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Asymptotic flexibility of globally hyperbolic manifolds

Flexibilité asymptotique des varietées globalment hyperboliques

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ABSTRACT

In this short Note, a question of patching together globally hyperbolic manifolds is addressed which appeared in the context of the construction of Hadamard states.

RÉSUMÉ

Dans cette Note, on regarde un problème de collage de deux varietées globalment hyperboliques qui surgit dans le contexte de la construction des états de Hadamard.

Often, for a normally hyperbolic field theory (as Maxwell or Klein–Gordon theory) on a globally hyperbolic manifold, one wishes to construct Hadamard states. Those are complex-linear functionals on the Weyl algebra (which, in turn, is a certain subalgebra on the algebra of smooth complex functions on a space of solutions to some field equation) satisfying additional properties, for details see [3]. The crucial point for this Note consists solely in the fact that, while the Hadamard property can be defined locally, to every state defined in, say, an open causal and thus globally hyperbolic neighborhood of some Cauchy surface we can associate a Hadamard state in all of the manifold. This procedure is called *propagation of the state* (to the future or the past). In ultrastatic globally hyperbolic manifolds, there is an easy and very explicit method for the construction of Hadamard states. Now if we know that we can modify the past of a Cauchy surface of a given manifold (M, g) in a way that the modified metric $(M, \tilde{g})$ is asymptotically ultrastatic (while staying globally hyperbolic) then we can define a Hadamard state in the past in $\tilde{g}$ and propagate it to the future. According to what has been said above, it will stay Hadamard. Then we consider the Hadamard state in the future in which g and $\tilde{g}$ coincide and propagate it back to the past of the original metric g. The state we have constructed this way is Hadamard for the original metric. Now the question arises if this construction can be performed for every globally hyperbolic manifold. Sometimes a slightly different procedure is done in which, for a given globally hyperbolic manifold (M, g), another one $(M, \tilde{g})$ is constructed which is ultrastatic in the past and contains an open neighborhood N of a Cauchy surface S of (M, g) (cf. [5], for example). But the size of N cannot be controlled due to the proof which works by Fermi coordinates around S. Thus the construction, although very useful for showing the existence of Hadamard spaces, leaves questions from Lorentzian geometry involving concepts as geodesic completeness unanswered. The following result answers the above question in the affirmative.

Definition 1. Two globally hyperbolic manifolds (M, g) and (N, h) are called **future-isometric** (resp. **past-isometric**) iff there is a Cauchy hypersurface S of (M, g) and T of (N, h) such that $I^+(S)$ is isometric to $I^+(T)$ (resp. $I^-(S)$ is isometric to $I^-(T)$). Let $J(g, h)$ be the set of globally hyperbolic manifolds past-isometric to g and future-isometric to h. Any metric in $J(g, h)$ is called an **asymptotic join of** g **and** h.

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We define a binary symmetric relation P of past-isometry (resp. F of future-isometry). Due to the following lemma, P and F are moreover transitive. The content of the lemma is elementary, well-known and frequently used and there can certainly be found a good reference for it. However, for the sake of self-containedness, here we include a proof:

Lemma 1. *For each two different topological Cauchy surfaces S_1, S_2 of a globally hyperbolic manifold (M, g), there is a smooth Cauchy surface in the past of both and a smooth Cauchy surface in the future of both.*

Proof. We use the existence of a smooth Cauchy temporal function t as established in [1,2] (for a shorter proof with a somewhat stronger conclusion see [4]). The function t induces an isometry I between (M, g) and $(\mathbb{R} \times N, -a^2\,\mathrm{d}t^2 + g_t)$ where a is a smooth function on $\mathbb{R} \times N$ and g_t is a smooth one-parameter family of Riemannian metrics on N. By the defining properties of Cauchy surfaces, and as the flow lines of $\operatorname{grad} t$ are timelike, we get immediately that the S_i are graphs of functions f_i: $S_i := \{(f_i(n), n) \mid n \in N\}$. As they are topological hypersurfaces, the f_i are continuous. Now choose a smooth function f_- with $f_-(n) < \min\{f_1(n), f_2(n)\}$ for all $n \in N$ and a smooth function f_+ with $f_+(n) > \max\{f_1(n), f_2(n)\}$ for all $n \in N$. Then the graphs $S_- := \{(f_-(n), n) \mid n \in N\}$, $S_+ := \{(f_+(n), n) \mid n \in N\}$ satisfy $S_- \subset I^-(S_1) \cap I^-(S_2)$, $S_+ \subset I^+(S_1) \cap I^+(S_2)$ as required. $\square$

Furthermore, it is clear right from the definition that for $g \in J(h, k)$ and $G \in J(H, K)$ we get $J(g, G) = J(h, K)$. Now let us prove another result used in the proof of the second theorem:

Theorem 1. *Let $\lambda \in C^\infty(\mathbb{R} \times N, (0, \infty))$ and let $g := -\mathrm{d}t^2 + g_t$ be a Lorentzian metric on $\mathbb{R} \times N$, where each g_t is a Riemannian metric on $\{t\} \times N$. Then there is an $f \in C^\infty(\mathbb{R} \times N, (0, \infty))$ such that $(M := \mathbb{R} \times N, h := -\lambda\,\mathrm{d}t^2 + f g_t)$ is globally hyperbolic. For each real r, by λ_r we denote the function on N given by $\lambda_r(n) := \lambda(r, n)$. If $\lambda_s = \lambda_u$, $g_s = g_u$ for any two $s, u \in (-\infty, 0)$, and if $(a, \infty) \times N$ is already globally hyperbolic, then f can be chosen such that $f_s = f_u$ for any two $s, u \in (-\infty, 0)$ as well and equal to one on (a, ∞).*

Proof. First choose a smooth function j on $S := t^{-1}(\{0\})$ such that jg_0 is complete. Let us denote the timelike future resp. past cone of a point p w.r.t. the metric h by $I_h^\pm(p)$, and for any point $x \in S$ we denote by $B_a(x)$ the ball of radius a around x w.r.t. the metric jg_0. Now we want to have $I_h^\pm((t, x)) \cap S \subset B_t(x)$ (which then ensures global hyperbolicity). As we can parametrize any causal curve c as $c(t) = (t, k(t))$ and as for the resulting curve k holds $f_t g_t(\dot{k}, \dot{k}) \leqslant \lambda_t$, it is sufficient that $f_t g_t \geqslant \max\{1, \lambda\} \cdot jg_0$. By compactness of the Euclidean spheres in each tangent space, there is a continuous function $\underline{f}$ satisfying this inequality, so we can choose a smooth function $f \geqslant \underline{f}$ with this property as well. The additional property is now obvious as the choice of $\underline{f}$ was pointwise. $\square$

We need a last theorem for the proof of the main result:

Theorem 2. *Let (M, g) be a spacetime and $\{U, V\}$ an open covering of M with the property that $(U, g|_U)$ and $(V, g|_V)$ are globally hyperbolic and have a Cauchy hypersurface in common, then (M, g) itself is globally hyperbolic, and every Cauchy hypersurface of U or of V is one of M.*

Proof. Let $S \subset U \cap V$ be the joint Cauchy surface and S' a Cauchy surface of either U or V. We have to show that every C^0-inextendable causal curve c in M meets S' exactly once. Now either $c^{-1}(U) \neq \emptyset$ or $c^{-1}(V) \neq \emptyset$. In the first case we note that $c_U := c|_{c^{-1}(U)}$ is a C^0-inextendable causal curve in U and use that S is a Cauchy surface of U which is in $U \cap V$ to show that $c^{-1}(V) \neq \emptyset$, in the other case we proceed conversely. Therefore c meets both subsets and therefore meets S' at least once. If it met S' twice, then either c_U or c_V would meet S' twice depending in which region S' lies in contradiction to the fact that S' was a Cauchy surface of that region. $\square$

Note that the last condition of the theorem is indispensable. An easy counterexample for the affirmation without the condition of the subsets sharing a Cauchy hypersurface is the following: Consider, in $\mathbb{R}^{1,1}$, the set $M := U \cup V$ with $U := D^o((0, 0), (0, 2))$ and $V := D^o((0, 1), (0, 3))$ where, for $p, q \in \mathbb{R}^{1,1}$, $D^o(p, q)$ denotes the causal diamond $I^+(p) \cap I^-(q)$. Then U and V being causal subsets are globally hyperbolic whereas M isn't.

Now we state and prove the main result:

Theorem 3. *Let (M, g) and (M, h) be globally hyperbolic, let the Cauchy hypersurfaces of g be diffeomorphic to those of h. Then $J(g, h)$ is nonempty. In particular, for any (M, g) globally hyperbolic, there is a globally hyperbolic ultrastatic metric u on M such that $J(g, u)$ is nonempty.*

Proof. Choose a metric splitting $(M, g) = (\mathbb{R} \times N, -s \cdot \mathrm{d}t^2 + g_t)$ by a smooth Cauchy time function t as in [4] and put $T := t^{-1}(\{0\})$ and $S := t^{-1}(\{1\})$. Then choose a smooth positive function f on $M = \mathbb{R} \times N$ such that $f|_{I^+(S)} = 1$ and $f = s^{-1}$ on $I^-(T)$. Via t, the metric $g^{(1)} := f \cdot g$ splits as $g^{(1)} = -l\,\mathrm{d}t^2 + f g_t$ and $l = -1$ in $I^-(T)$. Moreover, $F(g, fg)$. Now, for a smooth monotonously increasing function $\psi : \mathbb{R} \to [0, \infty)$ with $\psi(r) = 0\ \forall r \leqslant 0$, $\psi(r) = r\ \forall r \geqslant 1$, define a smooth function

