C. R. Acad. Sci. Paris, Ser. I 350 (2012) 365–370

Contents lists available at SciVerse ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

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Partial Differential Equations/Probability Theory

An analytic approach to the ergodic theory of a stochastic variational inequality [☆]

Une approche analytique de la théorie ergodique d'une inéquation variationnelle stochastique

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ARTICLE INFO

Article history:
Received 13 December 2011
Accepted after revision 15 March 2012
Available online 5 April 2012

Presented by Alain Bensoussan

ABSTRACT

In an earlier work made by the first author with J. Turi (Degenerate Dirichlet Problems Related to the Invariant Measure of Elasto-Plastic Oscillators, AMO, 2008), the solution of a stochastic variational inequality modeling an elasto-perfectly-plastic oscillator has been studied. The existence and uniqueness of an invariant measure have been proven. Nonlocal problems have been introduced in this context. In this work, we present a new characterization of the invariant measure. The key finding is the connection between nonlocal PDEs and local PDEs which can be interpreted with short cycles of the Markov process solution of the stochastic variational inequality.

RÉSUMÉ

Dans un travail précédent du premier auteur en collaboration avec Janos Turi (Degenerate Dirichlet Problems Related to the Invariant Measure of Elasto-Plastic Oscillators, AMO, 2008), la solution d'une inéquation variationnelle stochastique modélisant un oscillateur élastique-parfaitement-plastique a été étudiée. L'existence et l'unicité d'une mesure invariante ont été prouvées. Des problèmes nonlocaux ont été introduits dans ce contexte. Le point clé est le lien entre des EDPs nonlocales et des EDPs locales qui peuvent être interprétées comme les cycles courts du processus de Markov solution de l'inéquation variationnelle stochastique.

☆ This research was partially supported by the ANRT, a grant from CEA, Commissariat à l'énergie atomique and by the National Science Foundation under grant DMS-0705247. A large part of this work was completed while one of the authors was visiting the University of Texas at Dallas and the Hong Kong Polytechnic University. We wish to thank warmly these institutions for the hospitality and support.

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1 This research in the paper was supported by WCU (World Class University) program through the National Research Foundation of Korea funded by the Ministry of Education, Science and Technology (R31-20007).

1631-073X/$ – see front matter © 2012 Published by Elsevier Masson SAS on behalf of Académie des sciences.
http://dx.doi.org/10.1016/j.crma.2012.03.011

Version française abrégée

A. Bensoussan et J. Turi ont montré que la relation entre la vitesse et la composante élastique de l'oscillateur élastique-parfaitement-plastique est un processus de Markov ergodique qui satisfait une inéquation variationnelle stochastique (voir (1)). La solution admet une mesure invariante caractérisée par dualité à l'aide d'une équation aux dérivées partielles avec des conditions de bord non-locales (voir (2)). Dans ce travail, une nouvelle preuve de la théorie ergodique est présentée ainsi qu'une nouvelle caractérisation de l'unique distribution invariante. Dans ce contexte, nous déduisons des nouvelles formules reliant des équations aux dérivées partielles avec des conditions de bord non-locales à des problèmes locaux (voir (7)).

1. Introduction

A mathematical framework of stochastic variational inequalities (SVI) modeling an elasto-perfectly-plastic (EPP) oscillator with noise has been introduced by A. Bensoussan and J. Turi in [2]. Although SVI have been already studied in [1] to represent reflection–diffusion processes in convex sets, no connection with random vibration had been made so far. The inequality governs the relationship between the velocity $y(t)$ and the elastic deformation $z(t)$:

$$\mathrm{d}y(t) = -\big(c_0 y(t) + kz(t)\big)\,\mathrm{d}t + \mathrm{d}w(t), \qquad \big(\mathrm{d}z(t) - y(t)\,\mathrm{d}t\big)\big(\phi - z(t)\big) \geqslant 0, \quad \forall|\phi| \leqslant Y, \ \big|z(t)\big| \leqslant Y. \tag{1}$$

Here $c_0 > 0$ is the viscous damping coefficient, $k > 0$ the stiffness, w is a Wiener process and Y is an elasto-plastic bound. Let us introduce some notations.

