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# New boundary conditions on the time-like conformal infinity of the Anti-de Sitter universe

## *Conditions au bord conforme de genre temps de l'univers Anti-de Sitter*

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ABSTRACT

We investigate the possible propagations of the gravitational waves in the 5-dimensional Anti-de Sitter universe. We construct a large family of unitary dynamics with respect to some high order energies that are conserved and positive. These dynamics are associated with asymptotic conditions on the conformal time-like boundary of the universe. The key point is the introduction of a new Hilbert functional framework that contains the massless graviton which is not normalizable in $L^2$. The proof needs the study of the Klein–Gordon equation on $\mathbb{R}^{1+6}$ with a super-singular perturbation of $\delta_0$-type.



RÉSUMÉ

Nous étudions diverses propagations possibles des ondes gravitationnelles dans l'univers Anti-de Sitter pentadimensionnel. Nous construisons une grande famille de dynamiques unitaires par rapport à des énergies conservées d'ordre élevé et positives. Ces dynamiques sont associées à des conditions asymptotiques au bord conforme de genre temps de l'univers. Le point clef est l'introduction d'un nouveau cadre fonctionnel hilbertien qui contient le graviton sans masse, lequel n'est pas normalisable dans $L^2$. La preuve s'appuie sur l'étude de l'équation de Klein–Gordon dans $\mathbb{R}^{1+6}$ avec une perturbation super-singulière de type Dirac à l'origine.



### Version française abrégée

Les fluctuations gravitationnelles dans l'univers Anti-de Sitter de dimension 5 sont solutions de l'équation

$$\left(\partial_t^2-\Delta_{\mathbf{x}}-\partial_z^2+\frac{15}{4z^2}\right)\Phi=0,\quad \text{in } \mathbb{R}_t\times\mathbb{R}^3_{\mathbf{x}}\times\,]0,\infty[_z.$$

$z=0$ situe l'infini conforme de genre temps de cet espace-temps non globalement hyperbolique ce qui amène la question d'éventuelles conditions sur $\Phi$ à imposer sur ce bord. L'opérateur $-\Delta_{\mathbf{x}}-\partial_z^2+\frac{15}{4z^2}$ étant essentiellement auto-adjoint sur

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$C_0^\infty(\mathbb{R}^3_{\mathbf{x}}\times]0,\infty[_z)$ dans $L^2(\mathbb{R}^3_{\mathbf{x}}\times]0,\infty[_z)$, le problème de Cauchy est bien posé pour les champs d'énergie usuelle finie :

$$\mathbf{E}(\Phi):=\int_{\mathbb{R}^3}\int_0^\infty \left|\nabla_{t,\mathbf{x},z}\Phi(t,\mathbf{x},z)\right|^2+\frac{15}{4z^2}\left|\Phi(t,\mathbf{x},z)\right|^2\mathrm{d}\mathbf{x}\,\mathrm{d}z<\infty.$$

Cette contrainte implique en fait la condition de Dirichlet $\Phi(t,\mathbf{x},0)=0$. Malheureusement ce cadre étudié dans [2], ne prend pas en compte le graviton sans masse $\Phi_G(t,\mathbf{x},z):=z^{-\frac{3}{2}}\phi(t,\mathbf{x})$ où $\partial_t^2\phi-\Delta_{\mathbf{x}}\phi=0$, qui est une solution d'importance physique majeure. Dans cette Note, nous contruisons une famille continue de dynamiques unitaires (pour de nouvelles énergies d'ordre élevé), pour des solutions admettant une décomposition du type

$$\Phi(t,\mathbf{x},z)=\Phi_r(t,\mathbf{x},z)z^{\frac{5}{2}}+\phi_{-1}(t,\mathbf{x})\chi(z)z^{\frac{5}{2}}+\phi_0(t,\mathbf{x})\chi(z)z^{\frac{5}{2}}\log z+\phi_1(t,\mathbf{x})\chi(z)z^{\frac{1}{2}}+\phi_2(t,\mathbf{x})z^{-\frac{3}{2}}$$

où $\chi\in C_0^\infty(\mathbb{R})$, $\chi(z)=1$ sur un voisinage de 0 et $\Phi_r(t,\mathbf{x},0)=0$. La condition au bord prend la forme

$$\phi_{-1}(t,\mathbf{x})+\alpha_0\phi_0(t,\mathbf{x})+\alpha_1\phi_1(t,\mathbf{x})+\alpha_2\phi_2(t,\mathbf{x})=0,\quad t\in\mathbb{R},\ \mathbf{x}\in\mathbb{R}^3,$$

pour une grande variété de $(\alpha_0,\alpha_1,\alpha_2)\in\mathbb{R}^3$. La possibilité de prendre $\alpha_2=0$ permet de considérer le graviton sans masse, et l'énergie conservée est positive, ce qui exprime que la dynamique est stable. Par ailleurs ces dynamiques sont non triviales dans la mesure où tout champ initialement $C_0^\infty(\mathbb{R}^3_{\mathbf{x}}\times]0,\infty[_z)$ interagit avec le graviton en atteignant le bord de l'univers : $\phi_2\neq 0$ pour les ondes non nulles.

