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On $\mathbb{L}_2$-structure of bilinear models on $\mathbb{Z}^d$

Sur la structure $\mathbb{L}_2$ des modèles bilinéaires sur $\mathbb{Z}^d$

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ABSTRACT

One-dimensionally indexed bilinear (*BL*) models are widely used for modeling non-Gaussian dataset. Extending *BL* models to multidimensionally indexed (spatial) (*SBL*) one yields a novel class of models which are capable of taking into account the non-Gaussianity character and spatiality behavior. Hence, the main contribution here is to study the $\mathbb{L}_2$-structure of some *SBL* models which play an important role in spatial statistical analysis. So, we establish necessary and sufficient conditions for the existence of regular second order stationary and ergodic solutions in terms of its transfer functions. As a consequence, we observe that the second order structure is similar to a weak *ARMA* field, and that the variance of the best linear prediction error is always greater than the one obtained from an *SBL* model.

RÉSUMÉ

Les modèles bilinéaires (*BL*) classiques sont largement utilisés pour la modélisation des données non gaussiennes. Cependant, l'extension de ces modèles au cas spatial (*SBL*) donne une nouvelle classe de modèles susceptibles de prendre en considération la non gaussianité et le comportement spatial. Le but principal de cette Note consiste à étudier la structure $\mathbb{L}_2$ de certains modèles *SBL* qui jouent un rôle très important dans l'analyse statistique spatiale. Nous établissons des conditions nécessaires et suffisantes pour l'existence de solutions stationnaires aux seconds ordres, réguliers et ergodiques basées sur les fonctions de transferts. En utilisant la représentation ARMA spatiale, on montre que la variance de l'erreur de prédiction linéaire est toujours plus grande que celle obtenue par *SBL*.

Version française abrégée

Dans la présente Note, nous étudions la structure $\mathbb{L}_2$ des champs aléatoires non linéaires générés par l'équation récursive (3). Après avoir présenté les principaux résultats de la généralisation au cas $n - D$, nous donnons des conditions nécessaires et suffisantes (*CNS*) d'existence de solution unique, causale et stationnaire basées sur les fonctions de transferts associées. Nous dérivons également des *CNS* pour d'autre modèles en particulier pour les processus *GARCH* sur $\mathbb{Z}^d$. Nous terminons notre étude par une représentation *ARMA* sur $\mathbb{Z}^d$ du modèle (3) et nous concluons que la variance de l'erreur de prédiction linéaire est toujours plus grande que celle obtenue par (3).

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1. Introduction

The d-dimensionally indexed processes $(X(\mathbf{t}))_{\mathbf{t}\in\mathbb{Z}^d}$ where $\mathbb{Z}^d$ denotes the d-dimensional integer lattice arises naturally in modeling some spatial dataset. These processes are often assumed to be linear and may be Gaussian (see for instance [4] and the references therein). Recent studies have shown that such an assumption is very unrealistic. Hence, by extending some one-dimensionally indexed nonlinear models to multidimensionally indexed one, yields a novel random fields which are capable to taking into account the nonlinearity and spatiality dependence. Indeed, Amirmazlaghani and Amindavar [1] have used two-dimensional *GARCH* model for wavelet coefficients modeling in order to perform the image denoising, and to distilling a small number of clustered pixels. Doukhan and Truquet [3] have proposed a general random fields models with infinite interactions which encompass many commonly used models in the literature. Their contribution is focused on conditions ensuring the weak dependency with long memory solutions.

In this Note, we examine the $\mathbb{L}_2$-structure of bilinear processes (*SBL*) indexed by $\mathbb{Z}^d$. The study of such processes is motivated by the theoretical and practical demand of finding necessary and sufficient conditions ensuring the existence of regular stationary (in $\mathbb{L}_2$ sense) and ergodic solutions for *SBL* models. Under these conditions, the best linear predictor is obtained, and it is shown that the variance of the prediction errors is greater than the one obtained from *SBL* models. As a consequence, we observe that the $\mathbb{L}_2$-structure of *SBL* is the same as an *ARMA* model on $\mathbb{Z}^d$. Throughout, d denotes some positive integer, $\mathbf{0} = (0, \ldots, 0)$ and $\mathbf{1} = (1, \ldots, 1)$ are the zeros and the unity vectors of $\mathbb{Z}^d$ and for any $\mathbf{k} = (k_1, \ldots, k_d)$, $\mathbf{l} = (l_1, \ldots, l_d)$ belonging to $\mathbb{Z}^d$ and $\mathbf{z} = (z_1, \ldots, z_d) \in \mathbb{C}^d$ we write $\mathbf{k} \preccurlyeq \mathbf{l}$ (resp. $\mathbf{k} \prec \mathbf{l}$) iff $k_m \leqslant l_m$ (resp. $k_m < l_m$) for $m = 1, \ldots, d$ and $\mathbf{z}^{\mathbf{l}} = \prod_{j=1}^{d} z_j^{l_j}$. For $\mathbf{p} \in \mathbb{N}^d$, the following indexing subsets in $\mathbb{N}^d$ will be considered $\Gamma[\mathbf{p}] = \{\mathbf{x} \in \mathbb{N}^d / \mathbf{0} \preccurlyeq \mathbf{x} \preccurlyeq \mathbf{p}\}$, $\Gamma]\mathbf{p}] = \Gamma[\mathbf{p}] \setminus \{\mathbf{0}\}$.

