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Partial Differential Equations

Singular quasilinear elliptic equations and Hölder regularity

Régularité höldérienne pour des équations quasi-linéaires elliptiques singulières

Jacques Giacomoni [a], Ian Schindler [b], Peter Takáč [c]

[a] *Laboratoire de mathématiques et de leurs applications - UMR CNRS 5142, bâtiment IPRA, université de Pau et des Pays de l'Adour, avenue de l'université, BP 1155, 64013 Pau, France*

[b] *MIP-CEREMATH / bâtiment C, manufacture des tabacs, allée de Brienne 21, 31000 Toulouse, France*

[c] *Institut für Mathematik, Universität Rostock, Ulmenstraße 69, Haus 3, 18055 Rostock, Germany*

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ABSTRACT

We prove the Hölder regularity (Theorem 2.1) for weak solutions to singular quasilinear elliptic equations whose prototype is

$$\begin{cases} -\Delta_p u = \frac{K(x)}{u^\delta} + g(x) & \text{in } \Omega; \\ u|_{\partial\Omega} = 0, \ u > 0 & \text{in } \Omega, \end{cases} \tag{P}$$

where Ω is an open bounded domain with smooth boundary, $1 < p < \infty$, $\delta > 0$, $K \in L^\infty_{\mathrm{loc}}(\Omega)$ satisfies $0 \leqslant K(x) \leqslant \mathrm{const} \cdot \mathrm{dist}(x, \partial\Omega)^{-\omega}$ for a.e. $x \in \Omega$, $0 < \omega < 1 + (1-\delta)(1-\frac{1}{p})$, and $0 \leqslant g \in L^\infty(\Omega)$. Theorem 2.1 together with the Schauder fixed point theorem can be used to obtain the existence of weak solutions to the singular quasilinear elliptic system (PS) described in the Introduction.

RÉSUMÉ

Nous démontrons la régularité höldérienne (Théorème 2.1) des solutions faibles des équations quasi-linéaires elliptiques singulières de la forme suivante :

$$\begin{cases} -\Delta_p u = \frac{K(x)}{u^\delta} + g(x) & \text{in } \Omega; \\ u|_{\partial\Omega} = 0, \ u > 0 & \text{in } \Omega, \end{cases} \tag{P}$$

où Ω est un ouvert borné régulier, $1 < p < \infty$, $\delta > 0$, $K \in L^\infty_{\mathrm{loc}}(\Omega)$ satisfait $0 \leqslant K(x) \leqslant \mathrm{const} \cdot \mathrm{dist}(x, \partial\Omega)^{-\omega}$ pour p.p. $x \in \Omega$, $0 < \omega < 1 + (1-\delta)(1-\frac{1}{p})$ et $0 \leqslant g \in L^\infty(\Omega)$. Théorème 2.1 combiné avec le théorème du point fixe de Schauder permet de démontrer l'existence de solutions faibles de systèmes elliptiques quasi-linéaires singuliers de la forme (PS) (voir Introduction).

Version française abrégée

Nous démontrons dans cette Note la régularité höldérienne des solutions faibles d'équations quasi-linéaires elliptiques singulières de la forme suivante :

E-mail addresses: jacques.giacomoni@univ-pau.fr (J. Giacomoni), Ian.Schindler@univ-tlse1.fr (I. Schindler), peter.takac@uni-rostock.de (P. Takáč).

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$$\begin{cases} -\Delta_p u = \frac{K(x)}{u^\delta} + g(x) & \text{in } \Omega; \\ u|_{\partial\Omega} = 0,\ u > 0 & \text{in } \Omega, \end{cases} \tag{P}$$

où Ω est un ouvert borné régulier, $1 < p < \infty$, $\delta > 0$, $K \in L^\infty_{\text{loc}}(\Omega)$ satisfait $0 \leqslant K(x) \leqslant \text{const} \cdot \text{d}(x)^{-\omega}$ pour p.p. $x \in \Omega$ et où $\text{d}(x)$ désigne la distance d'un point $x \in \overline{\Omega}$ au bord $\partial\Omega$, ω est une constante positive ou nulle et $0 \leqslant g \in L^\infty(\Omega)$. Précisément, nous avons le résultat suivant (Théorème 1.1 dans la section 1) :