L'idée de la construction repose sur le fait que $\Phi$ est une onde gravitationnelle si et seulement si $u(t,Z):=|Z|^{-\frac{5}{2}}\mathcal{F}_{\mathbf{x}}\Phi(t,\boldsymbol{\xi},|Z|)$, $Z\in\mathbb{R}^6\setminus\{0\}$, est solution de $(\partial_t^2-\Delta_Z+|\boldsymbol{\xi}|^2)u=0$ dans $\mathbb{R}_t\times(\mathbb{R}^6_Z\setminus\{Z=0\})$. On cherche alors des extensions auto-adjointes du laplacien $\Delta_Z$ défini sur $C_0^\infty(\mathbb{R}^6_Z\setminus\{Z=0\})$. Puisque cet opérateur est essentiellement auto-adjoint sur $L^2(\mathbb{R}^6)$, on doit introduire une nouvel espace de Hilbert et donner un sens à une perturbation localisée à $Z=0$. On étudie alors l'équation de Klein–Gordon sur $\mathbb{R}^{1+6}$, avec une perturbation super-singulière au sens de [5]

$$\begin{aligned}
&\partial_t^2u-\Delta_Zu+m^2u-4\pi^3v_2\delta_0(Z)=0,\\
&u(t,Z)=V_r(t,Z)+v_0(t)\chi(Z)\log\big(|Z|\big)+v_1(t)\chi(Z)|Z|^{-2}+v_2(t)|Z|^{-4},\quad V_r(t,.)\in H^4\big(\mathbb{R}^6_Z\big),\\
&V_r(t,0)+\alpha_0v_0(t)+\alpha_1v_1(t)+\alpha_2v_2(t)=0.
\end{aligned}$$

Le détail des démonstrations est présenté dans [3].

## 1. Gravitational waves in Anti-de Sitter cosmology

The Poincaré patch $\mathcal{P}$ of the Anti-de Sitter universe $AdS^5$ is the Lorentzian manifold

$$\mathcal{P}:=\mathbb{R}_t\times\mathbb{R}^3_{\mathbf{x}}\times]0,\infty[_z,\qquad g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu=\frac{1}{z^2}\big(\mathrm{d}t^2-\mathrm{d}\mathbf{x}^2-\mathrm{d}z^2\big).$$

The crucial point is that $\mathcal{P}$ is not globally hyperbolic: the conformal infinity $\mathbb{R}_t\times\mathbb{R}_{\mathbf{x}}\times\{z=0\}$ is time-like. Therefore a difficulty arises to determine the dynamics of the gravitational waves that are solutions of the equation

$$\left(\partial_t^2-\Delta_{\mathbf{x}}-\partial_z^2+\frac{15}{4z^2}\right)\Phi=0,\quad \text{in }\mathbb{R}_t\times\mathbb{R}^3_{\mathbf{x}}\times]0,\infty[_z.\tag{1}$$

The usual way to attack this problem consists to invoke the fact that the Breitenlohner–Freedman condition is satisfied for the gravitational waves ([4,6] and Appendix of [1]), and so the Hamiltonian $-\Delta_{\mathbf{x}}-\partial_z^2+\frac{15}{4z^2}$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^3_{\mathbf{x}}\times]0,\infty[_z)$ in the Hilbert space $\mathcal{H}$ chosen to be $L^2(\mathbb{R}^3_{\mathbf{x}}\times]0,\infty[_z)$. As a consequence there exists a unique dynamics in the functional framework of the fields with finite conserved energy [2,4]:

$$\mathbf{E}(\Phi):=\int_{\mathbb{R}^3}\int_0^\infty \left|\nabla_{t,\mathbf{x},z}\Phi(t,\mathbf{x},z)\right|^2+\frac{15}{4z^2}\left|\Phi(t,\mathbf{x},z)\right|^2\mathrm{d}\mathbf{x}\,\mathrm{d}z<\infty.\tag{2}$$

In fact this constraint implies an implicit Dirichlet condition on the boundary of the universe, $\Phi(t,\mathbf{x},0)=0$, and these gravitational waves are called *Friedrichs solutions*. For this dynamics, the time-like conformal infinity acts as a perfect mirror as regards the propagation of the singularities [2].

