C. R. Acad. Sci. Paris, Ser. I 350 (2012) 375–378

Contents lists available at SciVerse ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Partial Differential Equations

A time-independent approach for the study of spectral shift function

Une approche indépendante du temps pour l'étude de la fonction de décalage spectral

Mouez Dimassi [a], Maher Zerzeri [b]

[a] IMB (UMR CNRS 5251), université de Bordeaux 1, 33405 Talence, France
[b] LAGA (UMR CNRS 7539), institut Galilée, université Paris nord 13, 93430 Villetaneuse, France

ARTICLE INFO

Article history:
Received 15 February 2012
Accepted 22 March 2012
Available online 24 April 2012

Presented by Jean-Michel Bony

ABSTRACT

In this Note, we give a new proof of a pointwise asymptotic expansion in powers of h of the derivative of the spectral shift function corresponding to the pair $(-h^2\Delta + V(x), -h^2\Delta)$, near a non-trapping energy. Here the potential V is smooth, real-valued and $\mathcal{O}(|x|^{-\delta})$ for some $\delta > n$, and $h > 0$ is a small parameter. This result is originally due to D. Robert and H. Tamura and their proof is based on the construction of a long-time parametrix for the time-dependent Schrödinger equation. Here we give a time-independent method.

RÉSUMÉ

Dans cette Note, on donne une nouvelle preuve pour l'asymptotique forte en puissances de h de la dérivée de la fonction de décalage spectral associée au couple $(-h^2\Delta + V(x), -h^2\Delta)$, près d'une énergie non captive. Ici le potentiel V est lisse, à valeurs réelles et $\mathcal{O}(|x|^{-\delta})$ pour un certain $\delta > n$, et $h > 0$ est un petit paramètre. Ce résultat est due à D. Robert et H. Tamura et leur preuve est basée sur la construction de paramétix pour des temps grands pour l'équation de Schrödinger dépendant du temps. Ici on donne une méthode indépendante du temps.

1. Introduction

We consider the semiclassical Schrödinger operator on $\mathbb{R}^n$, $n \geqslant 1$, $P_1 := -h^2\Delta + V(x)$, where V is smooth, real-valued potential and satisfies the following assumption:

(A1). There exist $\delta > n$ s.t. for all $\alpha \in \mathbb{N}^n$ there exist $C_\alpha > 0$: $|\partial_x^\alpha V(x)| \leqslant C_\alpha \langle x\rangle^{-\delta-|\alpha|}$, for all $x \in \mathbb{R}^n$.

Here $\langle x\rangle = (1 + |x|^2)^{\frac{1}{2}}$. We denote the semiclassical free Laplacian by $P_0 := -h^2\Delta$.

The operators P_0, P_1 are self-adjoint on $L^2(\mathbb{R}^n)$ with domain $H^2(\mathbb{R}^n)$. Under the assumption (A1), the operator $[f(P_1) - f(P_0)]$ belongs to the trace class for all $f \in \mathcal{C}_0^\infty(\mathbb{R})$. Following the general setup we define the spectral shift function, SSF, $\xi_h(\lambda) := \xi(\lambda; P_1, P_0)$ related to the pair (P_1, P_0) by $\langle \xi_h'(\cdot), f(\cdot)\rangle := -\operatorname{tr}(f(P_1) - f(P_0))$, $f \in \mathcal{C}_0^\infty(\mathbb{R})$. By this formula ξ_h is defined modulo a constant but for the analysis of the derivative $\xi_h'(\lambda)$ this is not important. The SSF may be considered as a generalization of eigenvalues counting function. Background information on the SSF theory can be found in [7] and the references given there.

E-mail addresses: mouez.dimassi@math.u-bordeaux1.fr (M. Dimassi), zerzeri@math.univ-paris13.fr (M. Zerzeri).