2. The Wiener–Itô representation

For any stationary Gaussian random field $(e(\mathbf{t}))_{\mathbf{t}\in\mathbb{Z}^d}$ with mean 0 and variance σ^2, we associate the spectral representation, i.e., $e(\mathbf{t}) = \int_\pi e^{i\mathbf{t}.\lambda}\, \mathrm{d}Z(\lambda)$ in which $\mathbf{t}.\lambda = \sum_{i=1}^{d} \lambda_i t_i$ for $\mathbf{t} = (t_1, \ldots, t_d) \in \mathbb{Z}^d$, $\lambda = (\lambda_1, \ldots, \lambda_d) \in \pi = [-\pi, \pi[^d$ and Z is a Gaussian orthogonal stochastic measure with $E\{\mathrm{d}Z(\lambda)\} = 0$ and spectral measure $\mathrm{d}F(\lambda) := E\{|\mathrm{d}Z(\lambda)|^2\} = \frac{\sigma^2}{(2\pi)^d}\, \mathrm{d}\lambda$ where $\mathrm{d}\lambda$ means the Lebesgue measure on $\mathbb{R}^d$. Let $\mathcal{H} = L_2(\pi, \mathcal{B}_\pi, F)$ denote the real Hilbert space consisting of the square integrable complex functions f satisfying $f(-\lambda) = \overline{f(\lambda)}$ for any $\lambda \in \pi$. For any $n \geqslant 1$, we associated three real Hilbert spaces based on $\mathcal{H}$, the first is $\mathcal{H}_n = \mathcal{H}^{\otimes n}$ the n-fold tensor product of $\mathcal{H}$ endowed by the inner product $\langle f_n, g_n \rangle_\otimes = \int_{\pi^n} f_n(\lambda_{(n)}) \overline{g_n(\lambda_{(n)})}\, \mathrm{d}F(\lambda_{(n)})$ where $\lambda_{(n)} = (\lambda_1, \ldots, \lambda_n) \in \pi^n$, $f_n(-\lambda_{(n)}) = \overline{f_n(\lambda_{(n)})}$, $\|f_n\|^2 < \infty$, $\mathrm{d}F(\lambda_{(n)}) = \prod_{i=1}^{n} \mathrm{d}F(\lambda_i)$ and $\mathrm{d}\lambda_{(n)} = \prod_{i=1}^{n} \mathrm{d}\lambda_i$. The second one is $\widehat{\mathcal{H}}_n = \mathcal{H}^{\oplus n} \subset \mathcal{H}_n$ the n-fold symmetrized tensor product of $\mathcal{H}$ defined by $f_n \in \widehat{\mathcal{H}}_n$ iff f_n is invariant under permutation of their arguments, i.e., $f_n(\lambda_{(n)}) = f_n(\lambda_{p(1)}, \ldots, \lambda_{p(n)})$ for all $p \in \mathcal{P}(n)$ where $\mathcal{P}(n)$ denotes the group of all permutations of the set $\{1, \ldots, n\}$ with an inner product $\langle f_n, g_n \rangle_\oplus = n! \langle f_n, g_n \rangle_\otimes$ for $f_n, g_n \in \widehat{\mathcal{H}}_n$. The third space is called Fock space over $\mathcal{H}$ denoted by $\Im(\mathcal{H})$ and defined by $\Im(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \widehat{\mathcal{H}}_n$ in which $\bigoplus$ denotes the direct orthogonal sum, whose elements are $f := (f_0, f_1, f_2, \ldots)$ with $f_n \in \widehat{\mathcal{H}}_n$, $\widehat{\mathcal{H}}_0 = \mathcal{H}_0 = \mathbb{R}$ and satisfying $\|f\|^2 = \sum_{n\geqslant 0} \frac{1}{n!} \|f_n\|^2 < +\infty$. Finally for any $f_n \in \mathcal{H}_n$, we define $sym\{f_n\}$ by $sym\{f_n(\lambda_{(n)})\} = \frac{1}{n!} \sum_{p\in\mathcal{P}(n)} f_n(\lambda_{(p(n))})$.