Notation 1. $D := \mathbb{R} \times (-Y, +Y)$, $D^+ := (0, \infty) \times \{Y\}$, $D^- := (-\infty, 0) \times \{-Y\}$, and the differential operators $A\zeta := -\frac{1}{2}\zeta_{yy} + (c_0 y + kz)\zeta_y - y\zeta_z$, $B_+\zeta := -\frac{1}{2}\zeta_{yy} + (c_0 y + kY)\zeta_y$, $B_-\zeta := -\frac{1}{2}\zeta_{yy} + (c_0 y - kY)\zeta_y$, where ζ is a smooth function on D.

In [2], it has been shown that the probability distribution of $(y(t), z(t))$ converges to an asymptotic probability measure on $D \cup D^+ \cup D^-$ namely ν. Moreover, ν is the unique invariant distribution of $(y(t), z(t))$. In addition, from [3] we know also that there exists a unique solution u_λ to the following partial differential equation (PDE):

$$\lambda u_\lambda + Au_\lambda = f \quad \text{in } D, \qquad \lambda u_\lambda + B_+ u_\lambda = f \quad \text{in } D^+, \qquad \lambda u_\lambda + B_- u_\lambda = f \quad \text{in } D^- \tag{2}$$

with the nonlocal boundary conditions given by the fact that $u_\lambda(y, Y)$ and $u_\lambda(y, -Y)$ are continuous, where $\lambda > 0$ and f is a bounded measurable function. The function u_λ satisfies $\|u_\lambda\|_\infty \leqslant \frac{\|f\|_\infty}{\lambda}$, u_λ is continuous and for all $(y, z) \in \bar{D}$, we have $\lim_{\lambda \to 0} \lambda u_\lambda(y, z) = \nu(f)$. We use the notation $u_\lambda(y, z; f)$.

Now, we introduce short cycles to provide a new proof of the ergodic theory for (1). In this context, we derive new formulas linking PDEs with nonlocal boundary conditions to local problems.

1.1. Short cycles

Let $\lambda > 0$, consider $v_\lambda(y, z)$ the solution of (2) with the local boundary conditions $v_\lambda(0^+, Y) = 0$ and $v_\lambda(0^-, -Y) = 0$. Also, if f is symmetric (resp. antisymmetric) then v_λ is symmetric (resp. antisymmetric). We use the notation $v_\lambda(y, z; f)$.

Note that similarly to what was done for u_λ in [3], the existence and uniqueness of v_λ can be obtained in an appropriate weighted Sobolev space. In addition, v_λ is not continuous at $(y, z) = (0, \pm Y)$. This is a direct consequence of the degeneracy of the operator. Indeed no boundary conditions are specified on $(-\infty, 0) \times \{Y\}$ and $(0, \infty) \times \{-Y\}$. The value of the solution on these boundaries are nonlocal and follow from the solution of the equations in the full domain. Therefore there is no reason why $v_\lambda(0^-, Y) = 0$ and $v_\lambda(0^+, -Y) = 0$. Besides, in this case v_λ would be also solution of problem (2) which is satisfied by the function u_λ. The latter characterizes the law related to the stochastic process solving (1). From a probabilistic point of view, that would mean that the solution is the process $(y(t), z(t))$ stopped at the first instant where $(y(t), z(t)) = (0, \pm Y)$. This is not possible because of the Wiener process.

As $\lambda \to 0$, $v_\lambda \to v$ with

$$Av = f \quad \text{in } D, \qquad B_+ v = f \quad \text{in } D^+, \qquad B_- v = f \quad \text{in } D^- \tag{P_v}$$

with the local boundary conditions $v(0^+, Y) = 0$ and $v(0^-, -Y) = 0$. We use the notation $v(y, z; f)$. We call $v(y, z; f)$ *a short cycle*. Existence and uniqueness of a solution to (P_v) are discussed in the next section.

