C. R. Acad. Sci. Paris, Ser. I 350 (2012) 407–410

Contents lists available at SciVerse ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Functional Analysis

On the binary relation $\leqslant_u$ on self-adjoint Hilbert space operators

Relation binaire $\leqslant_u$ sur un espace de Hilbert d'opérateurs auto-adjoints

M.S. Moslehian, S.M.S. Nabavi Sales, H. Najafi

Department of Pure Mathematics, Center of Excellence in Analysis on Algebraic Structures (CEAAS), Ferdowsi University of Mashhad, P.O. Box 1159, Mashhad 91775, Iran

ARTICLE INFO

Article history:
Received 25 November 2011
Accepted after revision 10 April 2012
Available online 4 May 2012

Presented by the Editorial Board

ABSTRACT

Given self-adjoint operators $A, B \in \mathbb{B}(\mathscr{H})$ it is said $A \leqslant_u B$ whenever $A \leqslant U^*BU$ for some unitary operator U. We show that $A \leqslant_u B$ if and only if $f(g(A)^r) \leqslant_u f(g(B)^r)$ for any increasing operator convex function f, any operator monotone function g and any positive number r. We present some sufficient conditions under which if $B \leqslant A \leqslant U^*BU$, then $B = A = U^*BU$. Finally we prove that if $A^n \leqslant U^*A^nU$ for all $n \in \mathbb{N}$, then $A = U^*AU$.

RÉSUMÉ

Soient $A, B \in \mathbb{B}(\mathscr{H})$ des opérateurs auto-adjoints donnés, on dit que $A \leqslant_u B$ si $A \leqslant U^*BU$, où U est un opérateur unitaire. On montre que $A \leqslant_u B$ si et seulement si $f(g(A)^r) \leqslant_u f(g(B)^r)$ pour toute fonction d'opérateurs f, convexe et croissante, toute fonction d'opérateurs g, monotone et tout nombre r positif. On donne des conditions nécessaires et suffisantes pour que $B \leqslant A \leqslant U^*BU$ implique $B = A = U^*BU$. Enfin on montre que si $A^n \leqslant U^*A^nU$ pour tout $n \in \mathbb{N}$ alors $A = U^*AU$.

1. Introduction

Let $\mathbb{B}(\mathscr{H})$ be the algebra of all bounded linear operators on a complex Hilbert space $\mathscr{H}$ with the identity I, let $\mathbb{B}_h(\mathscr{H})$ be the real linear space of all self-adjoint operators and let $\mathcal{U}(\mathscr{H})$ be the set of all unitary operators in $\mathbb{B}(\mathscr{H})$. By an orthogonal projection we mean an operator $P \in \mathbb{B}_h(\mathscr{H})$ such that $P^2 = P$. An operator $A \in \mathbb{B}(\mathscr{H})$ is called positive if $\langle Ax, x\rangle \geqslant 0$ for every $x \in \mathscr{H}$ and then we write $A \geqslant 0$. If A is a positive invertible operator we write $A > 0$. For $A, B \in \mathbb{B}_h(\mathscr{H})$ we say that $A \leqslant B$ if $B - A \geqslant 0$. The celebrated Löwner–Heinz inequality asserts that the operator inequality $T \geqslant S \geqslant 0$ implies $T^\alpha \geqslant S^\alpha$ for any $\alpha \in [0, 1]$, see [4, Theorem 3.2.1]. An operator T is called hyponormal if $T^*T \geqslant TT^*$.

Douglas [2] investigated the operator inequality $T^*HT \leqslant H$, with H Hermitian and showed that if P is a positive compact operator and A is a contraction such that $P \leqslant A^*PA$, then $P = A^*PA$; see also [3].