1631-073X/$ – see front matter
http://dx.doi.org/10.1016/j.crma.2012.03.016

We denote by $p_1(x,\zeta)=\zeta^2+V(x)$, $(x,\zeta)\in\mathbb{R}^{2n}$, the classical Hamiltonian associated to the operator P_1. The vector field $H_{p_1}=\partial_\zeta p_1\cdot\partial_x-\partial_x p_1\cdot\partial_\zeta=2\zeta\cdot\partial_x-\nabla V(x)\cdot\partial_\zeta$, is the Hamiltonian vector field associated to p_1. Integral curves $t\mapsto \exp tH_{p_1}(x,\zeta)$ of H_{p_1} are called classical trajectories or bicharacteristic curves, and p_1 is constant along such curves. Let $E_0>0$. We assume that:

(A2). The energy E_0 is non-trapping for the classical Hamiltonian p_1, i.e. for all compact $K\subset p_1^{-1}\{E_0\}$ there exists $T_K>0$ such that: $(x,\zeta)\in K\Longrightarrow \exp tH_{p_1}(x,\zeta)\in\mathbb{R}^{2n}\setminus K$, $\forall t>T_K$.

Notice that the non-trapping condition holds on $[E_0-\epsilon,E_0+\epsilon]$ for $\epsilon>0$ small enough.

Under the assumption (A2) a complete asymptotic expansion in powers of h of $\xi_h'(\lambda)$ has been obtained (see [8]). More precisely, we have the following well-known result (see [8, Theorem 0.1]):

Theorem 1.1 *(Pointwise asymptotic). Assume* (A1), (A2). *For $\epsilon>0$ sufficiently small, the following asymptotic expansion holds*: $\zeta_h'(\lambda)\sim h^{-n}\sum_{j\geqslant 0}\gamma_j(\lambda)h^j$ *as* $h\searrow 0$, *uniformly for* $\lambda\in[E_0-\epsilon,E_0+\epsilon]$. *The coefficients* $(\gamma_j(\cdot))_{j\geqslant 0}$ *are smooth function of* $\lambda\in[E_0-\epsilon,E_0+\epsilon]$. *In particular,*

$$\gamma_0(\lambda)=\frac{n\cdot\operatorname{vol}(\{x\in\mathbb{R}^n;\ |x|<1\})}{2(2\pi)^n}\int_{\mathbb{R}^n_x}\left[\lambda_+^{\frac{n}{2}-1}-\left(\lambda-V(x)\right)_+^{\frac{n}{2}-1}\right]\mathrm{d}x.$$

Here $\lambda_+=\max(\lambda,0)$. *Furthermore, this expansion has derivate in λ to any order.*

The proof of this theorem given by Robert and Tamura is based on the construction of a long-time parametrix for the time-dependent Schrödinger equation known as Isozaki and Kitada's constructions. The aim of this Note is to give a time-independent approach to prove this result. More precisely, our proof is based on the absorption limiting principle and the functional calculus due to Helffer–Sjöstrand. This stationary method is very useful when we treat situations where the spectral parameter is implicit, and when there is no really natural associated evolution equation, and we get a flexible tool which can be combined with Grushin reductions and effective Hamiltonians. For example in the case of SSF for perturbed periodic Schrödinger operators (see [3]) and also SSF for Stark Schrödinger operator (see [1]).

In this Note, we only give the proof in the case where $V\in\mathcal{C}_0^\infty(\mathbb{R}^n,\mathbb{R})$. For the general case see Remark 3.1.