Next, we introduce next $\pi^+(y, z)$ and $\pi^-(y, z)$ such that

$$A\pi^+ = 0 \quad \text{in } D, \qquad \pi^+ = 1 \quad \text{in } D^+, \qquad \pi^+ = 0 \quad \text{in } D^- \tag{3}$$

and

$$A\pi^- = 0 \quad \text{in } D, \qquad \pi^- = 0 \quad \text{in } D^+, \qquad \pi^- = 1 \quad \text{in } D^-. \tag{4}$$

Note that $\pi^+ + \pi^- = 1$. The existence of solutions to (3) and (4) between 0 and 1 can be deduced by the limit as λ goes to 0 of the functions $\pi_\lambda^\pm$ which are presented in the proof of the main result. A new formulation of the invariant distribution is given by the following theorem:

Theorem 1.1 *(New formulation of the invariant distribution ν). Let f be a bounded measurable function on $\bar{D}$, we have the following analytical characterization of the invariant distribution*:

$$\nu(f) = \frac{v(0^-, Y; f) + v(0^+, -Y; f)}{2v(0^+, Y; 1)}.$$

Denote

$$\nu_\lambda(f) := \frac{v_\lambda(0^-, Y; f) + v_\lambda(0^+, -Y; f)}{2v_\lambda(0^-, Y; 1)}.$$

As $\lambda \to 0$,

$$u_\lambda(y, z; f) - \frac{\nu_\lambda(f)}{\lambda} \to u(y, z; f), \quad \nu_\lambda(f) \to \nu(f) \tag{5}$$

where u satisfies

$$Au = f - \nu(f) \quad \text{in } D, \qquad B_+ u = f - \nu(f) \quad \text{in } D^+, \qquad B_- u = f - \nu(f) \quad \text{in } D^- \tag{6}$$

with the nonlocal boundary conditions given by the fact that

$$u(y, Y) \quad \text{and} \quad u(y, -Y) \quad \text{are continuous}.$$

Then, we obtain also the representation formula

$$u(y, z; f) = v(y, z; f) - \nu(f) v(y, z; 1) + \frac{\pi^+(y, z) - \pi^-(y, z)}{4\pi^-(0^-, Y)} \big(v\big(0^-, Y; f\big) - v\big(0^+, -Y; f\big)\big). \tag{7}$$

2. Analysis of the short cycles

We describe the solution of (P_v). We can write $v(y, z; f) = v_e(y, z; f) + v^+(y, z; f) + v^-(y, z; f)$ with v_e, v^+, v^- satisfying

$$A v_e = f(y, z) \quad \text{in } D, \qquad v_e = 0 \quad \text{in } D^+, \qquad v_e = 0 \quad \text{in } D^-, \tag{8}$$

$$A v^+ = 0 \quad \text{in } D, \qquad v^+(y, Y) = \varphi^+(y; f) \quad \text{in } D^+, \qquad v^+ = 0 \quad \text{in } D^-, \tag{9}$$

and

$$A v^- = 0 \quad \text{in } D, \qquad v^- = 0 \quad \text{in } D^+, \qquad v^-(y, -Y) = \varphi^-(y; f) \quad \text{in } D^-, \tag{10}$$

where $\varphi^+(y; f)$ and $\varphi^-(y; f)$ are defined by

$$-\frac{1}{2}\varphi^+_{yy} + (c_0 y + kY)\varphi^+_y = f(y, Y), \quad y > 0, \qquad \varphi^+\big(0^+; f\big) = 0 \tag{11}$$

and

$$-\frac{1}{2}\varphi^-_{yy} + (c_0 y - kY)\varphi^-_y = f(y, -Y), \quad y < 0, \qquad \varphi^-\big(0^-; f\big) = 0. \tag{12}$$