Nevertheless this result of uniqueness is not the end of the story because it depends deeply on the choice of the Hilbert space $\mathcal{H}$ (or the choice of the energy $\mathbf{E}(\Phi)$). On the other hand this framework does not take into account the massless graviton $\Phi_G(t,\mathbf{x},z):=z^{-\frac{3}{2}}\phi(t,\mathbf{x})$ where $\partial_t^2\phi-\Delta_{\mathbf{x}}\phi=0$, since it is not normalizable in the sense of this energy. In this Note, we introduce a rich variety of different unitary dynamics for the gravitational waves by changing the choice of the conserved energy. We construct a Hilbert space $\mathcal{H}$ such that $-\Delta_{\mathbf{x}}-\partial_z^2+\frac{15}{4z^2}$ is *not* essentially self-adjoint on $C_0^\infty(\mathbb{R}^3_{\mathbf{x}}\times]0,\infty[_z)$ and admits many self-adjoint extensions associated with different boundary conditions at $z=0$ of asymptotic type. The graviton is normalizable in the sense of the new Hilbert space. Moreover these dynamics are not trivial, i.e. any field localized far from $z=0$ at time $t=0$, interacts with the massless graviton: when the field hits the boundary $z=0$, a part of the scattered field is given by the graviton. Furthermore, many of these dynamics are stable in the sense that there is no growing mode and the conserved energy is positive.

Now we describe a very simple idea of the construction of these new dynamics. We can see that $\Phi$ is solution of (1) iff $\Psi(t,\mathbf{x},Z):=|Z|^{-\frac{5}{2}}\Phi(t,\mathbf{x},|Z|)$ is solution of

$$\big(\partial_t^2-\Delta_{\mathbf{x}}-\Delta_Z\big)\Psi=0,\quad \text{in } \mathbb{R}_t\times\mathbb{R}^3_{\mathbf{x}}\times\big(\mathbb{R}^6_Z\setminus\{Z=0\}\big), \tag{3}$$

and we have proved in [2] that $\Phi$ satisfies (2) iff $\Psi$ is solution of the free wave equation in the whole Minkowski space–time $\mathbb{R}_t\times\mathbb{R}^9_{\mathbf{x},Z}$. As a consequence, to obtain new dynamics for (1), it is sufficient to construct solutions of (3) that are not free waves in $\mathbb{R}_t\times\mathbb{R}^9_{\mathbf{x},Z}$. A partial Fourier transform with respect to $\mathbf{x}$ allows to reduce the study to the investigation of the Klein–Gordon equation $(\partial_t^2-\Delta_Z+m^2)u=0$ in $\mathbb{R}_t\times(\mathbb{R}^6_Z\setminus\{Z=0\})$. Therefore we look for some self-adjoint extensions of the Laplace operator $\Delta_Z$ defined on $C_0^\infty(\mathbb{R}^6_Z\setminus\{Z=0\})$. Since this operator is essentially self-adjoint in $L^2(\mathbb{R}^6)$, we must consider another Hilbert space and try to give a sense to a perturbation localized at $Z=0$.

## 2. Super-singular perturbations of the Klein–Gordon equation in $\mathbb{R}^{1+6}$

Given $m\geqslant 0$, we consider the abstract Klein–Gordon equation

$$\partial_t^2u+\mathbb{A}u+m^2u=0, \tag{4}$$

where $\mathbb{A}$ is a densely defined self-adjoint operator on a Hilbert space $\mathbb{H}_0$ of distributions on $\mathbb{R}^6$, such that

$$C_0^\infty\big(\mathbb{R}^6\setminus\{0\}\big)\subset \mathrm{Dom}(\mathbb{A}),\quad \forall\varphi\in C_0^\infty\big(\mathbb{R}^6\setminus\{0\}\big),\ \mathbb{A}\varphi=-\Delta\varphi.$$

In fact, we choose a very simple point-like interaction at the origin, so for all $u\in\mathrm{Dom}(\mathbb{A})$, $\mathbb{A}u$ has the form

$$\mathbb{A}u=-\Delta u+L(u)\delta_0, \tag{5}$$

where $L$ is a continuous linear form on $\mathbb{H}_0$, equal to zero on $C_0^\infty(\mathbb{R}^6\setminus\{0\})$. This constraint yields a character very singular to the perturbation and the Cauchy problem cannot be solved as usual in a scale of Sobolev spaces: if $u\in\bigcap_{k=0}^2 C^k(\mathbb{R}_t;H^{s-k}(\mathbb{R}^6))$ is solution of (4) and (5) with $L(u)\neq 0$, then $s<-1$ since $\delta_0\in H^\sigma(\mathbb{R}^6)$ iff $\sigma<-3$. Hence a contradiction appears since $C_0^\infty(\mathbb{R}^6\setminus\{0\})$ is dense in $H^s(\mathbb{R}^6)$, $s\leqslant 3$, and as a consequence $L(u)=0$. Therefore we have to introduce some functional spaces, in which $C_0^\infty(\mathbb{R}^6\setminus\{0\})$ is not dense. We want also to recover the static solutions $u_{stat}(t,Z)=|Z|^{-2}K_2(m|Z|)$ where $K_2$ is the classical modified Bessel function, that is solution of (4) and (5) with $L(u_{stat})=-4\pi^3$. The asymptotic expansion