Let $\Im = \Im(e) := \sigma(e(\mathbf{t}), \mathbf{t} \in \mathbb{Z}^d)$ be the σ-algebra generated by all $e(\mathbf{t})$, $\mathbf{t} \in \mathbb{Z}^d$, $\Im_{\mathbf{t}}(e) := \sigma(e(\mathbf{s}), \mathbf{s} \preccurlyeq \mathbf{t})$ and $\mathbb{L}_2(\Im)$ be the real Hilbert space of $\mathbb{L}_2$-functional of $e(\mathbf{t})$, $\mathbf{t} \in \mathbb{Z}^d$, endowed by the inner product $\langle X, Y \rangle = E\{XY\}$. It is well known (see [6] for further details) that $\mathbb{L}_2(\Im)$ is isometrically isomorphic to $\Im(\mathcal{H})$, i.e., for any stationary random field $(X(\mathbf{t}))_{\mathbf{t}\in\mathbb{Z}^d}$ of $\mathbb{L}_2(\Im)$ admits the so-called Wiener–Itô orthogonal representation, i.e.,

$$X(\mathbf{t}) = f_0 + \sum_{r\geqslant 1} \int_{\pi^r} f_r(\lambda_{(r)}) e^{i\mathbf{t}.\underline{\lambda}_{(r)}}\, \mathrm{d}Z(\lambda_{(r)}) \tag{1}$$

where $\underline{\lambda}_{(r)} := \sum_{i=1}^{r} \lambda_i$, $f_0 = E\{X(\mathbf{t})\}$, $\mathrm{d}Z(\lambda_{(r)}) = \prod_{i=1}^{r} \mathrm{d}Z(\lambda_i)$, $f_r \in \widehat{\mathcal{H}}_r$ are referred as d-dimensional transfer functions of $(X(\mathbf{t}))_{\mathbf{t}\in\mathbb{Z}^d}$ and uniquely determined and the integrals are the Wiener–Itô stochastic integrals with respect to Z.

Example 1. A general class of nonlinear random fields $(X(\mathbf{t}))_{\mathbf{t}\in\mathbb{Z}^d}$ which admits a regular solution are the Wiener fields, i.e.,

$$X(\mathbf{t}) = g_{\mathbf{0}} + \sum_{r=1}^{\infty} \sum_{\mathbf{k}_{(r)}\in(\mathbb{N}^d\setminus\{\mathbf{0}\})^r} \sum_{\mathbf{s}_{(r)}\in(\mathbb{N}^d_{\prec})^r} g_{\mathbf{k}_{(r)}}(\mathbf{s}_{(r)}) \prod_{j=1}^{r} h_{k_j}\big(e(\mathbf{t} - \mathbf{s}_j)\big) \tag{2}$$

where h_j denotes the j-th Hermite polynomial with leading coefficient 1, i.e., $h_j(x) = (-1)^j e^{\frac{x^2}{2}} \frac{\mathrm{d}^j}{\mathrm{d}x^j} e^{-\frac{x^2}{2}}$, $x \in \mathbb{R}$, $(\mathbb{N}^d_{\prec})^r := \{\mathbf{s}_{(r)} \in (\mathbb{N}^d)^r\colon\ \mathbf{0} \preccurlyeq \mathbf{s}_1 \prec \mathbf{s}_2 \prec \cdots \prec \mathbf{s}_r\}$ and where the Volterra's kernels $g_{\mathbf{k}_{(r)}}(\mathbf{s}_{(r)})$ are uniquely determined if there are assumed to be symmetric functions in their arguments. Hence, by applying Itô's formula, it is easily seen that $X(\mathbf{t})$ admits a Wiener–Itô orthogonal representation (1).

In the following section, various random fields satisfying the Wiener models (2) will be investigated.