We check easily the formula

$$\varphi^+(y; f) = 2\int_0^\infty d\xi \exp\big(-\big(c_0\xi^2 + 2kY\xi\big)\big) \int_\xi^{\xi+y} f(\zeta; Y) \exp\big(-2c_0\xi(\zeta - \xi)\big)\, d\zeta,$$

if $y \geqslant 0$ and also

$$\varphi_-(y; f) = 2\int_0^\infty d\xi \exp\big(-\big(c_0\xi^2 - 2kY\xi\big)\big) \int_{y-\xi}^{-\xi} f(\zeta; -Y) \exp\big(-2c_0\xi(\zeta - \xi)\big)\, d\zeta,$$

if $y \leqslant 0$.

2.1. Solution to problem (8)

The proof will be based on solving a sequence of Interior–Exterior Dirichlet problems and a fixed point argument. Thus, we need to state the two following lemmas as preliminary results. It is sufficient to consider $f = 1$, with no loss of generality.

2.1.1. Interior Dirichlet problem

We begin with the interior problem, let $D_1 := (-\bar{y}_1, \bar{y}_1) \times (-Y, Y)$, $D_1^+ := [0, \bar{y}_1) \times \{Y\}$, $D_1^- := (-\bar{y}_1, 0] \times \{-Y\}$. Let us consider the space C_1^+ of continuous functions on $[-Y, Y]$ which are 0 on Y and the space C_1^- of continuous functions on $[-Y, Y]$ which are 0 on $-Y$. Let $\varphi^+ \in C_1^+$ and $\varphi^- \in C_1^-$. We consider the problem

$$-\frac{1}{2}\zeta_{yy} + (c_0 y + kz)\zeta_y - y\zeta_z = 1 \quad \text{in } D_1, \qquad \zeta(y, Y) = 0 \quad \text{in } D_1^+, \qquad \zeta(y, -Y) = 0 \quad \text{in } D_1^- \tag{13}$$

with $\zeta(\bar{y}_1, z) = \varphi^+(z)$ and $\zeta(-\bar{y}_1, z) = \varphi^-(z)$, if $-Y < z < Y$.

Lemma 2.1. *There exists a unique bounded solution to Eq.* (13)*.*

Proof. It is sufficient to prove an a priori bound. Indeed, that leads to some priori estimates on the solution and then a regularization method shows the existence. For that we can assume $\varphi^+, \varphi^- = 0$. Consider $\lambda > 0$ and the function $\theta(y, z) = \exp(\lambda c_0(y^2 + kz^2))$ then $-\frac{1}{2}\theta_{yy} + (c_0 y + kz)\theta_y - y\theta_z = \theta(-\lambda c_0 + 2\lambda c_0^2 y^2(1-\lambda))$. Set next $H := -(\theta + \zeta)$ then

$$-\frac{1}{2}H_{yy} + (c_0 y + kz)H_y - yH_z = -1 + \theta\left(\lambda c_0 - 2\lambda c_0^2 y^2(1-\lambda)\right). \tag{14}$$

If we pick $\lambda > \max(1, \frac{1}{c_0})$ the right-hand side of (14) is positive. Therefore the minimum of H can occur only on the boundary $y = \bar{y}_1$ and $z = Y$ with $y > 0$ or $z = -Y$ with $y < 0$. It follows that $H(y, z) \geqslant -\exp(\lambda c_0(\bar{y}_1^2 + Y^2))$ and thus also $0 \leqslant \zeta \leqslant \exp(\lambda c_0(\bar{y}_1^2 + Y^2))$. □