$$|Z|^{-2}K_2\big(m|Z|\big)=\frac{2}{m^2}|Z|^{-4}-\frac{1}{2}|Z|^{-2}-\frac{m^2}{8}\log|Z|+O(1),\quad Z\to 0,$$

suggests to consider the Hilbert spaces of distribution

$$\mathbb{H}_k:=\big\{u=V_r+v_1\chi(Z)|Z|^{-2}+v_2|Z|^{-4},\ V_r\in H^{k+2}\big(\mathbb{R}^6_Z\big),\ v_j\in\mathbb{C}\big\},\quad k=-1,0,$$
$$\mathbb{H}_k:=\big\{u=V_r+v_0\chi(Z)\log\big(|Z|\big)+v_1\chi(Z)|Z|^{-2}+v_2|Z|^{-4},\ V_r\in H^{k+2}\big(\mathbb{R}^6_Z\big),\ v_j\in\mathbb{C}\big\},\quad k=1,2,$$

where $\chi\in C_0^\infty(\mathbb{R}^6_Z)$ satisfies $\chi(Z)=1$ for some $\rho>0$, when $|Z|\leqslant\rho$. It is clear that these spaces do not depend on the choice of $\chi$, and given $u$, the coordinates $v_j$ are also independent of $\chi$, and $\mathbb{H}_k$ is a Hilbert space for the norm

$$\|u\|^2_{\mathbb{H}_k}:=\|V_r\|^2_{H^{k+2}}+\sum_j|v_j|^2.$$

Since we want $\mathbb{A}u$ to belong to $L^1_{loc}(\mathbb{R}^6)$ when $u\in\mathbb{H}_2$, we take:

$$L(u):=-4\pi^3v_2.$$

To impose a constraint at $Z=0$ we introduce the set $\mathcal{A}$ of $\alpha=(\alpha_0,\alpha_1,\alpha_2)\in\mathbb{R}^3$ satisfying firstly

$$\alpha_0+\frac{\alpha_1}{\alpha_1+\sqrt{\alpha_1^2-4\alpha_2}}-\frac{\alpha_2}{(\alpha_1+\sqrt{\alpha_1^2-4\alpha_2}\,)^2}+\frac{1}{2}\log\Big|\alpha_1+\sqrt{\alpha_1^2-4\alpha_2}\,\Big|<\frac{3}{4}-\gamma,$$

where $\gamma$ is the Euler's constant, and secondly

$$\alpha_2 < 0, \quad \text{or} \quad (\alpha_2 = 0, \quad \text{and} \quad \alpha_1 > 0), \quad \text{or}$$
$$\begin{cases} 0 < \alpha_1, \qquad 0 < 4\alpha_2 < \alpha_1^2, \\ \alpha_0 + \dfrac{\alpha_1}{\alpha_1 - \sqrt{\alpha_1^2 - 4\alpha_2}} - \dfrac{\alpha_2}{(\alpha_1 - \sqrt{\alpha_1^2 - 4\alpha_2})^2} + \dfrac{1}{2}\log\left|\alpha_1 - \sqrt{\alpha_1^2 - 4\alpha_2}\right| > \dfrac{3}{4} - \gamma, \end{cases}$$

and we define

$$\mathbf{D}_\alpha := \{u \in \mathbb{H}_2;\ V_r(0) + \alpha_0 v_0 + \alpha_1 v_1 + \alpha_2 v_2 = 0\}.$$

By using the recent work of Kurasov on the super-singular perturbations [5], we prove that there exists an equivalent Hermitian product on $\mathbb{H}_0$ such that $\mathbb{A}$ with domain $\mathbf{D}_\alpha$ is self-adjoint, and we obtain the following

**Theorem 2.1.** *For any $\alpha \in \mathcal{A}$, $m \geqslant 0$, $f \in \mathbb{H}_1$, $g \in \mathbb{H}_0$, there exists a unique $u_\alpha$ satisfying*

$$u_\alpha \in C^2(\mathbb{R}_t; \mathbb{H}_{-1}) \cap C^1(\mathbb{R}_t; \mathbb{H}_0) \cap C^0(\mathbb{R}_t; \mathbb{H}_1) \cap \mathcal{D}'(\mathbb{R}_t; \mathbf{D}_\alpha),$$
$$\partial_t^2 u_\alpha - \Delta_Z u_\alpha + m^2 u_\alpha + L(u_\alpha)\delta_0 = 0, \quad u_\alpha(0, Z) = f(Z),\ \partial_t u_\alpha(0, Z) = g(Z).$$