Given operators $A, B \in \mathbb{B}_h(\mathscr{H})$ it is said that $A \leqslant_u B$ whenever $A \leqslant U^*BU$ for some $U \in \mathcal{U}(\mathscr{H})$; see [6,9]. This binary relation was investigated by Kosaki [6] by showing that

$$A \leqslant_u B \quad \Rightarrow \quad e^A \leqslant_u e^B. \tag{1}$$

E-mail addresses: moslehian@ferdowsi.um.ac.ir, moslehian@member.ams.org (M.S. Moslehian), sadegh.nabavi@gmail.com (S.M.S. Nabavi Sales), hamednajafi20@gmail.com (H. Najafi).

1631-073X/$ – see front matter
http://dx.doi.org/10.1016/j.crma.2012.04.004

Okayasu and Ueta [9] gave a sufficient condition for a triple of operators (A, B, U) with $A, B \in \mathbb{B}_h(\mathscr{H})$ and $U \in \mathcal{U}(\mathscr{H})$ under which $B \leqslant A \leqslant U^*BU$ implies $B = A = U^*BU$. In this note we use their idea and prove a similar result. It is known that $\leqslant_u$ satisfies the reflexive and transitive laws but not the antisymmetric law in general; cf. [9]. The antisymmetric law states that $A \leqslant_u B$ and $B \leqslant_u A \Rightarrow A, B$ are unitarily equivalent. We, among other things, study some cases in which the antisymmetric law holds for the relation $\leqslant_u$. Utilizing a result of [8] we show that $A \leqslant_u B$ if and only if $f(g(A)^r) \leqslant_u f(g(B)^r)$ for any increasing operator convex function f, any operator monotone function g and any positive number r. Recall that a real function f defined on an interval J is said to be operator convex if $f(\lambda A + (1-\lambda)B) \leqslant \lambda f(A) + (1-\lambda)f(B)$ for any $A, B \in \mathbb{B}_h(\mathscr{H})$ with spectra in J and $\lambda \in [0,1]$ and is called operator monotone if $f(A) \leqslant f(B)$ whenever $A \leqslant B$ for any $A, B \in \mathbb{B}_h(\mathscr{H})$ with spectra in J, see [11].

2. The results

First we give the following known lemmas that we need in the sequel. The first one is applied frequently without referring to it.

Lemma 2.1. *Let $A \in \mathbb{B}_h(\mathscr{H})$ and $U \in \mathcal{U}(\mathscr{H})$. Then $f(U^*AU) = U^*f(A)U$ for any function f which is continues on the spectra of A.*

Lemma 2.2. *(See [7, Theorems 2.1, 2.3].) Let $T \in \mathbb{B}(\mathscr{H})$ be hyponormal and $T = U|T|$ be the polar decomposition of T such that $U^{n_0} = I$ for some positive integer n_0, $U^{*n} \to I$ as $n \to \infty$ or $U^n \to I$ as $n \to \infty$, where the limits are taken in the strong operator topology. Then T is normal.*

Lemma 2.3. *Let $U, V \in \mathcal{U}(\mathscr{H})$ be two commuting operators such that $U^n \to I$ and $V^n \to I$ as $n \to \infty$. Then $(UV)^n \to I$ as $n \to \infty$, where all limits are taken in the strong operator topology.*

Theorem 2.4. *Let $U \in \mathcal{U}(\mathscr{H})$ such that any one of the following conditions holds*:

(i) *$U^{n_0} = I$ for some positive integer n_0,*
(ii) *$U^n \to I$ as $n \to \infty$ in which the limit is taken in the strong operator topology.*

Then $B \leqslant A \leqslant U^*BU$ implies that $B = A = U^*BU$ for any $A, B \in \mathbb{B}_h(\mathscr{H})$.