2.1.2. Exterior Dirichlet problems

Now, we proceed by considering two exterior Dirichlet problems. Let $0 < \bar{y} < \bar{y}_1$, we define $D_{\bar{y}<y} := \{y > \bar{y}, -Y < z < Y\}$, $D_{\bar{y}<y}^+ := \{y > \bar{y}, z = Y\}$ and $D_{y<-\bar{y}} := \{y < -\bar{y}, -Y < z < Y\}$, $D_{y<-\bar{y}}^- := \{y < -\bar{y}, z = -Y\}$ and consider

$$-\frac{1}{2}\eta_{yy}^+ + (c_0 y + kz)\eta_y^+ - y\eta_z^+ = 1 \quad \text{in } D_{\bar{y}<y}, \qquad \eta^+(y, Y) = 0 \quad \text{in } D_{\bar{y}<y}^+ \tag{15}$$

with the condition $\eta^+(\bar{y}, z) = \zeta(\bar{y}, z)$ if $-Y < z < Y$, and

$$-\frac{1}{2}\eta_{yy}^- + (c_0 y + kz)\eta_y^- - y\eta_z^- = 1 \quad \text{in } D_{y<-\bar{y}}, \qquad \eta^-(y, -Y) = 0 \quad \text{in } D_{y<-\bar{y}}^- \tag{16}$$

with the condition $\eta^-(-\bar{y}, z) = \zeta(-\bar{y}, z)$, if $-Y < z < Y$. We use the same notation $\eta(y, z)$ for the two problems (15), (16) for the convenience of the reader. We have

Lemma 2.2. *For any $\bar{y} > 0$ there exists a unique bounded solution of* (15)*,* (16)*.*

Proof. It is sufficient to prove the bound, we claim that $\|\zeta\|_\infty \leqslant \eta(y, z) \leqslant \|\zeta\|_\infty + \frac{Y-z}{\bar{y}}$, for $y > \bar{y}$ and $\|\zeta\|_\infty \leqslant \eta(y, z) \leqslant \|\zeta\|_\infty + \frac{Y+z}{\bar{y}}$, for $y < -\bar{y}$. Consider for instance $\rho(z) = \|\zeta\|_\infty + \frac{Y-z}{\bar{y}}$ for $y > \bar{y}$, $-Y < z < Y$ then $-\frac{1}{2}\rho_{yy} + \rho_y(c_0 y + kz) - y\rho_z = \frac{y}{\bar{y}} > 1$, $\rho(\bar{y}, z) = \|\zeta\|_{L^\infty} + \frac{Y-z}{\bar{y}} > \zeta(\bar{y}, z)$, $\rho(\bar{y}, z) = \|\zeta\|_{L^\infty} > 0$. So clearly $\eta(y, z) \leqslant \rho(z)$. So in all cases we can assert that $\|\eta\|_\infty \leqslant \|\zeta\|_\infty + \frac{2Y}{\bar{y}}$. □

2.1.3. Solution to problem (8)

Proposition 2.1. *There exists a unique bounded solution to problem* (8)*.*

Proof. Uniqueness comes from maximum principle. Setting $\Phi = (\varphi^+(z), \varphi^-(z))$ and using the notation $\Phi(\bar{y}_1, z) = \varphi^+(z)$, $\Phi(-\bar{y}_1, z) = \varphi^-(z)$, we can next define $\Gamma\Phi(\bar{y}_1, z) = \eta(\bar{y}_1, z)$ and $\Gamma\Phi(-\bar{y}_1, z) = \eta(-\bar{y}_1, z)$. We thus have defined a map Γ from C_1^+, C_1^- into itself. If Γ has a fixed point, then it is clear that the function

$$v_e(y, z) = \begin{cases} \zeta(y, z), & -\bar{y}_1 < y < \bar{y}_1, \\ \eta(y, z), & y > \bar{y},\ y < -\bar{y} \end{cases}$$

is a solution of (8) since $\zeta = \eta$ for $\bar{y} < y < \bar{y}_1$, $z \in (-Y, Y)$ and for $-\bar{y}_1 < y < -\bar{y}$, $z \in (-Y, Y)$ and the required regularity is available at boundary points $\bar{y}, \bar{y}_1, -\bar{y}, -\bar{y}_1$. The result will follow from the property: Γ is a contraction mapping. This property will be an easy consequence of the following result. Consider the exterior problem