3. Wiener–Itô solution for subdiagonal bilinear random fields

Let us consider the subdiagonal bilinear random field $(X(\mathbf{t}))_{\mathbf{t}\in\mathbb{Z}^d}$ defined on some probability space $(\Omega,\Im,P)$ (see [7] and [9] for structural properties) denoted by $SBL_d(\mathbf{p},\mathbf{q},\mathbf{P},\mathbf{Q})$ subject to the recursion equation

$$X(\mathbf{t}) = a_{\mathbf{0}} + \sum_{\mathbf{i}\in\Gamma]\mathbf{p}]} a_{\mathbf{i}} X(\mathbf{t}-\mathbf{i}) + \sum_{\mathbf{j}\in\Gamma[\mathbf{q}]} b_{\mathbf{j}} e(\mathbf{t}-\mathbf{j}) + \sum_{\mathbf{i}\in\Gamma[\mathbf{P}],\,\mathbf{j}\in\Gamma]\mathbf{Q}]}\sum c_{\mathbf{ji}} X(\mathbf{t}-\mathbf{i}-\mathbf{j}) e(\mathbf{t}-\mathbf{j}). \tag{3}$$

In (3) $(e(\mathbf{t}))_{\mathbf{t}\in\mathbb{Z}^d}$ is an $i.i.d(0,\sigma^2)$ Gaussian random field defined on the same probability $(\Omega,\Im,P)$ with $e(\mathbf{l})$ is independent of $X(\mathbf{k})$, $\mathbf{k}\prec\mathbf{l}$. Model (3) may be viewed as a special case of Wiener's field (2) and extend on *ARMA* random fields (see Yao and Brockwell [10]). Noting here that different SBL_d representations appear to depend on the lexicographic order chosen on $\mathbb{Z}^d$. Noting here that formally the *ARCH*-type bilinear random fields appear to be a particular case of more general bilinear random fields (see [5] for an extensive discussion). Our aim objective here is to seek necessary and sufficient conditions ensuring the existence of regular stationary solution with short memory of (3) in the Form (1). For this purpose define the transfer functions

$$\Theta(\lambda) = 1 - \sum_{\mathbf{i}\in\Gamma]\mathbf{p}]} a_{\mathbf{i}} e^{-i\mathbf{i}.\lambda}, \qquad \Phi(\lambda) = \sum_{\mathbf{j}\in\Gamma[\mathbf{q}]} b_{\mathbf{j}} e^{-i\mathbf{j}.\lambda}, \qquad \Psi_0(\lambda) = \sum_{\mathbf{j}\in\Gamma]\mathbf{Q}]} c_{\mathbf{j0}} e^{-i\mathbf{j}.\lambda},$$
$$\Psi(\lambda,\mu) = \sum_{\mathbf{i}\in\Gamma[\mathbf{P}]}\sum_{\mathbf{j}\in\Gamma]\mathbf{Q}]} c_{\mathbf{ji}} e^{-i(\mathbf{i}+\mathbf{j}).\lambda} e^{-i\mathbf{j}.\mu}$$

and assume the following:

Condition 3.1. All the characteristic roots of the polynomial $\Theta(\mathbf{z}) = 1 - \sum_{\mathbf{i}\in\Gamma]\mathbf{p}]} a_{\mathbf{i}}\mathbf{z}^{\mathbf{i}}$ are outside the unit circle, in the sense that $\Theta(\mathbf{z}) \neq 0$ for $|z_i| \leqslant 1$, $i = 1,\dots,d$ (see also Yao and Brockwell [10] for further discussions).

Lemma 3.2. *Assume that the SBL_d model* (3) *has regular stationary solution. Then the transfer functions are given by the symmetrization of the following functions defined recursively by*

$$f_0 = \sigma^2 \frac{\Psi_0(\mathbf{0}) + a_{\mathbf{0}}}{\Theta(\mathbf{0})}, \qquad f_1(\lambda) = \frac{\Phi^*(\lambda)}{\Theta(\lambda)}, \qquad f_r(\lambda_{(r)}) = \frac{\Psi(\underline{\lambda}_{(r-1)},\lambda_r)}{\Theta(\underline{\lambda}_{(r)})} f_{r-1}(\lambda_{(r-1)}) \quad \text{if } r \geqslant 2 \tag{4}$$

with $\Phi^*(\lambda) = \Phi(\lambda) + f_0\Psi(\mathbf{0},\lambda)$.