*The solution depends continuously of the initial data: there exist $C, K > 0$, depending of $\alpha$ but independent on $m$, such that*

$$\|u_\alpha(t)\|_{\mathbb{H}_1} + m\|u_\alpha(t)\|_{\mathbb{H}_0} + \|\partial_t u_\alpha(t)\|_{\mathbb{H}_0} \leqslant C\left(\|f\|_{\mathbb{H}_1} + m\|f\|_{\mathbb{H}_0} + \|g\|_{\mathbb{H}_0}\right)e^{(K-m^2)_+|t|},$$

*where $x_+ = x$ when $x > 0$ and $x_+ = 0$ when $x \leqslant 0$, and for all $\Theta \in C_0^\infty(\mathbb{R}_t)$ we have:*

$$\left\|\int \Theta(t) u_\alpha(t)\,\mathrm{d}t\right\|_{\mathbb{H}_2} \leqslant C\left(\|f\|_{\mathbb{H}_1} + m\|f\|_{\mathbb{H}_0} + \|g\|_{\mathbb{H}_0}\right)\int \left(|\Theta(t)| + |\Theta''(t)|\right)e^{(K-m^2)_+|t|}\,\mathrm{d}t.$$

*There exists a conserved energy, i.e. a non-trivial, continuous quadratic form $\mathcal{E}_\alpha$ defined on $\mathbb{H}_1 \oplus \mathbb{H}_0$, that satisfies:*

$$\forall t \in \mathbb{R}, \quad \mathcal{E}_\alpha\left(u_\alpha(t), \partial_t u_\alpha(t)\right) = \mathcal{E}_\alpha(f, g),$$

*and there exist $\lambda_j > 0$ such that $\mathcal{E}_\alpha$ is given on $C_0^\infty(\mathbb{R}^6 \setminus \{0\}) \oplus C_0^\infty(\mathbb{R}^6 \setminus \{0\})$ by:*

$$\mathcal{E}_\alpha(f, g) = \|\nabla f\|_{H^2}^2 + m^2\|f\|_{H^2}^2 + \|g\|_{H^2}^2, \quad \|v\|_{H^2} := \left\|(-\Delta + \lambda_1)^{\frac{1}{2}}(-\Delta + \lambda_2)^{\frac{1}{2}} v\right\|_{H^2}.$$

*In general $\mathcal{E}_\alpha$ is not positive, but it is strictly positive if $\alpha_2 < 0$ and non-negative if $\alpha_2 = 0$ and $\alpha_1 > 0$.*

*The dynamics is non-trivial: for all $f, g$ in $C_0^\infty(\mathbb{R}^6 \setminus \{0\})$, if $f$ and $g$ are spherically symmetric, then $L(u_\alpha(t)) \neq 0$ for some time $t$, except if $f = g = 0$.*

*If $\alpha \neq \alpha'$ the dynamics are different: given two spherically symmetric functions $f, g$ in $C_0^\infty(\mathbb{R}^6 \setminus \{0\})$, $(f, g) \neq (0, 0)$, the solutions $u_\alpha$ and $u_{\alpha'}$ are different.*

*The propagation is causal, i.e.*

$$\operatorname{supp}\left(u_\alpha(t, .)\right) \subset \{Z;\ |Z| \leqslant |t|\} + \left[\operatorname{supp}(f) \cup \operatorname{supp}(g)\right].$$

## 3. New dynamics for the gravitational fluctuations

In this section we look for the solutions of (1) that have an expansion of the following form

$$\Phi(t, \mathbf{x}, z) = \Phi_r(t, \mathbf{x}, z)z^{\frac{5}{2}} + \phi_{-1}(t, \mathbf{x})\chi(z)z^{\frac{5}{2}} + \phi_0(t, \mathbf{x})\chi(z)z^{\frac{5}{2}}\log z + \phi_1(t, \mathbf{x})\chi(z)z^{\frac{1}{2}} + \phi_2(t, \mathbf{x})z^{-\frac{3}{2}}, \tag{6}$$

where $\chi \in C_0^\infty(\mathbb{R})$, $\chi(z) = 1$ in a neighborhood of 0 and $\Phi_r(t, \mathbf{x}, 0) = 0$, and satisfy the Cauchy condition

$$\Phi(0, \mathbf{x}, z) = \Phi_0(\mathbf{x}, z), \qquad \partial_t \Phi(0, \mathbf{x}, z) = \Phi_1(\mathbf{x}, z). \tag{7}$$