$$-\frac{1}{2}\psi_{yy} + \psi_y(c_0 y + kz) - y\psi_z = 0 \quad \text{in } D_{\bar{y}<y}, \qquad \psi(y, Y) = 0 \quad \text{in } D_{\bar{y}<y}^+, \tag{17}$$

where $\psi(\bar{y}, z) = 1$ if $-Y < z < Y$, then $\sup_{-Y<z<Y} \psi(\bar{y}_1, z) \leqslant \rho < 1$.

Indeed if $\sup_{-Y<z<Y} \psi(\bar{y}_1, z) = 1$, then the maximum is attained on the line $y = \bar{y}_1$, and this is impossible because it cannot be at $z = Y$, nor at $z = -Y$, nor at the interior, by maximum principle considerations. □

2.2. Solution to problems (9) *and* (10)

We now consider the function φ^+ and φ^- solution of (11) and (12). Note that if $y<0$, we have $\varphi^-(y;1)=\varphi^+(-y;1)$. So it is sufficient to consider (11) and we easily see that

$$\varphi^+(y;1)=\int_0^\infty \exp\bigl(-\bigl(c_0\xi^2+2kY\xi\bigr)\bigr)\frac{1-\exp(-2c_0y\xi)}{2c_0\xi}\,\mathrm{d}\xi,\quad \text{if } y>0$$

and we have $\varphi^+(y;1)\leqslant \frac{1}{c_0}\log(\frac{c_0y+kY}{kY})$, if $y>0$. We next want to solve the problem (9). We proceed as follow. We extend φ^+ for $y<0$, by a function which is C^2 on $\mathbb{R}$ and with compact support on $y<0$. It is convenient to call $\varphi(y)$ the C^2 function on $\mathbb{R}$, with compact support for $y<0$ and $\varphi(y)=\varphi^+(y;1)$ for $y>0$. We set $w^+(y,z)=v^+(y,z)-\varphi(y)$ then we obtain the problem

$$Aw^+=g \quad \text{in } D,\qquad w^+(y,Y)=0 \quad \text{in } D^+,\qquad w^+(y,-Y)=-\varphi(y)\quad \text{in } D^- \tag{18}$$

with $g(y,z)=-(-\frac{1}{2}\varphi_{yy}+(c_0y+kz)\varphi_y)$.

But, $g(y,z)=\mathbf{1}_{\{y>0\}}(-1+k(Y-z)\varphi_y(y))+\mathbf{1}_{\{y<0\}}(-(-\frac{1}{2}\varphi_{yy}+(c_0y+kz)\varphi_y))$ and thus, taking into account the definition of φ when $y<0$, we can assert that $g(y,z)$ is a bounded function. Again, from the definition of $\varphi(y)$ when $y<0$, we obtain that on the boundary, w^+ is bounded. It follows from what was done for problem (8) that (18) has a unique solution. So we can state the following proposition:

Proposition 2.2. *There exists a unique solution to* (9) *of the form* $v^+(y,z)=\varphi^+(y)\mathbf{1}_{\{y>0\}}+\tilde{v}^+(y,z)$ *where* $\tilde{v}^+(y,z)$ *is bounded. Similarly, there exists a unique solution to* (10) *of the form* $v^-(y,z)=\varphi^-(y)\mathbf{1}_{\{y<0\}}+\tilde{v}^-(y,z)$ *where* $\tilde{v}^-(y,z)$ *is bounded.*