Proof. The proof follows essentially the same arguments as in Terdik and Subba Rao [8]. □

Remark 1. It not difficult to see that the symmetrized transfer functions are given by

$$sym\{f_0\} = f_0, \qquad sym\{f_1(\lambda)\} = f_1(\lambda),$$
$$sym\{f_r(\lambda_{(r)})\} = \Theta^{-1}(\underline{\lambda}_{(r)}) \sum_{\mathbf{j}\in\Gamma[\mathbf{Q}]} e^{-i\mathbf{j}.\underline{\lambda}_{(r)}} \sum_{\mathbf{i}\in\Gamma]\mathbf{P}]} c_{\mathbf{ij}}\, sym\{f_{r-1}(\lambda_{(r-1)}) e^{-i\mathbf{i}.\underline{\lambda}_{(r-1)}}\}, \quad \text{if } r \geqslant 2.$$

Lemma 3.3. *For any $f_n \in \mathcal{H}_n$, we have $\|f_n\|^2 \leqslant n!\|sym\{f_n\}\|^2 \leqslant 2\|f_n\|^2$ for any $n \geqslant 1$.*

Proof. The proof follows from standard arguments (cf. Terdik and Subba Rao [8]). □

Theorem 3.4. *A necessary and sufficient condition* (*NSD*) *for the existence of regular stationary solution for $SBL_d(\mathbf{p},\mathbf{q},\mathbf{P},\mathbf{Q})$ model* (3) *is that*

$$\sum_{r\geqslant 0} \|f_r\|^2 < +\infty \tag{5}$$

where the transfer functions $f_r(\lambda_{(r)})$ are given by (4).

Proof. To prove Theorem 3.4, we use Lemmas 3.2, 3.3 and the fact that $Var(X(\mathbf{t}))$ is finite iff the condition (5) holds true. □

Lemma 3.5. *A simple sufficient condition for* (5) *is* $\frac{\sigma^2}{(2\pi)^d}\int_\pi |\frac{\Psi(\lambda,\mu)}{\Theta(\lambda+\mu)}|^2\,\mathrm{d}\mu = c < 1$, $\lambda \in \pi$.

Proof. Since

$$\int_{\pi^r} \left|f_r(\lambda_{(r)})\right|^2 \mathrm{d}F(\lambda_{(r)}) = \int_{\pi^{r-1}} \left\{ \int_{\pi} \left| \frac{\Psi(\lambda,\mu)}{\Theta(\lambda+\mu)} \right|^2 \mathrm{d}F(\mu) \right\} \left|f_{r-1}(\lambda_{(r-2)}, \lambda - \underline{\lambda}_{(r-2)})\right|^2 \mathrm{d}F(\lambda_{(r-2)})\,\mathrm{d}F(\lambda)$$

then, if there exists c such that $\frac{\sigma^2}{(2\pi)^d} \int_\pi |\frac{\Psi(\lambda,\mu)}{\Theta(\lambda+\mu)}|^2 \,\mathrm{d}\mu = c < 1$ for any $\lambda \in \pi$, we get

$$\int_{\pi^r} \left|f_r(\lambda_{(r)})\right|^2 \mathrm{d}F(\lambda_{(r)}) \leqslant c \int_{\pi^r} \left|f_{r-1}(\lambda_{(r-1)})\right|^2 \mathrm{d}F(\lambda_{(r-1)}) \leqslant \frac{\sigma^2 c^{r-1}}{(2\pi)^d} \int_\pi \left| \frac{\Phi^*(\lambda)}{\Theta(\lambda)} \right|^2 \mathrm{d}\lambda$$

so the condition (5) holds true. □

Corollary 3.6 *(Superdiagonal model). A sufficient condition for the superdiagonal random field (i.e., $c_{\mathbf{i},\mathbf{j}} = 0$ for $\mathbf{i} \preccurlyeq \mathbf{j}$ in (3)) to have a regular stationary solution is that $\frac{\sigma^2}{(2\pi)^d} \int_\pi |\frac{\Psi(\lambda,\mu)}{\Theta(\lambda+\mu)}|^2 \,\mathrm{d}\lambda \leqslant K < 1$, $\mu \in \pi$.*

Proof. In this case $f_0 = a_{\mathbf{0}}$, $f_1(\lambda) = \frac{\Phi^*(\lambda)}{\Theta(\lambda)}$, $f_r(\lambda_{(r)}) = \frac{\Psi(\underline{\lambda}_{(r-1)},\lambda_r)}{\Theta(\underline{\lambda}_{(r)})} f_{r-1}(\lambda_{(r-1)})$, if $r \geqslant 2$, so, we obtain $f_r(\lambda_{(r)}) = \prod_{s=2}^r \frac{\Psi(\underline{\lambda}_{(s-1)},\lambda_s)}{\Theta(\underline{\lambda}_{(s)})} \Psi_0(\lambda_1)$. Using the last expression of $f_r(\lambda_{(r)})$, the condition of stationarity is thus

$$\sum_{r=1}^{\infty} \left(\frac{\sigma^2}{(2\pi)^d} \right)^r \int_{\pi^r} \left|f_r(\lambda_{(r)})\right|^2 \mathrm{d}\lambda_{(r)} < +\infty. \tag{6}$$