Théorème 1. *Soit u une solution positive de* (P) *et supposons que les conditions énoncées précédemment soient satisfaites.*

(i) *Si* $0 < \delta + \omega < 1$ *alors* $u \in C^{1,\alpha}(\overline{\Omega})$ *pour un* $0 < \alpha < 1$.
(ii) *Si* $1 \leqslant \delta + \omega < 2 - \frac{1-\delta}{p}$ *alors* $u \in C^{0,\alpha}(\overline{\Omega})$ *pour un* $0 < \alpha < 1$.

La preuve du Théorème 1 est une conséquence du théorème plus général suivant démontré dans la section 2 :

Théorème 2. *Supposons que* $\mathbf{a}(x,\eta)$ *satisfait les conditions structurelles* (2) *jusqu'à* (5)*, et que* $f(x)$ *satisfait l'hypothèse* (6)*. Soit* $u \in W^{1,p}_0(\Omega)$ *l'unique solution faible du problème* (1) *satisfaisant*

$$0 \leqslant u(x) \leqslant C\, \text{d}(x)^{\varepsilon'} \quad \textit{pour presque tout } x \in \Omega,$$

où C et ε' sont des constantes telles que $0 \leqslant C < \infty$ *et* $0 < \varepsilon' < \varepsilon$*. Finalement, soit* $\beta \in (0,\alpha)$ *telle que* $0 < \beta < \frac{p}{p-1+(\varepsilon/\varepsilon')} < 1$*. Alors il existe une quantité* $M \in (0,\infty)$*, dépendant seulement de* Ω, N, p*, des constantes* γ, Γ, α *dans* (3) *jusqu'à* (5)*, des constantes* c, ε *dans* (6)*, des constantes* C, ε' *dans* (7)*, et de la constante* β*, mais* pas *de* $\kappa \in [0,1]$*, telle que u satisfait* $u \in C^{0,\beta}(\overline{\Omega})$ *et*

$$\|u\|_{C^{0,\beta}(\overline{\Omega})} \leqslant M.$$

Théorème 2 (Théorème 2.1) dont la preuve utilise l'approche de [7] basée sur des estimations dans les espaces de Campanato, combiné avec le théorème du point fixe de Schauder permet de démontrer l'existence de solutions faibles de systèmes elliptiques quasi-linéaires singuliers de la forme (PS) (voir Introduction).

1. Introduction

In this Note we prove Hölder regularity of weak solutions to (P). Problem (P) arises, for instance, in models of pseudoplastic flows, chemical reactions, morphogenesis (Gierer–Meinhardt system [5]), population dynamics, and many other applications. The existence of a weak solution to (P), that is, a function $u \in W^{1,p}_0(\Omega)$ locally uniformly positive and satisfying (P) in the sense of distributions, can be obtained by means of suitable sub- and supersolutions combined with the theory of monotone operators. Our aim in this Note is to show the following:

Theorem 1.1. *Let u be a positive weak solution to* (P) *and* $0 \leqslant K(x) \leqslant \text{const} \cdot \text{d}(x)^{-\omega}$ *for* $x \in \Omega$*, where* $\text{d}(x)$ *denotes the distance from a point* $x \in \overline{\Omega}$ *to the boundary* $\partial\Omega$ *and* $\omega > 0$ *is a constant.*

(i) *If* $0 < \delta + \omega < 1$ *then* $u \in C^{1,\alpha}(\overline{\Omega})$ *for some* $0 < \alpha < 1$.
(ii) *If* $1 \leqslant \delta + \omega < 2 - \frac{1-\delta}{p}$ *then* $u \in C^{0,\alpha}(\overline{\Omega})$ *for some* $0 < \alpha < 1$.