Proof. Let $A, B \in \mathbb{B}_h(\mathscr{H})$ such that $B \leqslant A \leqslant U^*BU$. There exist $\lambda > 0$ such that $B + \lambda > 0$. Put $T = (B+\lambda)^{\frac{1}{2}}U$. By our assumption we have

$$TT^* = B + \lambda \leqslant A + \lambda \leqslant U^*(B+\lambda)U = T^*T. \tag{2}$$

Thus T is a hyponormal operator. Obviously $|T| = U^*(B+\lambda)^{\frac{1}{2}}U = U^*T$. Let $T = V|T|$ be the polar decomposition of T. Hence $T = VU^*T$. It follows from the invertibility of T that $I = VU^*$, that is, $U = V$. Thus T satisfies the conditions of Lemma 2.2. Therefore T turns out to be normal. Then (2) yields that $B = A = U^*BU$. □

Corollary 2.5. *Let $U, V \in \mathcal{U}(\mathscr{H})$ be two commuting operators satisfying any one of the following conditions*:

(i) *$U^{n_0} = I$ and $V^{n_0} = I$ for some positive integer n_0,*
(ii) *$U^n \to I$ and $V^n \to I$ as $n \to \infty$,*

where all limits are taken in the strong operator topology. If $A, B \in \mathbb{B}_h(\mathscr{H})$ such that $A \leqslant U^*BU$ and $B \leqslant V^*AV$, then $A = U^*BU$ and $B = V^*AV$.

Proof. By Lemma 2.3, the unitary operator UV satisfies the conditions of Theorem 2.4. □

The following lemmas are used in the proof of Theorem 2.8:

Lemma 2.6. *(See [4, Theorem 3.2.3.1].) If $0 < A \leqslant B$, then $\log(A) \leqslant \log(B)$.*

Lemma 2.7. *(See [8, Theorem 2.6].) Let $A, B \in \mathbb{B}(\mathscr{H})$ be two positive operators. Then $B^2 \leqslant A^2$ if and only if for each operator convex function f on $[0,\infty)$ with $f'_+(0) \geqslant 0$ it holds that $f(B) \leqslant f(A)$.*

If f is an increasing operator convex function, then $f'_+(0) \geqslant 0$. The converse is also true. In fact f can be represented as $f(t) = f(0) + \beta t + \gamma t^2 + \int_0^\infty \frac{\lambda t^2}{\lambda + t}\,\mathrm{d}\mu(\lambda)$, where $\gamma \geqslant 0$, $\beta = f'_+(0)$ and μ is a positive measure on $[0,\infty)$; see [1, Chapter V].

Hence if $f'_+(0)$, then $f'(t) = f'_+(0) + 2\gamma t + \int_0^\infty \frac{2\lambda t+\lambda t^2}{(\lambda+t)^2}\, d\mu(\lambda) \geqslant 0$ for each $t \in [0,\infty)$. Now we are ready to state our next result.

Theorem 2.8. *Let A and B be two positive operators. Then $A \leqslant_u B$ if and only if $f(g(A)^r) \leqslant_u f(g(B)^r)$ for any increasing operator convex function f, any operator monotone function g and any positive number r.*

Proof. First we assume that $0 < A \leqslant U^*BU$ for some operator $U \in \mathcal{U}(\mathscr{H})$. Then $0 < g(A) \leqslant g(U^*BU) = U^*g(B)U$ for any operator monotone function g. Let r be a positive number. By Lemma 2.6 we have $\log(g(A)) \leqslant U^* \log(g(B))U$. Hence $\log(g(A)^{2r}) \leqslant \log(U^*g(B)^{2r}U)$. Thus by Kosaki's result (1) there is an operator $V \in \mathcal{U}(\mathscr{H})$ such that $e^{\log g(A)^{2r}} \leqslant V^*e^{\log(U^*g(B)^{2r}U)}V$, that is $g(A)^{2r} \leqslant V^*U^*g(B)^{2r}UV = (V^*U^*g(B)UV)^{2r}$. From which and Lemma 2.7 we conclude that

$$f\big(g(A)^r\big) \leqslant f\big(\big(V^*U^*g(B)UV\big)^r\big) = V^*U^*f\big(g(B)^r\big)UV \tag{3}$$

for any increasing operator convex function f. This means that $f(g(A)^r) \leqslant_u f(g(B)^r)$ as desired.

For the general case note that the condition $0 \leqslant A \leqslant U^*BU$ ensures $0 < A + \varepsilon \leqslant U^*(B + \varepsilon)U$ for all $\varepsilon > 0$. Now the general result is deduced from the paragraph above and a limit argument by letting ε tend to 0.