Proof. We just define $\varphi(y)$ extension of $\varphi^+(y)$ for $y<0$ as explained before and consider $w^+(y,z)$ solution of (18). We know that $w^+(y,z)$ is bounded and we have $v^+(y,z)=\varphi(y)+w^+(y,z)=\varphi^+(y)\mathbf{1}_{\{y>0\}}+\varphi(y)\mathbf{1}_{\{y<0\}}+w^+(y,z)$ which is of the form (9) with $\tilde{v}^+(y,z)=\varphi(y)\mathbf{1}_{\{y<0\}}+w^+(y,z)$. □

2.3. The complete problem (P_v)

Finally, we consider the complete problem (P_v), we can state

Theorem 2.1. *There exists a unique solution of* (P_v) *of the form* $v(y,z;f)=\varphi^+(y;f)\mathbf{1}_{\{y>0\}}+\varphi^-(y;f)\mathbf{1}_{\{y<0\}}+\tilde{w}(y,z)$ *where* $\tilde{w}(y,z)$ *is a bounded function which can be written as* $\tilde{w}=v_e+w^++w^-$.

Proof. We just collect the results of Propositions 2.1 and 2.2. □

3. Ergodic theorem

Proof of Theorem 1.1. We first prove the result when f is symmetric. In that case, we can write

$$u_\lambda(y,z;f)=v_\lambda(y,z;f)+\frac{v_\lambda(0^-,Y;f)}{v_\lambda(0^-,Y;1)}\left(\frac{1}{\lambda}-v_\lambda(y,z;1)\right). \tag{19}$$

Indeed, we know that $u_\lambda(y,z;f)$ and $v_\lambda(y,z;f)$ are symmetric. Setting $\tilde{u}_\lambda(y,z;f)=u_\lambda(y,z;f)-v_\lambda(y,z;f)$, we obtain

$$\lambda\tilde{u}_\lambda+A\tilde{u}_\lambda=0 \quad \text{in } D,\qquad \lambda\tilde{u}_\lambda+B_+\tilde{u}_\lambda=0 \quad \text{in } D^+,\qquad \lambda\tilde{u}_\lambda+B_-\tilde{u}_\lambda=0 \quad \text{in } D^- \tag{20}$$

with the boundary conditions $\tilde{u}_\lambda(0^+,Y;f)-\tilde{u}_\lambda(0^-,Y;f)=v_\lambda(0^-,Y;f)$ and $\tilde{u}_\lambda(0^+,-Y;f)-\tilde{u}_\lambda(0^-,-Y;f)=-v_\lambda(0^+,-Y;f)$. This last condition is automatically satisfied, thanks to the previous one and the symmetry. The function $\frac{1}{\lambda}-v_\lambda(y,z;1)$ satisfies the three partial differential equations on D, D^+ and D^-. So, $\tilde{u}_\lambda=C(\frac{1}{\lambda}-v_\lambda(y,z;1))$ and writing the first boundary condition, we have

$$\tilde{u}_\lambda\bigl(0^+,Y;f\bigr)-\tilde{u}_\lambda\bigl(0^-,Y;f\bigr)=-C\bigl(v_\lambda\bigl(0^+,Y;1\bigr)-v_\lambda\bigl(0^-,Y;1\bigr)\bigr)=Cv_\lambda\bigl(0^-,Y;1\bigr).$$

Hence,

$$C=\frac{v_\lambda(0^-,Y;f)}{v_\lambda(0^-,Y;1)}$$

and formula (19) has been obtained. Now, we have $\nu_\lambda(f)\to\nu(f)=\frac{v(0^-,Y;f)}{v(0^-,Y;1)}$, as $\lambda\to 0$. If we define

$$u^\star_\lambda(y,z;f) = u_\lambda(y,z;f) - \frac{\nu_\lambda(f)}{\lambda} = v_\lambda(y,z;f) - \nu_\lambda(f) v_\lambda(y,z;1).$$

The function

$$u^\star_\lambda(y,z;f) \to v(y,z;f) - \nu(f) v(y,z;1) = v\big(y,z;f-\nu(f)\big), \quad \lambda \to 0.$$