Part (i) of this theorem is proved in the present authors' work [3, Theorem B.1, pp. 147–148]. $C^{1,\alpha}(\overline{\Omega})$-regularity can be used for proving existence of multiple solutions to problems related to (P); see H. Brézis and L. Nirenberg [1]. Concerning part (ii), a more general result, Theorem 2.1, is stated and proved in the next section. Conditions (6) and (7) are satisfied thanks to the existence of sub- and supersolutions, respectively, to problem (P) of the form $c\phi^a_{1,p}$, where $\phi_{1,p}$ denotes the normalized positive eigenfunction associated with the principal eigenvalue $\lambda_{1,p}$ of $-\Delta_p$, and $c > 0$ and $a \in (0,1]$ are suitable constants; cf. [9, Theorem 5, p. 200]. One may choose $\varepsilon' = a = \frac{p-\omega}{p-1+\delta}$ and $\varepsilon = \omega + a\delta$ $(> \varepsilon')$. A closely related result for elliptic problems with the (Dirichlet–) Laplace operator ($p = 2$) and a singular nonlinearity of our type is established in [6, Theorem 1.1]. However, their methods differ from ours significantly in that they take advantage of the Green function for the Laplace operator. In contrast, we are forced to estimate the Campanato norms in Hölder spaces (see, e.g., [4]).

Theorem 1.1 (or Theorem 2.1 below) can be used also to prove the existence of weak solutions to the following singular quasilinear elliptic system:

$$\begin{cases} -\Delta_p u = u^{-a_1} v^{-b_1} & \text{in } \Omega; \qquad u|_{\partial\Omega} = 0,\ u > 0 \quad \text{in } \Omega; \\ -\Delta_q v = v^{-a_2} u^{-b_2} & \text{in } \Omega; \qquad v|_{\partial\Omega} = 0,\ v > 0 \quad \text{in } \Omega, \end{cases} \tag{PS}$$

where $1 < p,\ q < \infty$, and the numbers $a_1, a_2, b_1, b_2 > 0$ satisfy certain upper bounds. The proof employs the Schauder fixed point theorem in a conical shell of the form $[m\phi^a_{1,p}, \frac{1}{m}\phi^a_{1,p}] \times [m\phi^b_{1,q}, \frac{1}{m}\phi^b_{1,q}]$, where $0 < a,\ b \leqslant 1$ depending on $a_1, a_1, a_2, b_1, b_2 > 0$.

2. A general regularity result

In this section we extend our earlier regularity result [3, Theorem B.1, pp. 147–148] for the following quasilinear elliptic boundary value problem:

$$-\nabla\cdot\big(\mathbf{a}(x,\nabla u)\big)\stackrel{\text{def}}{=}-\sum_{i=1}^{N}\frac{\partial}{\partial x_i}a_i\big(x,\nabla u(x)\big)=f(x)\quad\text{in }\Omega;\qquad u=0\quad\text{on }\partial\Omega. \tag{1}$$

We assume that Ω is a (nonempty) bounded domain in $\mathbb{R}^N$ whose boundary $\partial\Omega$ is a compact C^2-manifold. We denote by u the unknown function of $x=(x_1,\dots,x_N)\in\Omega$, where $u\in W_0^{1,p}(\Omega)$ for $p\in(1,\infty)$. The components a_i of the vector field $\mathbf{a}:\Omega\times\mathbb{R}^N\to\mathbb{R}^N$ are functions of x and $\eta=\nabla u\in\mathbb{R}^N$, such that $a_i\in C^0(\Omega\times\mathbb{R}^N)$ and $\partial a_i/\partial\eta_j\in C^0(\Omega\times(\mathbb{R}^N\setminus\{0\}))$. We assume that $\mathbf{a}$ satisfies the following *ellipticity* and *growth conditions*:

(**H1**) There exist some constants $\kappa\in[0,1]$, $\gamma,\Gamma\in(0,\infty)$, and $\alpha\in(0,1)$, such that

$$a_i(x,0)=0;\quad i=1,\dots,N, \tag{2}$$

$$\sum_{i,j=1}^{N}\frac{\partial a_i}{\partial\eta_j}(x,\eta)\cdot\xi_i\xi_j\geqslant\gamma\cdot\big(\kappa+|\eta|\big)^{p-2}\cdot|\xi|^2, \tag{3}$$

$$\sum_{i,j=1}^{N}\left|\frac{\partial a_i}{\partial\eta_j}(x,\eta)\right|\leqslant\Gamma\cdot\big(\kappa+|\eta|\big)^{p-2}, \tag{4}$$

$$\sum_{i=1}^{N}\big|a_i(x,\eta)-a_i(y,\eta)\big|\leqslant\Gamma\cdot\big(1+|\eta|\big)^{p}\cdot|x-y|^{\alpha}, \tag{5}$$

for all $x,y\in\Omega$, all $\eta\in\mathbb{R}^N\setminus\{0\}$, and all $\xi\in\mathbb{R}^N$.

Finally, we impose the following *growth condition* on the function $f\in L^\infty_{\text{loc}}(\Omega)$:

(**H2**) There exist constants $c,\varepsilon\in(0,\infty)$ such that

$$0\leqslant f(x)\leqslant c\,\mathrm{d}(x)^{-\varepsilon}\quad\text{holds for almost all }x\in\Omega. \tag{6}$$

We will show the following analogue of a well-known regularity result for problem (1) due to [7, Theorem 1, p. 1203] (regularity near the boundary). Interior regularity was established earlier independently in [2, Theorem 2, p. 829] and [8, Theorem 1, p. 127].

Theorem 2.1. *Assume that* $\mathbf{a}(x,\eta)$ *satisfies the structural hypotheses* (2) *through* (5)*, and* $f(x)$ *satisfies hypothesis* (6)*. Let* $u\in W_0^{1,p}(\Omega)$ *be the* (*unique*) *weak solution of problem* (1) *satisfying*

$$0\leqslant u(x)\leqslant C\,\mathrm{d}(x)^{\varepsilon'}\quad\text{for almost all }x\in\Omega, \tag{7}$$

where C *and* ε' *are constants, such that* $0\leqslant C<\infty$ *and* $0<\varepsilon'<\varepsilon$*. Finally, let* $\beta\in(0,\alpha)$ *be an arbitrary number, such that* $0<\beta<\frac{p}{p-1+(\varepsilon/\varepsilon')}<1$*. Then there exists a constant* $M\in(0,\infty)$*, depending solely on* Ω*,* N*,* p*, on the constants* γ*,* Γ*,* α *in* (3) *through* (5)*, on the constants* c*,* ε *in* (6)*, on the constants* C*,* ε' *in* (7)*, and the constant* β*, but* not *on* $\kappa\in[0,1]$*, such that* u *satisfies* $u\in C^{0,\beta}(\overline{\Omega})$ *and*

$$\|u\|_{C^{0,\beta}(\overline{\Omega})}\leqslant M. \tag{8}$$

Proof. We "flatten" the boundary $\partial\Omega$ locally by a C^2 diffeomorphism Φ. Such a local transformation of coordinates, $\tilde{x}=\Phi(x)$, leaves all structural conditions for a_i unchanged. The same remark is valid also for f and u in inequalities (6) and (7). In particular, we can adjust this transformation (by rotation and translation of coordinate axes) in order to achieve $\mathrm{d}(\tilde{x})=\tilde{x}_N$ for all $\tilde{x}\in\mathbb{R}^N$ from an open ball centered at the origin and such that $\tilde{x}_N\geqslant 0$. Therefore, writing $x=(x_1,\dots,x_N)\in\mathbb{R}^N$, $x'=(x_1,\dots,x_{N-1})\in\mathbb{R}^{N-1}$, and $x=(x',x_N)$, let us consider only an open ball

$$B_r(y)\stackrel{\text{def}}{=}\big\{x\in\mathbb{R}^N:\ |x-y|<r\big\}\quad\text{for some }y\in\mathbb{R}^N\text{ and }0<r<\infty$$

and the corresponding open half-ball $B_r^+(y)\stackrel{\text{def}}{=}\{x\in B_r(y):\ x_N-y_N>0\}$ with the flat boundary portion $B_r^0(y)\stackrel{\text{def}}{=}\{x\in B_r(y):\ x_N-y_N=0\}$ and the half-sphere boundary portion

$$S_r^+(y) \stackrel{\text{def}}{=} \partial B_r^+(y) \setminus B_r^0(y) = \{x \in \mathbb{R}^N\colon\ |x-y| = r \text{ and } x_N - y_N \geqslant 0\}.$$