The reverse is clear by taking $f(x) = x$ and $r = 1$. □

From Theorem 2.8 one can see that if $A \leqslant_u B$, then there exists a sequence $\{U_n\}_{n\in\mathbb{N}} \subset \mathcal{U}(\mathscr{H})$ such that $A^n \leqslant U_n^*B^nU_n$. An interesting problem is finding an operator $U \in \mathcal{U}(\mathscr{H})$ such that $A^n \leqslant U^*B^nU$ for any positive integer n. If there exist a sequence $\{U_n\}_{n\in\mathbb{N}} \subset \mathcal{U}(\mathscr{H})$ such that $A^n \leqslant U_n^*B^nU_n$ and in the strong operator topology $\{U_n\}$ converges to an operator $U \in \mathcal{U}(\mathscr{H})$, then U is the desired unitary operator. To see this let $\xi \in \mathscr{H}$, $n \in \mathbb{N}$ and $m > n$. Then

$$\big\langle A^n\xi, \xi\big\rangle \leqslant \big\langle U_m^*B^nU_m\xi, \xi\big\rangle = \big\langle B^nU_m\xi, U_m\xi\big\rangle. \tag{4}$$

Note that in the inequality of (4) we used $\alpha = \frac{m}{n}$ in the Löwner–Heinz inequality. By our assumption we have $\langle B^nU_m\xi, U_m\xi\rangle \to \langle B^nU\xi, U\xi\rangle = \langle U^*B^nU\xi, \xi\rangle$ as $m \to \infty$, which by (4) implies that $A^n \leqslant U^*B^nU$ as requested.

The next theorem is related to the problem above. First we need to introduce our notation. For any two positive operators A and B and any positive integer n let $\mathcal{K}_{n,A,B} = \{U \in \mathcal{U}(\mathscr{H})\colon\ A^n \leqslant U^*B^nU\}$. This set is compact in the case when $\mathscr{H}$ is finite dimensional. Further, $A \leqslant U^*BU$ for some unitary matrix U if $\lambda_j(A) \leqslant \lambda_j(B)$ $(1 \leqslant j \leqslant n)$, where $\lambda_1(\cdot) \geqslant \cdots \geqslant \lambda_n(\cdot)$ denotes eigenvalues arranged in the decreasing order with their multiplicities counted. Thus $\mathcal{K}_{n,A,B}$ can be nonempty. Our next result reads as follows.

Theorem 2.9. *Suppose that A and B are two positive operators such that $\mathcal{K}_{n_0,A,B}$ is a nonempty set, which is either compact in the strong operator topology or closed in the weak operator topology for some positive integer n_0. Then there exists an operator $U \in \mathcal{U}(\mathscr{H})$ such that $A^n \leqslant U^*B^nU$ for every positive integer n.*

Proof. First assume that $\mathcal{K}_{n_0,A,B}$ is a nonempty strongly compact set for some positive integer n_0. Without loss of generality we may assume that $n_0 = 1$. Let us set $\mathcal{K}_n$ instead of $\mathcal{K}_{n,A,B}$ for the sake of simplicity. Using the Löwner–Heinz inequality one easily see that for any positive integer n

$$\mathcal{K}_{n+1} \subseteq \mathcal{K}_n. \tag{5}$$

We show that the sets $\mathcal{K}_n$ are strongly closed. To achieve this aim, fix n and let $\{U_\alpha\}$ be a net in $\mathcal{K}_n$ such that $U_\alpha \to U$ in which the limit is taken in the strong operator topology. Since $\mathcal{K}_n \subseteq \mathcal{K}_1$ and $\mathcal{K}_1$ is assumed to be a strongly compact set, we conclude that $U \in \mathcal{K}_1$ which implies that $U \in \mathcal{U}(\mathscr{H})$. Let $\xi \in \mathscr{H}$. We have

$$\big\langle A^n\xi, \xi\big\rangle \leqslant \big\langle U_\alpha^*B^nU_\alpha\xi, \xi\big\rangle = \big\langle B^nU_\alpha\xi, U_\alpha\xi\big\rangle. \tag{6}$$