2. Preliminaries

The operators P_0, P_1 are bounded from below then we may choose $\lambda_0\in\mathbb{R}_-$ away from the spectrum of P_1. From the assumption (A1) the operator $[(P_j-\lambda_0)^{-m}(z-P_j)^{-1}]_0^1$ is trace class for $m>\frac{n}{2}$. Here we use the notation $[a_j]_0^1=a_1-a_0$. We introduce the following function: $\sigma(z)=(z-\lambda_0)^m\operatorname{tr}[(P_j-\lambda_0)^{-m}(z-P_j)^{-1}]_0^1$, $z\in\mathbb{C}\setminus\mathbb{R}$. Let $f\in\mathcal{C}_0^\infty(\mathbb{R})$ and let $\tilde f$ be an almost analytic extension of f, i.e. $\tilde f\in\mathcal{C}_0^\infty(\mathbb{C})$, $\tilde f_{|\mathbb{R}}=f$ and $\bar\partial\tilde f(z)=\mathcal{O}(|\Im(z)|^N)$ for all $N\in\mathbb{N}$, where $\bar\partial=\frac{\partial}{\partial\bar z}$. The functional calculus due to Helffer–Sjöstrand (see for instance [2, Chapter 8]) yields

$$\langle\xi_h',f\rangle=\frac{1}{\pi}\int\bar\partial\tilde f(z)\sigma(z)\,L(\mathrm{d}z),\tag{1}$$

where $L(\mathrm{d}z):=\mathrm{d}x\,\mathrm{d}y$ denotes the Lebesgue measure on $\mathbb{C}$, $z=x+iy$, $(x,y)\in\mathbb{R}^2$. Here we used the fact that $\bar\partial[\tilde f(\cdot)(\cdot-\lambda_0)^m](z)=(z-\lambda_0)^m\bar\partial\tilde f(z)$. Thus,

$$\langle\xi_h',f\rangle=\lim_{\varepsilon\searrow 0}\left[\frac{1}{\pi}\int_{\{\Im(z)>0\}}\bar\partial\tilde f(z)\sigma(z+i\varepsilon)\,L(\mathrm{d}z)+\frac{1}{\pi}\int_{\{\Im(z)<0\}}\bar\partial\tilde f(z)\sigma(z-i\varepsilon)\,L(\mathrm{d}z)\right].$$

Using that $\sigma(z+i\varepsilon)$ (respectively $\sigma(z-i\varepsilon)$) is holomorphic on $\{z\in\mathbb{C},\Im(z)>0\}$ (respectively $\{z\in\mathbb{C},\Im(z)<0\}$) and applying Green formula, we obtain the following:

Proposition 2.1. *Assume* (A1), *we have* $\xi_h'(\lambda)=\frac{1}{\pi}\Im\sigma(\lambda+i0)$ *in* $\mathcal{D}'$. *More precisely, for all* $f\in\mathcal{C}_0^\infty(\mathbb{R})$, *we have* $\langle\xi_h',f\rangle=\lim_{\varepsilon\searrow 0}\frac{1}{\pi}\int_{\mathbb{R}}f(\lambda)\Im\sigma(\lambda+i\varepsilon)\,\mathrm{d}\lambda$, *where the limit is taken in the sense of distributions. Here $\Im\sigma$ denotes the imaginary part of σ.*

From now on fix $E_0>0$ and $\epsilon>0$ small enough such that the interval $[E_0-\epsilon,E_0+\epsilon](\subset]0,+\infty[)$ is contained in the non-trapping energy region. Then by Mourre's commutator method, we have the following (see [5] and also [4]):

Proposition 2.2. *For $p\in\mathbb{N}^*$. There exists $h_0(p)>0$ small enough such that for all $0<h<h_0(p)$, we have* $\sup_{z\in[E_0-\epsilon,E_0+\epsilon]}\|\langle x\rangle^{-\alpha}\times((z\pm i0)-P_j)^{-p}\langle x\rangle^{-\alpha}\|_{L^2(\mathbb{R}^n)}=\mathcal{O}(h^{-p})$, *for all* $\alpha>p-\frac{1}{2}$.