This means that we have replaced a general domain Ω by an open half-ball; we fix and normalize this half-ball to be $B_1^+(0) \subset \Omega$.

We recall that the function $f: B_1^+(0) \to \mathbb{R}^N$ verifies (6), i.e.,

$$0 \leqslant f(x) \leqslant c x_N^{-\varepsilon} \quad \text{holds for almost all } x = (x', x_N) \in B_1^+(0). \tag{9}$$

Hypothesis (7) for u reads

$$0 \leqslant u(x) \leqslant C x_N^{\varepsilon'} \quad \text{for almost all } x \in B_1^+(0). \tag{10}$$

Constants c and C in (9) and (10) above are possibly different from those in (6) and (7), respectively.

Following [7], we will prove that the (unique) weak solution $u \in W^{1,p}(B_1^+(0))$ of the problem

$$-\nabla \cdot \mathbf{a}(x, \nabla u) = f(x) \quad \text{in } B_1^+(0); \qquad u = 0 \quad \text{on } B_1^0(0), \tag{11}$$

which is assumed to obey (10), satisfies $u \in C^{0,\beta}(\overline{B_{1/2}^+(0)})$ for some $\beta \in (0, \alpha)$. We do not specify the boundary data of u on the half-sphere $S_1^+(0)$, but assume $u \in W^{1,p}(B_1^+(0))$ instead. The proof is based on a standard perturbation argument using the Dirichlet problem

$$\begin{cases} -\nabla \cdot \mathbf{a}(x, \nabla v) = 0 \quad \text{in } B_R^+(y); \\ v = 0 \quad \text{on } B_R^0(y), \qquad v = u \quad \text{on } S_R^+(y), \end{cases} \tag{12}$$

for any $y \in B_{1/2}^+(0)$ and any $0 < R < 1/2$; notice that $B_R^+(y) \subset B_1^+(0)$. This problem possesses a unique weak solution $v \equiv v_R$ in $W^{1,p}(B_R^+(y))$.

In order to establish the desired estimate for the Campanato expression for $u - v$ in $B_R^+(y)$, it suffices to consider the "normalized" case $y = 0 \in \mathbb{R}^N$ and $0 < R < 1$. In other words, the Dirichlet boundary value problem (12) becomes

$$\begin{cases} -\nabla \cdot \mathbf{a}(x, \nabla v) = 0 \quad \text{in } B_R^+(0); \\ v = 0 \quad \text{on } B_R^0(0), \qquad v = u \quad \text{on } S_R^+(0), \end{cases} \tag{13}$$

with a unique weak solution $v \in W^{1,p}(B_R^+(0))$, for any $0 < R < 1$.

First of all, we have $0 \leqslant v \leqslant u$ in $B_R^0(0)$, by the weak comparison principle. Hypothesis (10) on u thus forces

$$0 \leqslant u(x) - v(x) \leqslant C x_N^{\varepsilon'} \quad \text{for all } x = (x', x_N) \in B_R^+(0). \tag{14}$$