Since $\{U_\alpha\}$ converges strongly to U we obtain

$$\big\langle B^nU_\alpha\xi, U_\alpha\xi\big\rangle \to \big\langle B^nU\xi, U\xi\big\rangle = \big\langle U^*B^nU\xi, \xi\big\rangle. \tag{7}$$

Applying (6) and (7) we get $A^n \leqslant U^*B^nU$. Thus $U \in \mathcal{K}_n$. Hence $\mathcal{K}_n$ is closed. Now Theorem 2.8 shows that the sets $\mathcal{K}_n$ are nonempty and (5) shows that $\bigcap_{n\in F} \mathcal{K}_n = \mathcal{K}_{\max F} \neq \phi$ for any arbitrary finite subset F of $\mathbb{N}$. Hence $\bigcap_{n\in\mathbb{N}} \mathcal{K}_n \neq \phi$ because the $\mathcal{K}_n$ are closed subsets of $\mathcal{K}_1$ and $\mathcal{K}_1$ is compact.

Second, assume that $\mathcal{K}_{n_0}$ is a weakly closed nonempty set for some positive integer n_0. Due to the unit ball of $\mathbb{B}(\mathscr{H})$ is weakly compact, we can repeat the first argument and reach to the desired consequence. □

Now we aim to prove our last result. We state some lemmas which are interesting on their own right.

Lemma 2.10. *Let $P \in \mathbb{B}(\mathscr{H})$ be an orthogonal projection and $U \in \mathcal{U}(\mathscr{H})$ such that $P \leqslant U^*PU$. Then $P = U^*PU$.*

Proof. Let $ran(P) = \mathscr{H}_1$ and let I_1 and I_2 be the identity operators on $\mathscr{H}_1$ and $\mathscr{H}_1^{\perp}$, respectively. Therefore $P = I_1 \oplus 0$ and $U = \left(\begin{smallmatrix} U_1 & U_2 \\ U_3 & U_4 \end{smallmatrix}\right)$ on $\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_1^{\perp}$. From $P \leqslant U^*PU$ we reach to the following inequality:

$$\begin{pmatrix} I_1 & 0 \\ 0 & 0 \end{pmatrix} \leqslant \begin{pmatrix} U_1^*U_1 & U_1^*I_1U_2 \\ U_2^*I_1U_1 & U_2^*I_1U_2 \end{pmatrix}, \tag{8}$$

which implies that $I_1 \leqslant U_1^*U_1$. Since $U^*U = I$ hence

$$\begin{pmatrix} U_1^*U_1 + U_3^*U_3 & U_1^*U_3 + U_3^*U_4 \\ U_2^*U_1 + U_4^*U_3 & U_2^*U_2 + U_4^*U_4 \end{pmatrix} = \begin{pmatrix} I_1 & 0 \\ 0 & I_2 \end{pmatrix}. \tag{9}$$

From (8) and (9) we see that $I_1 = U_1^*U_1$, $U_3 = 0$ and $U_2^*U_1 = 0$. Thus $U_2^* = U_2^*U_1U_1^* = 0$ and this ensures that $U^*PU = \left(\begin{smallmatrix} U_1^*U_1 & 0 \\ 0 & 0 \end{smallmatrix}\right) = P$ as desired. □

In the sequel we need to use the structure of the spectral family $\{E_\lambda(A)\}$ corresponding to an operator $A \in \mathbb{B}_h(\mathscr{H})$; cf. [5]. Recall that $E_\lambda(A)$ can be defined as the strong operator limit $\varphi_\lambda(A)$ of the sequence $\{\varphi_{\lambda,n}(A)\}$, where $\{\varphi_{\lambda,n}\}$ is a sequence of decreasing nonnegative continuous functions on the real line pointwise converging to the following function defined on the spectrum $\mathrm{sp}(A)$ of A:

$$\varphi_\lambda(t) = \begin{cases} 1 & \text{if } -\infty < t \leqslant \lambda, \\ 0 & \text{if } \lambda < t < \infty. \end{cases}$$

Remark 2.11. It follows from Lemma 2.1 that if A is a positive operator and $U \in \mathcal{U}(\mathscr{H})$, then $E_\lambda(U^*AU) = U^*E_\lambda(A)U$ for every $\lambda \in \mathbb{R}$.