It is easy to see that the series (6) is dominated by a geometrically converged series. □

Corollary 3.7. *Consider the separable model (see [9]), i.e.,*

$$X(\mathbf{t}) = a_{\mathbf{0}} + \sum_{\mathbf{i}\in\Gamma]\mathbf{p}]} a_{\mathbf{i}} X(\mathbf{t}-\mathbf{i}) + \sum_{\mathbf{j}\in\Gamma[\mathbf{q}]} b_{\mathbf{j}} e(\mathbf{t}-\mathbf{j}) + \sum_{\mathbf{i}\in\Gamma]\mathbf{Q}]} c_{\mathbf{i}}^{(1)} e(\mathbf{t}-\mathbf{i}) \sum_{\mathbf{j}\in\Gamma[\mathbf{P}]} c_{\mathbf{j}}^{(2)} X(\mathbf{t}-\mathbf{i}-\mathbf{j}) \tag{7}$$

then an NSD for the existence of regular stationary solution of the process $(X(\mathbf{t}))_{\mathbf{t}\in\mathbb{Z}^d}$ generated by (7) is that $\frac{\sigma^2}{(2\pi)^d} \int_\pi |\frac{\Psi_1(\lambda)\Psi_2(\lambda)}{\Theta(\lambda)}|^2 \,\mathrm{d}\lambda < 1$ where $\Psi_1(\lambda) = \sum_{\mathbf{i}\in\Gamma]\mathbf{Q}]} c_{\mathbf{i}}^{(1)} e^{-\mathrm{i}\mathbf{i}.\lambda}$ and $\Psi_2(\lambda) = \sum_{\mathbf{j}\in\Gamma[\mathbf{P}]} c_{\mathbf{j}}^{(2)} e^{-\mathrm{i}\mathbf{j}.\lambda}$.

Proof. For any $r \geqslant 2$ we have

$$\begin{aligned}
\int_{\pi^r} \left|f_r(f\lambda_{(r)})\right|^2 \mathrm{d}F(\lambda_{(r)}) &= \int_{\pi^r} \left| \prod_{l=1}^r \Theta^{-1}(\underline{\lambda}_{(l)}) \Psi(\underline{\lambda}_{(l-1)}, \lambda_l) \right|^2 \left|\Theta^{-1}(\lambda_1)\Phi^*(\lambda_1)\right|^2 \mathrm{d}F(\lambda_{(r)}) \\
&= \int_{\pi^r} \left| \prod_{l=1}^r \Theta^{-1}(\underline{\lambda}_{(l)}) \Psi_1(\underline{\lambda}_{(l)}) \Psi_2(\underline{\lambda}_{(l-1)}) \right|^2 \left|\Theta^{-1}(\lambda_1)\Phi^*(\lambda_1)\right|^2 \mathrm{d}F(\lambda_{(r)}) \\
&= \int_\pi \left|\Theta^{-1}(\lambda)\Psi_1(\lambda)\right|^2 \mathrm{d}F(\lambda) \left[\int_\pi \left|\Theta^{-1}(\lambda)\Psi_1(\lambda)\Psi_2(\lambda)\right|^2 \mathrm{d}F(\lambda) \right]^{r-2} \\
&\quad \times \int_\pi \left|\Theta^{-1}(\lambda_1)\Phi^*(\lambda_1)\Psi_2(\lambda)\right|^2 \mathrm{d}F(\lambda).
\end{aligned}$$

The result follows by Theorem 3.4 iff $\int_\pi |\Theta^{-1}(\lambda)\Psi_1(\lambda)\Psi_2(\lambda)|^2 \,\mathrm{d}F(\lambda) < 1$. □

Corollary 3.8. *Consider the model*

$$X(\mathbf{t}) = a_{\mathbf{0}} + \sum_{\mathbf{i}\in\Gamma]\mathbf{p}]} a_{\mathbf{i}} X(\mathbf{t}-\mathbf{i}) + \sum_{\mathbf{j}\in\Gamma[\mathbf{q}]} b_{\mathbf{j}} e(\mathbf{t}-\mathbf{j}) + \sum_{\mathbf{j}\in\Gamma]\mathbf{P}]} c_{\mathbf{j}} X(\mathbf{t}-\mathbf{j}-\mathbf{l}) e(\mathbf{t}-\mathbf{j}) \tag{8}$$

where $\mathbf{l} \in \mathbb{N}^d$. Then the NSD for the existence of regular stationary solution for (8) is that

$$\frac{\sigma^2}{(2\pi)^d} \int_\pi \left| \frac{\Psi_1(\lambda)}{\Theta(\lambda)} \right|^2 \mathrm{d}\lambda < 1. \tag{9}$$