Let $\rho \in [1, \infty)$; this number will be specified later. Subtracting Eq. (13) from (11), multiplying the difference by the test function $\phi = |u - v|^{\rho-1}(u - v)$, and finally integrating over $B_R^+(0)$, we arrive at

$$\int_{B_R^+(0)} \left[\mathbf{a}(x, \nabla u) - \mathbf{a}(x, \nabla v)\right] \cdot \nabla\left[|u - v|^{\rho-1}(u - v)\right] \mathrm{d}x = \int_{B_R^+(0)} f(x)\phi \,\mathrm{d}x \leqslant c\,C \int_{B_R^+(0)} x_N^{\rho\varepsilon' - \varepsilon} \,\mathrm{d}x = c_1 R^{N + \rho\varepsilon' - \varepsilon}, \tag{15}$$

for any $0 < R < 1$, by (9) and (14). The constant $c_1 = c\,C\,c_0 \geqslant 0$ has been obtained using

$$\int_{B_R^+(0)} x_N^{\rho\varepsilon' - \varepsilon} \,\mathrm{d}x = R^{N + \rho\varepsilon' - \varepsilon} \int_{B_1^+(0)} z_N^{\rho\varepsilon' - \varepsilon} \,\mathrm{d}z \equiv c_0 R^{N + \rho\varepsilon' - \varepsilon}.$$

We estimate the left-hand side of inequality (15) from below as follows, applying ellipticity condition (3). For almost every $x \in B_R^+(0)$ we have

$$\left[\mathbf{a}(x, \nabla u) - \mathbf{a}(x, \nabla v)\right] \cdot \nabla\left[|u - v|^{\rho-1}(u - v)\right] \geqslant \rho\gamma |u - v|^{\rho-1} \left(\int_0^1 \left|\nabla\left(v + \theta(u - v)\right)\right|^{p-2} \mathrm{d}\theta\right) \left|\nabla(u - v)\right|^2. \tag{16}$$

If $2 \leqslant p < \infty$, we obtain immediately

$$\left[\mathbf{a}(x, \nabla u) - \mathbf{a}(x, \nabla v)\right] \cdot \nabla(u - v) \geqslant \gamma\kappa_p \left|\nabla(u - v)\right|^p, \tag{17}$$

where $\kappa_p \stackrel{\text{def}}{=} \min_{\mathbf{v}, \mathbf{w} \in \mathbb{R}^N |, \mathbf{w}| = 1} \int_0^1 |\mathbf{v} + \theta\mathbf{w}|^{p-2} \,\mathrm{d}\theta > 0$ is the constant from the inequality $\kappa_p |\mathbf{w}|^{p-2} \leqslant \int_0^1 |\mathbf{v} + \theta\mathbf{w}|^{p-2} \,\mathrm{d}\theta$ for all $\mathbf{v}, \mathbf{w} \in \mathbb{R}^N$. We combine (15), (16), and (17) to get $\int_{B_R^+(0)} |\nabla(u - v)|^p |u - v|^{\rho-1} \,\mathrm{d}x \leqslant c_2 R^{N + \rho\varepsilon' - \varepsilon}$ with the constant $c_2 = (\rho\gamma\kappa_p)^{-1} c_1 \geqslant 0$ independent from $0 < R < 1$, which simplifies to

$$\int_{B_R^+(0)} |\nabla w|^p \, dx \leqslant c_2 \left(\frac{\rho - 1}{p} + 1 \right)^p R^{N+\rho\varepsilon' - \varepsilon} \tag{18}$$

after the substitution

$$w \stackrel{\text{def}}{=} |u - v|^{(\rho-1)/p} (u - v). \tag{19}$$

If $1 < p < 2$, we use the (trivial) inequality $(|\mathbf{v}| + |\mathbf{w}|)^{p-2} \leqslant \int_0^1 |\mathbf{v} + \theta \mathbf{w}|^{p-2} \, d\theta$ for all $\mathbf{v}, \mathbf{w} \in \mathbb{R}^N$ to obtain

$$\big[\mathbf{a}(x, \nabla u) - \mathbf{a}(x, \nabla v)\big] \cdot \nabla(u - v) \geqslant \gamma \big(|\nabla v| + \big|\nabla(u - v)\big|\big)^{p-2} \big|\nabla(u - v)\big|^2. \tag{20}$$