Therefore we introduce the following Hilbert spaces endowed with the natural norms:

$$\mathfrak{h}_k := \left\{\psi(z);\ |Z|^{-\frac{5}{2}}\psi(|Z|) \in \mathbb{H}_k\right\},$$
$$\mathfrak{H}_0 := L^2\left(\mathbb{R}^3_{\mathbf{x}}; \mathfrak{h}_0\right), \qquad \mathfrak{H}_1 := \left\{\Phi \in L^2\left(\mathbb{R}^3_{\mathbf{x}}; \mathfrak{h}_1\right);\ \nabla_{\mathbf{x}}\Phi \in \mathfrak{H}_0\right\}, \qquad \mathfrak{H}_2 := \left\{\Phi \in L^2\left(\mathbb{R}^3_{\mathbf{x}}; \mathfrak{h}_2\right);\ \nabla_{\mathbf{x}}\Phi \in \mathfrak{H}_1\right\}.$$

In particular an element $\Phi \in \mathfrak{H}_2$ can uniquely be expressed as

$$\Phi(\mathbf{x}, z) = \Phi_r(\mathbf{x}, z)z^{\frac{5}{2}} + \phi_{-1}(\mathbf{x})\chi(z)z^{\frac{5}{2}} + \phi_0(\mathbf{x})\chi(z)z^{\frac{5}{2}}\log z + \phi_1(\mathbf{x})\chi(z)z^{\frac{1}{2}} + \phi_2(\mathbf{x})z^{-\frac{3}{2}},$$

with $\phi_j \in L^2(\mathbb{R}^3_{\mathbf{x}})$, $\Phi_r(\mathbf{x}, |Z|) \in L^2(\mathbb{R}^3_{\mathbf{x}}; H^4(\mathbb{R}^6_Z))$, $\Phi_r(\mathbf{x}, 0) = 0$. The behaviour of the field on the boundary of the universe is assumed to be for some $\alpha = (\alpha_0, \alpha_1, \alpha_2) \in \mathbb{R}^3$:

$$\phi_{-1}(t,\mathbf{x})+\alpha_0\phi_0(t,\mathbf{x})+\alpha_1\phi_1(t,\mathbf{x})+\alpha_2\phi_2(t,\mathbf{x})=0,\quad t\in\mathbb{R},\ \mathbf{x}\in\mathbb{R}^3. \tag{8}$$

To take into account the constraint (8), we introduce the subspace:

$$\mathfrak{D}_\alpha:=\left\{\Phi\in\mathfrak{H}_2;\ \phi_{-1}(\mathbf{x})+\alpha_0\phi_0(\mathbf{x})+\alpha_1\phi_1(\mathbf{x})+\alpha_2\phi_2(\mathbf{x})=0\right\}.$$

The previous theorem allows to obtain the main result of this Note:

**Theorem 3.1.** *For any $\alpha\in\mathcal{A}$, $\Phi_0\in\mathfrak{H}_1$, $\Phi_1\in\mathfrak{H}_0$, the Cauchy problem* (1), (7) *has a unique solution*

$$\Phi_\alpha\in C^0(\mathbb{R}_t;\mathfrak{H}_1)\cap C^1(\mathbb{R}_t;\mathfrak{H}_0)\cap C^2(\mathbb{R}_t;\mathfrak{H}_{-1})\cap\mathcal{D}'(\mathbb{R}_t;\mathfrak{D}_\alpha).$$

*Moreover there exist $C,\kappa>0$ independent of $\Phi_j$ such that*:

$$\left\|\partial_t\Phi_\alpha(t)\right\|_{\mathfrak{H}_0}+\left\|\Phi_\alpha(t)\right\|_{\mathfrak{H}_1}\leqslant C\left(\|\Phi_1\|_{\mathfrak{H}_0}+\|\Phi_0\|_{\mathfrak{H}_1}\right)e^{\kappa|t|}, \tag{9}$$

*and for all $\Theta\in C_0^\infty(\mathbb{R}_t)$ we have*:

$$\left\|\int\Theta(t)\Phi_\alpha(t)\,\mathrm{d}t\right\|_{\mathfrak{H}_2}\leqslant C\left(\|\Phi_1\|_{\mathfrak{H}_0}+\|\Phi_0\|_{\mathfrak{H}_1}\right)\int\left(\left|\Theta(t)\right|+\left|\Theta''(t)\right|\right)e^{\kappa|t|}\,\mathrm{d}t. \tag{10}$$

*When $\Phi_0,\Phi_1\in C_0^\infty(\mathbb{R}^3_{\mathbf{x}}\times]0,\infty[_z)$, $(\Phi_0,\Phi_1)\neq(0,0)$, then $\phi_2\neq 0$, hence $\Phi_\alpha$ is not the Friedrichs solution, moreover $\Phi_\alpha\neq\Phi_{\alpha'}$ if $\alpha\neq\alpha'$.*