Using the previous absorption limiting principle and the cyclicity of the trace we prove the following:

Proposition 2.3. *Assume* (A1), (A2) *and let* $N \in \mathbb{N}$. *There exist* $h_0(N) > 0$ *small enough such that for all* $0 < h < h_0(N)$ *the following estimate holds*:

$$\zeta_h^{(N)}(\lambda) = \mathcal{O}\big(h^{-n-N}\big), \quad \textit{uniformly for } \lambda \in [E_0 - \epsilon, E_0 + \epsilon]. \tag{2}$$

We finish this section by stating the following proposition which is the main step of the proof of Theorem 1.1. Let $\theta \in \mathcal{C}_0^\infty(\mathbb{R})$. We denote by $[\mathcal{F}_h^{-1}\theta]$ the h-Fourier inverse of the function θ given by

$$\big[\mathcal{F}_h^{-1}\theta\big](\lambda) = \frac{1}{2\pi h}\int_{\mathbb{R}} e^{\frac{i}{h}t\lambda}\theta(t)\,\mathrm{d}t.$$

Proposition 2.4. *Let* $\theta \in \mathcal{C}_0^\infty(]\frac{1}{2}, 1[; \mathbb{R})$, $\rho > 0$ *and* $k \geqslant 0$. *We have*:

$$\big\langle \xi_h'(\cdot), \big[\mathcal{F}_h^{-1}\theta_\mu\big](\lambda - \cdot) f(\cdot)\big\rangle = \mathcal{O}\big(h^\infty\big), \quad \textit{uniformly for } \lambda \in [E_0 - \epsilon, E_0 + \epsilon].$$

Here $\theta_\mu(t) = \theta(\frac{t}{\mu})$ *and* $\mu = \mu(h) = \rho h^{-k}$.

Remark that we obtain the same result for $\theta \in \mathcal{C}_0^\infty(]-1, -\frac{1}{2}[; \mathbb{R})$.

Outline of the proof of Proposition 2.4. Let $\theta \in \mathcal{C}_0^\infty(]\frac{1}{2}, 1[; \mathbb{R})$, $\mu = \mu(h) = \rho h^{-k}$ with $\rho > 0$ and $k \geqslant 0$. Let $\lambda \in [E_0 - \epsilon, E_0 + \epsilon]$.

Case $k > 0$: A simple computation gives: for any integer N,

$$\big[\mathcal{F}_h^{-1}\theta_\mu\big](\lambda) = (-i)^N \rho^{-N+1} h^{(1+k)N-k} \frac{\mathrm{d}^N}{\mathrm{d}\lambda^N}\big[\big(\mathcal{F}_h^{-1} g_N\big)(\mu \cdot \lambda)\big], \quad \text{where } g_N(t) = \frac{\theta(t)}{t^N} \in \mathcal{C}_0^\infty.$$

Integrating by parts the quantity $\langle \xi_h'(\cdot), [\mathcal{F}_h^{-1}\theta_\mu](\lambda - \cdot) f(\cdot)\rangle$, and using (2) and the last equality, we obtain, for all integer N,

$$\big\langle \xi_h'(\cdot), \big[\mathcal{F}_h^{-1}\theta_\mu\big](\lambda - \cdot) f(\cdot)\big\rangle = \mathcal{O}\big(h^{(N-1)k-n}\big), \tag{3}$$

which yields the proposition, since N is arbitrary and k is positive.

Case $k = 0$: This case is more involved. From now on we fix $\mu > 0$ independent on h. Let $\psi(t) \in \mathcal{C}_0^\infty(\mathbb{R}; [0, 1])$ be equal to 1 for $|t| \leqslant 1$ and be equal to 0 for $|t| \geqslant 2$. Let $M > 0$ be a sufficiently large constant. For $z \in \mathbb{C}$, we put $\psi_{a(h)}(z) = \psi(\frac{\Im z}{a(h)})$ with $a(h) = \frac{M}{\mu} h \ln(\frac{1}{h})$. Here $\Im z$ is the imaginary part of z. Recalling that if $\tilde{f}$ is an almost analytic extension of $f \in \mathcal{C}_0^\infty$ then $\bar{\partial}\tilde{f}(z) = \mathcal{O}(|\Im(z)|^\infty)$. This property and the construction of $\psi_{a(h)}$ we deduce that

$$\bar{\partial}(\tilde{f}\psi_{a(h)})(z) = a(h)\left(\mathbf{1}_{[-2,-1]}\left(\frac{\Im z}{a(h)}\right) + \mathbf{1}_{[1,2]}\left(\frac{\Im z}{a(h)}\right)\right) + \mathcal{O}\big(\psi_{a(h)}(z)|\Im z|^\infty\big). \tag{4}$$