Proof. In this case, $\Psi_1(\lambda) = \sum_{\mathbf{i}\in\Gamma]\mathbf{Q}]} c_{\mathbf{i}} e^{-i\lambda.\mathbf{i}}$, $\Psi_2(\lambda) = e^{-i\lambda.\mathbf{l}}$ and $\Psi_0(\lambda) = \Psi_1(\lambda)\delta_{\mathbf{0}}^{\mathbf{l}}$. So the *NSD* for the regular stationary solution reduces to (9). □

Corollary 3.9. *Consider the model*

$$X(\mathbf{t}) = a_{\mathbf{0}} + \sum_{\mathbf{i}\in\Gamma]\mathbf{p}]} a_{\mathbf{i}} X(\mathbf{t}-\mathbf{i}) + \sum_{\mathbf{j}\in\Gamma[\mathbf{q}]} b_{\mathbf{j}} e(\mathbf{t}-\mathbf{j}) + \sum_{\mathbf{j}\in\Gamma]\mathbf{P}]} c_{\mathbf{j}} X(\mathbf{t}-\mathbf{j}-\mathbf{l}) e(\mathbf{t}-\mathbf{l}), \tag{10}$$

where $\mathbf{l}\in\mathbb{N}^d\setminus\{\mathbf{0}\}$. *Then the NSD for the existence of regular stationary solution for* (10) *is that*

$$\frac{\sigma^2}{(2\pi)^d}\int_{\pi}\left|\frac{\Psi_2(\lambda)}{\Theta(\lambda)}\right|^2 \mathrm{d}\lambda < 1. \tag{11}$$

Proof. In this case, $\Psi_1(\lambda) = e^{-i\mathbf{l}.\lambda}$ and $\Psi_2(\lambda) = \sum_{\mathbf{i}\in\Gamma]\mathbf{Q}]} c_{\mathbf{i}} e^{-i\mathbf{i}.\lambda}$, so the *NSD* for the regular stationary solution reduces to (11). □

Corollary 3.10 *(SGARCH). Consider the GARCH*($\mathbf{p},\mathbf{q}$) *random field defined by*

$$X(\mathbf{t}) = \eta(\mathbf{t})\sqrt{h(\mathbf{t})} \quad \text{and} \quad h(\mathbf{t}) = a_{\mathbf{0}} + \sum_{\mathbf{i}\in\Gamma]\mathbf{p}]} c_{\mathbf{i}} X^2(\mathbf{t}-\mathbf{i}) + \sum_{\mathbf{j}\in\Gamma]\mathbf{q}]} a_{\mathbf{j}} h(\mathbf{t}-\mathbf{j}) \tag{12}$$

where $(c_{\mathbf{i}}, \mathbf{i}\in\Gamma]\mathbf{p}])$ *and* $(a_{\mathbf{i}}, \mathbf{i}\in\Gamma[\mathbf{q}])$ *are nonnegative constants with* $a_{\mathbf{0}} > 0$ *and* $(\eta(\mathbf{t}))_{\mathbf{t}\in\mathbb{Z}^d}$ *is a Gaussian random field with zero mean and variance* 1. *Then the model* (12) *has a regular stationary solution iff* $\sum_{\mathbf{i}\in\Gamma]\mathbf{p}]} a_{\mathbf{i}} + \sum_{\mathbf{j}\in\Gamma]\mathbf{q}]} c_{\mathbf{j}} < 1$. *Hence* $E\{X(\mathbf{t})\} = 0$, $Cov(X(\mathbf{t}), X(\mathbf{s})) = a_{\mathbf{0}}\delta_{\mathbf{s}}^{\mathbf{t}}(1 - \sum_{\mathbf{i}\in\,\Gamma]\mathbf{p}]} a_{\mathbf{i}} - \sum_{\mathbf{j}\in\Gamma]\mathbf{q}]} c_{\mathbf{j}})^{-1}$.