*There exists a conserved energy, i.e. a non-trivial, continuous quadratic form $\mathbf{E}_\alpha$ defined on $\mathfrak{H}_1\oplus\mathfrak{H}_0$, that satisfies*:

$$\forall t\in\mathbb{R},\quad \mathbf{E}_\alpha\left(\Phi_\alpha(t),\partial_t\Phi_\alpha(t)\right)=\mathcal{E}_\alpha(\Phi_0,\Phi_1).$$

*This energy is not positive definite in general but $\mathbf{E}_\alpha$ is equivalent to $\|\Phi_0\|^2_{\mathfrak{H}_1}+\|\Phi_1\|^2_{\mathfrak{H}_0}$ on $C_0^\infty(\mathbb{R}^3_{\mathbf{x}}\times]0,\infty[_z)\oplus C_0^\infty(\mathbb{R}^3_{\mathbf{x}}\times]0,\infty[_z)$. When $\alpha$ satisfies $\alpha_2<0$ or $\alpha_2=0$ and $\alpha_1>0$, $\mathbf{E}_\alpha$ is positive on $\mathfrak{H}_1\oplus\mathfrak{H}_0$.*

*There exists $M>0$ such that if $\hat{\Phi}_j(\boldsymbol{\xi},z)=0$ for all $|\boldsymbol{\xi}|\leqslant M$, then we can take $\kappa=0$ in the estimates* (9), (10) *and $\mathbf{E}_\alpha(\Phi_0,\Phi_1)>0$.*

*When $\alpha_2=0$, $\alpha_1>0$, the massless graviton $\Phi_G(t,\mathbf{x},z):=\phi_{[0]}(t,\mathbf{x})z^{-\frac{3}{2}}$ where $\phi_{[0]}\in C^0(\mathbb{R}_t;H^2(\mathbb{R}^3_{\mathbf{x}}))$ is solution of $\partial_t^2\phi_{[0]}-\Delta_{\mathbf{x}}\phi_{[0]}=0$, is a solution of* (1) *in $C^0(\mathbb{R}_t;\mathfrak{D}_\alpha)$, and its energy is given by*

$$\mathbf{E}_\alpha(\Phi_G,\partial_t\Phi_G)=\left\|z^{-\frac{3}{2}}\right\|_0^2\int_{\mathbb{R}^3_{\mathbf{x}}}\left|\nabla_{t,\mathbf{x}}\phi_{[0]}(t,\mathbf{x})\right|^2\mathrm{d}\mathbf{x}.$$

The proof is detailed in [3]. We achieve this Note by some comments. (1) The expression of the conserved energy is very complicated but it is given for $\Phi_j\in C_0^\infty(\mathbb{R}^3_{\mathbf{x}}\times]0,\infty[_z)$ by

$$\begin{aligned}\mathbf{E}_\alpha(\Phi_0,\Phi_1)={}&\|P_1P_2\Phi_0\|^2_{L^2}+(\lambda_1+\lambda_2)\|P_2\Phi_0\|^2_{L^2}+\lambda_1\lambda_2\|P_1\Phi_0\|^2_{L^2}+\|\nabla_{\mathbf{x}}P_2\Phi_0\|^2_{L^2}\\&+(\lambda_1+\lambda_2)\|\nabla_{\mathbf{x}}P_1\Phi_0\|^2_{L^2}+\lambda_1\lambda_2\|\nabla_{\mathbf{x}}\Phi_0\|^2_{L^2}+\|P_2\Phi_1\|^2_{L^2}+(\lambda_1+\lambda_2)\|P_1\Phi_1\|^2_{L^2}+\lambda_1\lambda_2\|\Phi_1\|^2_{L^2},\end{aligned}$$

for some $\lambda_j>0$, where $P_1:=\frac{\partial}{\partial z}-\frac{5}{2z}$, $P_2:=-\frac{\partial^2}{\partial z^2}+\frac{15}{4z^2}$.