λ and a Lorentzian metric $k := -\mathrm{d}t^2 + k_t$ as in Theorem 1 by $\lambda_t := l_{\psi(t)}$ and $k_t := f_{\psi(t)} \cdot g_{\psi(t)}$. Note that λ_t and k_t are constant for $t \leqslant 0$. Then apply the first theorem to (λ, k) and get a smooth function ϕ on $\mathbb{R} \times N$, that is to say, a smooth one-parameter family of smooth functions ϕ_t on N, such that $(\mathbb{R} \times N, \gamma := -\lambda_t \,\mathrm{d}t^2 + \phi_t k_t)$ is globally hyperbolic, and ϕ_t can be chosen equal to 1 on $[1, \infty)$ and such that $\phi_x = \phi_y$ for all $x, y \in (-\infty, 0]$. Then $F(g, \gamma)$, and $P(\gamma, u)$ where u is the ultrastatic metric $-\mathrm{d}t^2 + \phi_0 g_0$. Therefore $\gamma \in J(g, u)$, which proves the last affirmation of the theorem. An important detail to keep in mind is the fact that, as the ultrastatic metric u constructed in this way is globally hyperbolic according to Theorem 1, the Riemannian metric of its standard Cauchy surfaces is necessarily complete.

If we have two different globally hyperbolic metrics g and h, the strategy is to join g to the future with an ultrastatic metric u_g as in the first step and to join h to the past with an ultrastatic metric u_h and finally to interpolate between the two ultrastatic metrics. So we construct the ultrastatic metrics $u_g = -\mathrm{d}t^2 + k_0$ and $u_h = -\mathrm{d}t^2 + k_1$ as above, in other words we construct $\gamma_g \in J(g, u_g)$ and $\gamma_h \in J(u_h, h)$ and smooth Cauchy temporal functions t_g for γ_g and t_h for γ_h such that $\gamma_g = \mathrm{d}t_g^2 + u_g$ whenever $t_g \geqslant 0$ and $\gamma_h = \mathrm{d}t_h^2 + u_h$ whenever $t_h \leqslant 10$. Finally, we define $\gamma \in J(\gamma_g, \gamma_h) = J(g, h)$ via an interpolation between u_g and u_h by the metric $u_{gh} := -\mathrm{d}t^2 + k_{\theta(t)}$ on $[0, 10]$ where $k_r := rk_1 + (1 - r)k_0$ and $\theta : [0, 10] \to [0, 10]$ smooth and monotonously nondecreasing with $\theta([0, 7/2]) = \{0\}$ and $\theta([13/2, 10)) = \{10\}$. More precisely, using the diffeomorphisms $I_g : \mathbb{R} \times N \to M$ induced by t_g and $I_h : \mathbb{R} \times N \to M$ induced by t_h we define γ on $\mathbb{R} \times N$ by $\gamma(t, n) := (I_g^* \gamma_g)(t, n)$ for $t \leqslant 1$, $\gamma(t, n) = u_{gh}(t, n)$ for $0 \leqslant t \leqslant 10$ and $\gamma(t, n) := (I_h^* \gamma_h)(t, n)$ for $t \geqslant 9$. Moreover, we write $S(r) := t^{-1}(\{r\})$ for any $r \in \mathbb{R}$.

It remains to show that (M, γ) is globally hyperbolic. Due to Theorem 2, it is enough to show that $I^-(S(6))$ and $I^+(S(4))$ are globally hyperbolic and share the Cauchy surface $S(5)$. Due to the same theorem, it is enough to show that $I^-(S(3))$ and $I^+(S(1)) \cap I^-(S(6))$ are each globally hyperbolic and share the Cauchy hypersurface $S(2)$ (and analogously we proceed for the other part $I^+(S(4))$). The subset $I^-(S(3))$ is isometric to the past of a Cauchy hypersurface in γ_g which is a causally convex subset and therefore globally hyperbolic. $I^+(S(1)) \cap I^-(S(6))$ is globally hyperbolic as it is stably causal due to the existence of a time function t and as its light cones are narrower than the ones of the metric $-\mathrm{d}t^2 + (1 - \theta(6))k_0$ on $I^+(S(1)) \cap I^-(S(6))$. The latter metric is globally hyperbolic as k_0 was complete as metric on the standard Cauchy surfaces of the ultrastatic metric u_g. In the other case $t(p) > 4$ the same fact follows from the comparison with the complete metric $\theta(4)k_1$ which is complete as metric on the standard Cauchy surfaces of the ultrastatic metric u_h.

Thus, as there is an open covering by two globally hyperbolic manifolds with a joint Cauchy surface, the entire manifold $(\mathbb{R} \times N, \gamma)$ is globally hyperbolic. Moreover, by construction we have $\gamma \in J(g, h)$, and the claim follows. □

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