Next, by Hölder's inequality, we have

$$\begin{aligned}\int_{B_R^+(0)} \big|\nabla(u - v)\big|^p |u - v|^{\rho-1} \, dx \leqslant{}& \left(\int_{B_R^+(0)} \big(|\nabla v| + \big|\nabla(u - v)\big|\big)^{p-2} \big|\nabla(u - v)\big|^2 |u - v|^{\rho-1} \, dx \right)^{p/2} \\ &\times \left(\int_{B_R^+(0)} \big(|\nabla v| + \big|\nabla(u - v)\big|\big)^p |u - v|^{\rho-1} \, dx \right)^{(2-p)/2},\end{aligned}$$

and then, applying (20) followed by Minkowski's inequality and (15),

$$\begin{aligned}&\int_{B_R^+(0)} \big|\nabla(u - v)\big|^p |u - v|^{\rho-1} \, dx \\ &\quad \leqslant \left(\frac{c_1 R^{N+\rho\varepsilon' - \varepsilon}}{\rho\gamma} \right)^{p/2} \left[\left(\int_{B_R^+(0)} |\nabla v|^p |u - v|^{\rho-1} \, dx \right)^{1/p} + \left(\int_{B_R^+(0)} \big|\nabla(u - v)\big|^p \, |u - v|^{\rho-1} \, dx \right)^{1/p} \right]^{(2-p)p/2}.\end{aligned}$$

With the substitution (19) and the notation

$$J(w; R) = \int_{B_R^+(0)} |\nabla w|^p \, dx = \left(\frac{\rho - 1}{p} + 1 \right)^p \int_{B_R^+(0)} \big|\nabla(u - v)\big|^p |u - v|^{\rho-1} \, dx,$$

$$\hat{J}(v; R) = \int_{B_R^+(0)} |\nabla v|^p |u - v|^{\rho-1} \, dx,$$

the last inequality simplifies to

$$J(w; R)^{\frac{2}{(2-p)p}} = J(w; R)^{\frac{1}{p} + \frac{1}{2-p}} \leqslant \big(c_3 \, R^{N+\rho\varepsilon' - \varepsilon}\big)^{1/(2-p)} \left[\left(\frac{\rho - 1}{p} + 1 \right)^{-1} J(w; R)^{1/p} + \hat{J}(v; R)^{1/p} \right] \tag{21}$$

whenever $0 < R < 1$, where $c_3 = (\rho\gamma)^{-1}, c_1 \geqslant 0$. We apply (14) to estimate $\hat{J}(v; R) = \int_{B_R^+(0)} |\nabla v|^p |u - v|^{\rho-1} \, dx \leqslant C \int_{B_R^+(0)} |\nabla v|^p x_N^{\varepsilon'(\rho-1)} \, dx \leqslant C R^{\varepsilon'(\rho-1)} \int_{B_R^+(0)} |\nabla v|^p \, dx = C R^{\varepsilon'(\rho-1)} J(v; R)$. Consequently, inequality (21) yields

$$J(w; R)^{\frac{2}{(2-p)p}} \leqslant \big(c_4 R^{N+\rho\varepsilon' - \varepsilon}\big)^{1/(2-p)} \big[J(w; R)^{1/p} + c_5^{1/p} R^{\varepsilon'(\rho-1)/p} J(v; R)^{1/p} \big] \tag{22}$$