Since $\psi_{a(h)}(z) = 1$ for z real, it follows from (1) that:

$$\big\langle \xi_h'(\cdot), \big[\mathcal{F}_h^{-1}\theta_\mu\big](\lambda - \cdot) f(\cdot)\big\rangle = \frac{1}{\pi}\int_{\mathbb{C}} \bar{\partial}(\tilde{f}\psi_{a(h)})(z)\big[\mathcal{F}_h^{-1}\theta_\mu\big](\lambda - z)\sigma(z)\,L(\mathrm{d}z) = I_- + I_+, \tag{5}$$

where $I_\pm = \frac{1}{\pi}\int_{\{z \in \mathbb{C};\ \pm\Im z > 0\}} \bar{\partial}(\tilde{f}\psi_{a(h)})(z)[\mathcal{F}_h^{-1}\theta_\mu](\lambda - z)\sigma(z)\,L(\mathrm{d}z)$.

$$\text{The Paley–Wiener estimates yields:} \quad \big[\mathcal{F}_h^{-1}\theta_\mu\big](\lambda - z) = \begin{cases} \mathcal{O}(\frac{\mu}{h} e^{\frac{\mu \Im z}{h}}) & \text{for } \Im z > 0, \\ \mathcal{O}(\frac{\mu}{h} e^{\frac{\mu \Im z}{2h}}) & \text{for } \Im z < 0. \end{cases} \tag{6}$$

In the domain $\Im z < 0$, the corresponding estimations in (4) and (6) imply

$$I_- = \mathcal{O}\big(h^{\frac{M}{2}-1-n}\big), \quad \text{uniformly for } \lambda \in [E_0 - \epsilon, E_0 + \epsilon]. \tag{7}$$

It remains to prove that there exist $L = L(M)$ large enough such that $I_+ = \mathcal{O}(h^L)$, uniformly for $\lambda \in [E_0 - \epsilon, E_0 + \epsilon]$. We use a complex scaling which allows us to control I_+ by the values of the integrant in the lower complex half-plane. This idea is used among others to determine the resonance free region (see [9] and also [6]).

Let $F: \mathbb{R}^n \to \mathbb{R}^n$ be a smooth vector field, such that $F(x) = 0$ in a neighborhood of $\mathrm{supp}(V)$ and $F(x) = x$ for $|x|$ large enough. For $\nu \in \mathbb{R}$ small enough, we denote $U_\nu: L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ the unitary operator defined by $U_\nu \varphi(x) = |\det(1 + \nu\,\mathrm{d}F(x))|^{\frac{1}{2}}\varphi(x + \nu F(x))$, for $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$. Then the operator $\widetilde{P}_j = U_\nu P_j (U_\nu)^{-1}$ $(j = 0, 1)$ is a differential operator with analytic coefficients with respect to ν, and can be analytically continued to small enough complex values of ν. For $\nu \in \mathbb{C}$ such that $|\nu|$ small enough, we set $\sigma_\nu(z) = (z - \lambda_0)^m\,\mathrm{tr}[(\widetilde{P}_j - \lambda_0)^{-m}(z - \widetilde{P}_j)^{-1}]_0^1$, $\Im z > 0$. Note that $\sigma_0(z) = \sigma(z)$.

