in the sense of quantum groups are amenable (which implies that the Cowling–Haagerup constant is 1 thanks to [10, Thm. 3.8]). Thus $*_i(C(G_i))$ is the dual of a non-cocommutative discrete quantum group with Cowling–Haagerup constant equal to 1.

Example 2. The free orthogonal quantum groups $\widehat{A_o(F)}$ have Cowling–Haagerup constant equal to 1 for any $F \in GL(2, \mathbb{C})$ such that $F\overline{F} = Id$ since they are amenable. Moreover, for any $F \in GL(2, \mathbb{C})$ the free unitary quantum group $\widehat{A_u(F)}$ is a quantum subgroup of $\mathbb{Z} * \widehat{A_o(F)}$, thus $\Lambda_{cb}(\widehat{A_u(F)}) = 1$. It follows that any reduced free product of some 2-dimensional free quantum groups with duals of compact groups has Cowling–Haagerup constant equal to 1.

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