Lemma 2.12. *(See [10, Theorem 3].) Let A and B be positive operators in $\mathbb{B}(\mathscr{H})$. Then $A^n \leqslant B^n$ for $n \in \mathbb{N}$ if and only if $E_\lambda(A) \leqslant E_\lambda(B)$ for every $\lambda \in \mathbb{R}$.*

Theorem 2.13. *Let A be a positive operator and $U \in \mathcal{U}(\mathscr{H})$ such that $A^n \leqslant U^*A^nU$ for all $n \in \mathbb{N}$. Then $A = U^*AU$.*

Proof. From Lemma 2.12 and Remark 2.11 we see $E_\lambda(A) \leqslant E_\lambda(U^*AU) = U^*E_\lambda(A)U$. Thus from Lemma 2.10 we have $E_\lambda(A) = U^*E_\lambda(A)U$, which implies that $UE_\lambda(A) = E_\lambda(A)U$ for every $\lambda \in \mathbb{R}$. Hence $UA = AU$, or equivalently $A = U^*AU$. □

Corollary 2.14. *Suppose that A and B are two positive operators such that $\mathcal{K}_{n_0,A,B}$ and $\mathcal{K}_{m_0,B,A}$ are either strongly compact or weakly closed nonempty sets for some positive integers n_0 and m_0, respectively. Then A and B are unitarily equivalent.*

Proof. By Theorem 2.9 there exist operators $U, V \in \mathcal{U}(\mathscr{H})$ such that $A^n \leqslant U^*B^nU$ and $B^n \leqslant V^*A^nV$ for all $n \in \mathbb{N}$. Thus $A^n \leqslant U^*B^nU \leqslant U^*V^*A^nVU$. Now the result is obtained from Theorem 2.13. □

Acknowledgements

The authors would like to thank the referee for several useful comments. This research was supported by a grant from the Ferdowsi University of Mashhad (No. MP91266MOS).

References

[1] R. Bhatia, Matrix Analysis, Springer, New York, 1997.
[2] R.G. Douglas, On the operator equation $S^*XT = X$ and related topics, Acta Sci. Math. (Szeged) 30 (1969) 19–32.
[3] B.P. Duggal, The operator inequality $P \leqslant A^*PA$, Proc. Amer. Math. Soc. 109 (3) (1990) 697–698.
[4] T. Furuta, Invitation to Linear Operators, Taylor and Francis, London, New York, 2001.
[5] G. Helmberg, Introduction to Spectral Theory in Hilbert Space, John Wiley & Sons, New York, 1969.
[6] H. Kosaki, On some trace inequality, Proc. Center Math. Anal. Austral. Nat. Univ. 29 (1992) 129–134.
[7] M.S. Moslehian, S.M.S. Nabavi Sales, Some conditions implying normality of operators, C. R. Acad. Sci. Paris, Ser. I 349 (2011) 251–254.
[8] M.S. Moslehian, H. Najafi, Around operator monotone functions, Integral Equations Operator Theory 71 (4) (2011) 575–582.
[9] T. Okayasu, Y. Ueta, A condition under which $B = A = U^*BU$ follows from $B \geqslant A \geqslant U^*BU$, Proc. Amer. Math. Soc. 135 (5) (2007) 1399–1403.
[10] M.Ph. Olson, The selfadjoint operators of a Von Neumann algebra form a conditionally complete lattice, Proc. Amer. Math. Soc. 28 (2) (1971) 537–544.
[11] M. Uchiyama, Operator monotone functions which are defined implicitly and operator inequalities, J. Funct. Anal. 175 (2) (2000) 330–347.