Since U_ν is unitary for ν real and small enough, it follows that $\sigma_\nu(z) = \sigma(z)$ for $\Im z > 0$ and ν real and small enough. On the other hand, for $\Im z > a(h)$ the function $\nu \mapsto \sigma_\nu(z)$ is analytic near $\nu = 0$. Thus, for $\Im z > a(h)$ the function $\nu \mapsto \sigma_\nu(z)$ is a constant near $\nu = 0$ and equal to $\sigma_0(z) = \sigma(z)$. From now on we take $\nu_0 = is_0$ with $s_0 \sim a(h)$ fixed. Next, using the assumption (A2) and repeating the arguments in [9, Section 4] (see also [6]), we construct an escape function $G \in \mathcal{C}_0^\infty(\mathbb{R}^{2n}; \mathbb{R})$ such that $(z - \widetilde{P}_{j,a(h)})^{-1} := (z - e^{\frac{a(h)}{h} G^w(x,hD_x)} \widetilde{P}_j e^{-\frac{a(h)}{h} G^w(x,hD_x)})^{-1}$ exists, holomorphic and $\|(z - \widetilde{P}_{j,a(h)})^{-1}\| = \mathcal{O}(a(h)^{-1})$, for $\Im z > -3a(h)$ and $\Re z \in [E_0 - \epsilon, E_0 + \epsilon]$. We set

$$\tilde{\sigma}_{a(h)}(z) = (z - \lambda_0)^m \operatorname{tr}\big[(\widetilde{P}_{j,a(h)} - \lambda_0)^{-m}(z - \widetilde{P}_{j,a(h)})^{-1}\big]_0^1, \quad \Im z > -3a(h). \tag{8}$$

Applying the cyclicity of the trace and using the above construction, we deduce that

$$\tilde{\sigma}_{a(h)}(z) = \sigma(z), \quad \Im z > a(h). \tag{9}$$

Now we pass to the analysis of the integral I_+. Using (6), (8), (9) and the fact that the function $z \mapsto \tilde{\sigma}_{a(h)}(z)$ is holomorphic in a neighborhood of $\operatorname{supp}(\tilde{f}\psi_{a(h)})$, we get

$$\begin{aligned} I_+ &= \int_{\{\Im z > a(h)\}} \bar{\partial}(\tilde{f}\psi_{a(h)})(z)\big[\mathcal{F}_h^{-1}\theta_\mu\big](\lambda - z)\sigma(z)\, L(\mathrm{d}z) + \mathcal{O}\big(h^\infty\big) \\ &= \int_{\{\Im z > a(h)\}} \bar{\partial}(\tilde{f}\psi_{a(h)})(z)\big[\mathcal{F}_h^{-1}\theta_\mu\big](\lambda - z)\tilde{\sigma}_{a(h)}(z)\, L(\mathrm{d}z) + \mathcal{O}\big(h^\infty\big) \\ &= \int_{\{\Im z < -a(h)\}} \bar{\partial}(\tilde{f}\psi_{a(h)})(z)\big[\mathcal{F}_h^{-1}\theta_\mu\big](\lambda - z)\tilde{\sigma}_{a(h)}(z)\, L(\mathrm{d}z) + \mathcal{O}\big(h^\infty\big). \end{aligned} \tag{10}$$

In the domain $\Im z < -a(h)$, we repeat the arguments used to prove (7) and conclude that the first term of the right-hand side of (10) is $\mathcal{O}(h^{\frac{M}{2}-1-n})$, which together with (7) ends the proof of Proposition 2.4. We recall that $M > 0$ is arbitrary. □

3. Proof of Theorem 1.1

Let $\theta \in \mathcal{C}_0^\infty(]-\frac{1}{C}, \frac{1}{C}[; \mathbb{R})$ be equal to 1 near 0 and C is a large constant. Fix $\mu = h^{-k}$ which $k \in \mathbb{N}$ arbitrary large, and let $f \in \mathcal{C}_0^\infty(]E_0 - \epsilon, E_0 + \epsilon[; \mathbb{R})$, $\equiv 1$ near E_0. According to Proposition 2.4, we have:

$$\big\langle \xi_h'(\cdot), \big[\mathcal{F}_h^{-1}\theta\big](\lambda - \cdot) f(\cdot)\big\rangle = \big\langle \xi_h'(\cdot), \big[\mathcal{F}_h^{-1}\theta_\mu\big](\lambda - \cdot) f(\cdot)\big\rangle + \mathcal{O}\big(h^\infty\big). \tag{11}$$