(2) If we expand the strong solution as

$$\Phi_\alpha(t,\mathbf{x},z)=\phi_r(t,\mathbf{x},z)+\phi_0(t,\mathbf{x})\chi(z)z^{\frac{5}{2}}\log z+\phi_1(t,\mathbf{x})\chi(z)z^{\frac{1}{2}}+\phi_2(t,\mathbf{x})z^{-\frac{3}{2}},$$

we can see that Eq. (1) is equivalent to a system of coupled PDEs (we denote $\Box:=\partial_t^2-\Delta_{\mathbf{x}}$):

$$\Box\phi_2+4\phi_1=0,\qquad \Box\phi_1-4\phi_0=0,$$

$$\left[\Box-\partial_z^2+\frac{15}{4z^2}\right]\left(\phi_r+\chi(z)z^{\frac{5}{2}}\log(z)\phi_0\right)=-4\chi(z)z^{\frac{1}{2}}\phi_0+\left(\chi''(z)z^{\frac{1}{2}}+\chi'(z)z^{-\frac{1}{2}}+4\left(1-\chi(z)\right)z^{-\frac{3}{2}}\right)\phi_1,$$

supplemented by the boundary constraint at the time-like horizon:

$$\lim_{z\to 0}z^{-\frac{5}{2}}\phi_r(t,\mathbf{x},z)+\alpha_0\phi_0(t,\mathbf{x})+\alpha_1\phi_1(t,\mathbf{x})+\alpha_2\phi_2(t,\mathbf{x})=0.$$

We note that $\phi_2$ is not a free wave in the Minkowski space–time (see point (4) below for a link with the massless graviton), and $\Phi_F:=\phi_r+\chi(z)z^{\frac{5}{2}}\log(z)\phi_0$ is a Friedrichs solution of the inhomogeneous wave equation of the gravitational fluctuations, i.e. $\Phi_F$ satisfies (2).

(3) A particularly significant family of constraints on the boundary of the Anti-de Sitter universe is given by the condition $\alpha_2=0$, $\alpha_1>0$. In this case the energy is positive and $\sqrt{\mathbf{E}_\alpha(\Phi_0,\Phi_1)}$ is a norm on $\mathfrak{H}_1\times\mathfrak{H}_0$. Hence we can consider the Hilbert space $(\mathfrak{K}_1\times\mathfrak{H}_0,\langle.,.\rangle_1+\langle.,.\rangle_0)$ defined as the completion of this space for this norm. We remark that $\mathfrak{H}_1\neq\mathfrak{K}_1$ since

$$\left\|\phi(\mathbf{x})z^{-\frac{3}{2}}\right\|_1^2=\left\|z^{-\frac{3}{2}}\right\|_0^2\int_{\mathbb{R}^3_{\mathbf{x}}}\left|\nabla_{\mathbf{x}}\phi(\mathbf{x})\right|^2\mathrm{d}\mathbf{x},\qquad \left\|\phi(\mathbf{x})z^{-\frac{3}{2}}\right\|_{\mathfrak{H}_1}^2=\left\|\phi(\mathbf{x})z^{-\frac{3}{2}}\right\|_1^2+\left\|z^{-\frac{3}{2}}\right\|_{\mathfrak{h}_1}^2\int_{\mathbb{R}^3_{\mathbf{x}}}\left|\phi(\mathbf{x})\right|^2\mathrm{d}\mathbf{x}.$$

Then the Cauchy problem is well posed in $\mathfrak{K}_1\times\mathfrak{H}_0$ and the solution is given by a unitary group.

(4) Finally, in the previous case, we can split the solution $\Phi_\alpha$ into a massless graviton $\Phi_G$ and an orthogonal part $\Phi^\perp$, solutions of (1) satisfying:

$$\begin{aligned}&\Phi_\alpha=\Phi_G+\Phi^\perp,\quad \Phi_G(t,\mathbf{x},z)=\phi_{[0]}(t,\mathbf{x})z^{-\frac{3}{2}},\\&\partial_t^2\phi_{[0]}-\Delta_{\mathbf{x}}\phi_{[0]}=0,\quad \phi_{[0]}(0,\mathbf{x})=\left\|z^{-\frac{3}{2}}\right\|_0^{-2}\left\langle\Phi_0(\mathbf{x},.);z^{-\frac{3}{2}}\right\rangle_0,\ \partial_t\phi_{[0]}(0,\mathbf{x})=\left\|z^{-\frac{3}{2}}\right\|_0^{-2}\left\langle\Phi_1(\mathbf{x},.);z^{-\frac{3}{2}}\right\rangle_0,\\&\forall t\in\mathbb{R},\quad \left\langle\Phi^\perp(t,\mathbf{x},.);z^{-\frac{3}{2}}\right\rangle_0=0,\quad \text{a.e. }\mathbf{x}\in\mathbb{R}^3.\end{aligned}$$

We conjecture that

$$\lim_{|t|\to\infty}\left\|\nabla_{t,\mathbf{x}}\phi_{[0]}(t,.)-\nabla_{t,\mathbf{x}}\phi_2(t,.)\right\|_{L^2(\mathbb{R}^3_{\mathbf{x}})}=0,$$

that is to say, the more singular part of the gravitational wave is asymptotically given by a massless graviton. Last but not least, we let open the deep question on the "physically true" constraint on the boundary on the Anti-de Sitter universe, among the large family of the boundary conditions that we have introduced in this work.

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