Also from its definition the function $u^\star_\lambda(y,Y;f)$ and $u^\star_\lambda(y,-Y;f)$ are continuous. From the choice of $\nu(f)$ the function $v(y,Y;f-\nu(f))$ is continuous. Now, since $f-\nu(f)$ is symmetric

$$v\big(0^+,-Y;f-\nu(f)\big) - v\big(0^-,-Y;f-\nu(f)\big) = v\big(0^+,Y;f-\nu(f)\big) - v\big(0^-,Y;f-\nu(f)\big) = 0.$$

So the result is proven when f is symmetric. We now consider the situation when f is antisymmetric. We know that $u_\lambda(y,z;f)$ is antisymmetric. Similarly $v_\lambda(y,z;f)$ is antisymmetric. Consider π^-_λ and π^+_λ defined by

$$\lambda\pi^+_\lambda + A\pi^+_\lambda = 0 \quad \text{in } D, \qquad \lambda\pi^+_\lambda + B_+\pi^+_\lambda = 0 \quad \text{in } D^+, \qquad \pi^+_\lambda = 0 \quad \text{in } D^- \tag{21}$$

with the boundary condition $\pi^+_\lambda(0^+,Y) = 1$ and

$$\lambda\pi^-_\lambda + A\pi^-_\lambda = 0 \quad \text{in } D, \qquad \pi^-_\lambda = 0 \quad \text{in } D^+, \qquad \lambda\pi^-_\lambda + B_+\pi^-_\lambda = 0 \quad \text{in } D^- \tag{22}$$

with the boundary condition $\pi^-_\lambda(0^-,-Y) = 1$. We have $\pi^-_\lambda(y,z) = \pi^-_\lambda(-y,-z)$, we then state the formula

$$u_\lambda(y,z;f) = v_\lambda(y,z;f) - \frac{(\pi^+_\lambda(y,z) - \pi^-_\lambda(y,z)) v_\lambda(0^+,-Y;f)}{1 - \pi^+_\lambda(0^-,Y) + \pi^+_\lambda(0^+,-Y)}.$$

So we see that $u_\lambda(y,z;f)$ converges as $\lambda \to 0$, without substracting a number $\frac{\nu_\lambda(f)}{\lambda}$. The function $u_\lambda(y,z;f)$ converges pointwise to

$$u(y,z;f) = v(y,z;f) - \frac{(\pi^+(y,z) - \pi^-(y,z)) v(0^+,-Y;f)}{2\pi^-(0^-,Y)}.$$

So when f is antisymmetric, the results (5)–(6) hold with $\nu_\lambda(f) = 0$ and $\nu(f) = 0$. For the general case, we can write $f = f_{\mathrm{sym}} + f_{\mathrm{asym}}$ with

$$f_{\mathrm{sym}}(y,z) = \frac{f(y,z) + f(-y,-z)}{2}, \qquad f_{\mathrm{asym}}(y,z) = \frac{f(y,z) - f(-y,-z)}{2}.$$

We have $\nu(f_{\mathrm{sym}}) = \frac{v(0^-,Y;f_{\mathrm{sym}})}{v(0^-,Y;1)}$ and thus $\nu(f_{\mathrm{sym}}) = \frac{v(0^-,Y;f) + v(0^+,-Y;f)}{2v(0^+,-Y;1)}$. Since $\nu(f_{\mathrm{asym}}) = 0$, we deduce $\nu(f) = \nu(f_{\mathrm{sym}}) = \frac{v(0^-,Y;f) + v(0^+,-Y;f)}{2v(0^+,-Y;1)}$. We obtain also the representation formula

$$u(y,z;f) = v(y,z;f) - \nu(f) v(y,z;1) + \frac{\pi^+(y,z) - \pi^-(y,z)}{4\pi^-(0^-,Y)} \big(v\big(0^-,Y;f\big) - v\big(0^+,-Y;f\big)\big)$$

and the result is obtained. □

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