In fact we can represent the function $(\theta - \theta_\mu)$ as a finite sum $(\sim \mathcal{O}(h^{-k}))$ of functions of the type appearing in Proposition 2.4. As in (3), integrating by parts the first term of the right-hand side of (11) and using (2), we obtain $\langle \xi_h'(\cdot), [\mathcal{F}_h^{-1}\theta_\mu](\lambda - \cdot) f(\cdot)\rangle = \xi_h'(\lambda) + \mathcal{O}(h^{k-n-1})$, which together with (11) yield

$$\xi_h'(\lambda) = \big\langle \xi_h'(\cdot), \big[\mathcal{F}_h^{-1}\theta\big](\lambda - \cdot) f(\cdot)\big\rangle + \mathcal{O}\big(h^{k-n-1}\big). \tag{12}$$

On the other hand, it is well known that the first term of the right-hand side of (12) has a complete asymptotic expansion in powers of h (see [2, Chapter 12] for a time independent method). This ends the proof of Theorem 1.1.

Remark 3.1 *(Sketch of the proof in the general case).* The main change is the proof of Proposition 2.4 in the case where $k = 0$. For that, we proceed as follows. Let $E_0 > 0$ and $\mu = \rho > 0$ be fixed as in Proposition 2.4. We write $V = V_{\mathrm{comp}} + V_\infty$, where $V_{\mathrm{comp}} \in \mathcal{C}_0^\infty(\mathbb{R}^n, \mathbb{R})$ and $V_\infty \in \mathcal{C}^\infty$, satisfying:

$$\rho \cdot \sup_{x \in \mathbb{R}^n} \big|\partial_x^\alpha V_\infty(x)\big| \ll 1, \quad \forall \alpha \in \mathbb{N}^n \text{ such that } |\alpha| \leqslant 2n + 1. \tag{13}$$

For V_{comp} the result follows from the Proposition 2.4. For V_∞, we use the same arguments as in the proof of [2, p. 141, Proposition 12.4]. Note that, the proof of Proposition 12.4 in [2] is time independent, and holds under the assumption (13) and for $\theta \in \mathcal{C}_0^\infty(]\frac{1}{2}, 1[; \mathbb{R})$, $\theta_\mu(t) = \theta(\frac{t}{\mu})$ with $\mu = \rho > 0$.

References

[1] M. Dimassi, S. Fujiié, Spectral shift function and resonances in the non-trapping case for Stark Hamiltonian, preprint.
[2] M. Dimassi, J. Sjöstrand, Spectral Asymptotics in the Semi-Classical Limit, London Mathematical Society Lecture Note Series, vol. 268, Cambridge University Press, Cambridge, 1999.
[3] M. Dimassi, M. Zerzeri, A time-independent approach for the study of spectral shift function and applications, in preparation.
[4] C. Gérard, A proof of the abstract limiting absorption principle by energy estimates, J. Funct. Anal. 254 (11) (2008) 2707–2724.
[5] C. Gérard, A. Martinez, Principe d'absorption limite pour des opérateurs de Schrödinger à longue portée, C. R. Acad. Sci. Paris, Ser. I 306 (3) (1988) 121–123.
[6] A. Martinez, Resonance free domains for non globally analytic potentials, Ann. Henri Poincaré 3 (4) (2002) 739–756.
[7] D. Robert, Semiclassical asymptotics for the spectral shift function, Amer. Math. Soc. Transl. 189 (2) (1999) 187–203.
[8] D. Robert, H. Tamura, Semi-classical asymptotics for local spectral densities and time delay problems in scattering processes, J. Funct. Anal. 80 (1) (1988) 124–147.
[9] J. Sjöstrand, M. Zworski, Fractal upper bounds on the density of semiclassical resonances, Duke Math. J. 137 (3) (2007) 381–459.