whenever $0 < R < 1$, where $c_4 = c_3(\frac{\rho-1}{p} + 1)^{-(2-p)}$ and $c_5 = C(\frac{\rho-1}{p} + 1)^p$. Substituting $J(w; R) = c_4 R^{N+\rho\varepsilon' - \varepsilon} \tilde{J}(w; R)$ and $J(v; R) = c_4 R^{N+\rho\varepsilon' - \varepsilon} \tilde{J}(v; R)$ in the last inequality, we obtain $\tilde{J}(w; R)^{\frac{1}{p} + \frac{1}{2-p}} \leqslant \tilde{J}(w; R)^{1/p} + c_5^{1/p} R^{\varepsilon'(\rho-1)/p} \tilde{J}(v; R)^{1/p}$ whenever $0 < R < 1$. Examining the alternatives $\tilde{J}(w; R) \geqslant c_5 R^{\varepsilon'(\rho-1)} \tilde{J}(v; R)$ and $\tilde{J}(w; R) \leqslant c_5 R^{\varepsilon'(\rho-1)} \tilde{J}(v; R)$, from this inequality we further deduce $\tilde{J}(w; R) \leqslant \max\{2^{2-p}, 2^{(2-p)p/2} (c_5 R^{\varepsilon'(\rho-1)})^{(2-p)/2} \tilde{J}(v; R)^{(2-p)/2}\}$, whenever $0 < R < 1$, and consequently $J(w; R) \leqslant c_4 R^{N+\rho\varepsilon' - \varepsilon} \cdot \max\{2^{2-p}, 2^{(2-p)p/2} (c_4^{-1} c_5 R^{-N+\varepsilon - \varepsilon'})^{(2-p)/2} J(v; R)^{(2-p)/2}\}$ which yields

$$J(w; R) \leqslant c_6 R^{N+\rho\varepsilon' - \varepsilon} \cdot \max\big\{1, \big(R^{\varepsilon - \varepsilon' - N} J(v; R)\big)^{(2-p)/2}\big\} \tag{23}$$

where $c_6 \geqslant 0$ is a constant independent from $0 < R < 1$.

Applying certain estimates on suitable norms of v from [7, Lemma 5, p. 1211], Lieberman has derived the following inequality for $J(v; \cdot) : (0, 1) \to [0, \infty)$, see [7, Ineq. (3.6), p. 1212]:

$$J(v;r) \leqslant C_0\big\{R^N + (r/R)^N J(v;R)\big\} \quad \text{for all } 0 < r < R \leqslant R_0, \tag{24}$$

where $C_0 \geqslant 0$ and $0 < R_0 < 1$ are constants independent from both r and R. By Lemma B.1 in [3, pp. 157–158], this implies

$$\sup_{0<R\leqslant R_0} R^{-N+\eta} J(v;R) = \sup_{0<R\leqslant R_0} \frac{1}{R^{N-\eta}} \int_{B_R^+(0)} |\nabla v|^p \, dx \equiv C(\eta) < \infty \tag{25}$$

for any number $0 < \eta < N$. Finally, we apply this inequality to (23), thus arriving at

$$J(w;R) \leqslant c_7 R^{N+\rho\varepsilon'-\varepsilon} \cdot \max\big\{1, R^{(\varepsilon-\varepsilon'-\eta)(2-p)/2}\big\} = c_7 R^{N+\rho\varepsilon'-\varepsilon} = c_7 R^{N+\mu} \tag{26}$$

where η is chosen such that $\varepsilon - \varepsilon' - \eta \geqslant 0$, $\rho \geqslant 1$ needs to be chosen such that $\mu = \rho\varepsilon' - \varepsilon$ satisfies $0 < \mu < p$, and $c_7 \equiv c_7(R_0) \geqslant 0$ is a constant independent from $0 < R \leqslant R_0$.

Inequality (25) holds for any $1 < p < \infty$. We summarize inequalities (18) and (26) to obtain $J(w;R) \leqslant c_7 R^{N+\mu}$ where $0 < \mu < p$, and $c_7 \equiv c_7(R_0) \geqslant 0$ is a constant independent from $0 < R \leqslant R_0$. The proof of regularity of w, i.e., $w \in C^{1,\mu/p}(\overline{B^+_{1/2}(0)})$, can now be completed exactly as in [2, p. 849] or [7, p. 1213], again with a help from certain estimates on suitable norms of v obtained in [7, Lemma 5, p. 1211].

We finish our proof by the observation that the function $u - v = |w|^{\beta-1} w$ with $\beta = \frac{p}{p-1+\rho}$ (cf. (19)) satisfies $u - v \in C^{0,\beta}(\overline{B^+_{1/2}(0)})$; notice that $0 < \varepsilon' < \varepsilon$ implies $\rho > 1$ and $0 < \beta < \frac{p}{p-1+(\varepsilon/\varepsilon')} < 1$. $\square$

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