Proof. Since the volatility process in (12) can be regarded as a special case of model (8) with $\mathbf{l} = \mathbf{0}$, then the proof follows thus from Corollary 3.8 and the positivity of the coefficients. □

4. Applications

Theorem 4.1 *(ARMA representation). Assume that the field* $(X(\mathbf{t}))_{\mathbf{t}\in\mathbb{Z}^d}$ *defined by* (3) *is stationary, there exists an uncorrelated sequence of random fields* $(\xi(\mathbf{t}))_{\mathbf{t}\in\mathbb{Z}^d}$ *with zero mean and finite variance such that*

$$X(\mathbf{t}) = a_{\mathbf{0}} + \sum_{\mathbf{i}\in\Gamma]\mathbf{p}]} a_{\mathbf{i}} X(\mathbf{t}-\mathbf{i}) + \sum_{\mathbf{j}\in\Gamma[\mathbf{q}^*]} b_{\mathbf{j}}^* \xi(\mathbf{t}-\mathbf{j}), \qquad b_{\mathbf{0}}^* = b_{\mathbf{0}} = 1, \tag{13}$$

where the coefficients $(b_{\mathbf{j}}^*, \mathbf{j}\in\Gamma[\mathbf{q}^*])$ *are functions of* $(a_{\mathbf{j}}, \mathbf{j}\in\Gamma]\mathbf{p}])$, $(b_{\mathbf{j}}, \mathbf{j}\in\Gamma|\mathbf{q}])$ *and* $(c_{\mathbf{ji}}, \mathbf{j}\in\Gamma]\mathbf{Q}], \mathbf{i}\in\Gamma[\mathbf{P}])$. *The field* $(\xi(\mathbf{t}))_{\mathbf{t}\in\mathbb{Z}^d}$ *is not Gaussian nor a martingale difference sequence when the* $c_{\mathbf{ij}}$*'s are not equal to zero.*

Proof. The proof follows essentially the same arguments as in Bibi [2]. □

The above theorem implies that the spectral density of the field $(X(\mathbf{t}))_{\mathbf{t}\in\mathbb{Z}^d}$ is given by

$$f(\lambda) = \frac{\sigma^2}{(2\pi)^d}\frac{|\widetilde{\Phi}(\lambda)|^2}{|\Theta(\lambda)|^2} \tag{14}$$

where $\widetilde{\Phi}(\lambda) = \sum_{\mathbf{j}\in\Gamma[\mathbf{q}^*]} b_{\mathbf{j}}^* e^{-i\mathbf{j}.\lambda}$ such that $|\widetilde{\Phi}(\lambda)|^2 = |\Phi(\lambda)|^2 + \sigma^2|\Psi_0(\lambda)|^2 + |D(\lambda)|^2$ for some transfer function $D(\lambda)$. Hence, the second order properties of every bilinear random field $(X(\mathbf{t}))_{\mathbf{t}\in\mathbb{Z}^d}$ satisfying Eq. (3) are similar to an *ARMA*($\mathbf{p},\mathbf{q}^*$). So, one has to look to higher order moments and higher order cumulant spectra for further information on the process. The best linear predictor of $X(\mathbf{t}+\mathbf{h})$ given by $\{X(\mathbf{s}), \mathbf{s}\preccurlyeq\mathbf{t}\}$ where $(X(\mathbf{t}))_{\mathbf{t}\in\mathbb{Z}^d}$ satisfies (13) is now given:

Theorem 4.2. *Let* $(X(\mathbf{t}))_{\mathbf{t}\in\mathbb{Z}^d}$ *be a stationary random field satisfying* (13) *and assume that the polynomial* $\widetilde{\Phi}(\mathbf{z}) = \sum_{\mathbf{j}\in\Gamma[\mathbf{q}^*]} b_{\mathbf{j}}^*\mathbf{z}^{\mathbf{j}} \neq 0$ *for all* $\mathbf{z}\in\mathbb{C}^d$: $|z_i|\leqslant 1$, $i = 1,\dots,d$. *Let* $\widehat{X}_{\mathbf{h}}(\mathbf{t})$ *be the best linear predictor of* $X(\mathbf{t}+\mathbf{h})$, $\mathbf{0}\preccurlyeq\mathbf{h}\preccurlyeq\mathbf{1}$ *and* $\mathbf{h}\neq\mathbf{0}$ *when* $\{X(\mathbf{s}), \mathbf{s}\preccurlyeq\mathbf{t}\}$ *is given. Then*

$$\widehat{X}_{\mathbf{h}}(\mathbf{t}) = \left(1 - \frac{\Theta(\mathbf{B})}{\widetilde{\Phi}(\mathbf{B})}\right) X(\mathbf{t}+\mathbf{h})$$

where $\mathbf{B}$ *is the backward shift operator, i.e.,* $\mathbf{B}^{\mathbf{i}}X(\mathbf{t}) = X(\mathbf{t}-\mathbf{i})$ *and* $\sigma_\xi^2 = Var\{\xi(\mathbf{t})\} > Var\{e(\mathbf{t})\} = \sigma^2$.

Proof. The first assertion rests standard. The second follows essentially the same arguments as in Bibi [2]. □

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