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Algebraic Geometry

On Euler characteristics for large Kronecker quivers

Sur la caractéristique d'Euler de l'espace des représentations stables d'un grand carquois de Kronecker

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ABSTRACT

We study Euler characteristics of moduli spaces of stable representations of m-Kronecker quivers for $m \gg 0$. In particular, we study an asymptotic log formula of Euler characteristics and a normalized asymptotic log formula of Euler characteristic, motivated by so-called Douglas conjecture.

RÉSUMÉ

Nous étudions la caractéristique d'Euler des espaces de modules de représentations stables des m-carquois de Kronecker pour m grand. En particulier, nous étudions une formule log asymptotique pour la caractéristique d'Euler et une formule asymptotique normalisée, motivées par la conjecture de Douglas.

1. Introduction

For each positive integer m, let K^m be the m-Kronecker quiver which consists of two vertices and m arrows from one to the other. For generic non-trivial stability conditions [1] on the category of representations of K^m and moduli spaces of stable representations $M(K^m(a,b))$ of coprime dimension vectors (a,b) [5], we study Euler characteristics $\chi(K^m(a,b))$.

We give some more details in the later section and we go on as follows. Notice that for the Euler form $\langle\cdot,\cdot\rangle$ and a symplectic form $\{\cdot,\cdot\}$, which is an anti-symmetrization of the Euler form, we may take a non-trivial stability condition on the category of representations of K^m such that for representations E, F of K^m and the slope function μ, we have $\mu(E) > \mu(F)$ if and only if $\{E, F\} > 0$.

For objects to study in terms of wall-crossings, stability conditions such that the positivity of the difference of slopes coincides with that of symplectic forms on the Grothendieck group have been commonly called Denef's stability conditions in physics [2]. We employ these special stability conditions and the terminology.

Euler characteristics $\chi(K^m(a,b))$ have been studied extensively. In particular, formulas of Kontsevich–Soibelman and Reineke [6,10,12] have been known. In this article, we would like to study quantitative questions for $m \gg 0$.

To analyze further, for each coprime a, b and $m > 0$, let us define the bipartite quiver $Q^m(a,b)$ which consists of a source vertices and b terminal vertices with m arrows from each source vertex to each terminal vertex. On representations of $Q^m(a,b)$, we have Denef's stability conditions (see Section 2).

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We denote $M(Q^m(a,b))$ to be the moduli space of stable representations of dimension vectors being one on each vertex of $Q^m(a,b)$ and $\chi(Q^m(a,b))$ to be the corresponding Euler characteristic. We have the following:

Theorem 1. *For each coprime a, b, and $m \gg 0$, we have*

$$\chi\big(Q^1(a,b)\big) \sim \frac{a!b!}{m^{a+b-1}} \chi\big(K^m(a,b)\big).$$

We would like to mention that in Theorem 1, Euler characteristics in the left-hand and right-hand sides are discussed in terms of blackhole counting in supergravity [7] and Witten index in superstring theory [3] (see also [15]).

Key tools to obtain Theorem 1 are the recently obtained formula in Theorem 3 on $\chi(K^m(a,b))$ by Manschot, Pioline and Sen [7] (MPS formula for short, see also [8,9,14])[1] and our Lemma 2.1. We realize that by taking m to be a variable, MPS formula provides the polynomial expansion of $\chi(K^m(a,b))$ whose coefficients involve Euler characteristics of bipartite quivers such as $Q^1(a,b)$. Indeed, we are dealing with nothing but the first-order approximation of $\chi(K^m(a,b))$ for $m \gg 0$.

By Theorem 1, to compute $\chi(Q^1(a,b))$, we can take the advantage of $\chi(K^m(a,b))$. Since the explicit formula of $\chi(K^m(a,a+1))$ has been provided in [16], we can obtain $\chi(Q^1(a,a+1))$ as in Corollary 5. Let us mention that for the cases of $a=1$ and arbitrary b, we see that Stirling formula explains Theorem 1.

Douglas has conjectured the following [4,11,16]. For coprime $a, b \gg 0$ such that $\frac{b}{a} \sim r$ and each m, we have that $\frac{\ln(\chi(K^m(a,b)))}{a}$ is a continuous function of r. In particular, the conjecture gives an asymptotic closed formula of $\ln(\chi(K^m(a,b)))$. Allowing m to be large, we have the following:

Corollary 2. *For each coprime a, b and $m \gg 0$, we have*

$$\ln\big(\chi\big(K^m(a,b)\big)\big) \sim (a+b-1)\ln(m).$$

In particular, for $a, b \gg 0$ such that $\frac{b}{a} \sim r$ and large enough m depending on a, b, we have

$$\frac{\ln(\chi(K^m(a,b)))}{a} \sim (1+r)\ln(m).$$

2. Proofs

Let us expand and introduce notions. For each a, let $\bar{a}$ denote a partition of a such that for non-negative integers a_l of $l \geqslant 1$, we have $\sum_l la_l = a$. We put $S_{\bar{a}} = \sum a_l$ for our convenience. When $a_1 = a$, we simply write a for $\bar{a}$. For a quiver Q and representations E, F of Q, on the Grothendieck group of the category of representations of Q, let $\langle E, F\rangle_Q$ be the Euler form and $\{E, F\}_Q$ be the symplectic form $\langle F, E\rangle_Q - \langle E, F\rangle_Q$. For a dimension vector d, we call a partition $d^1, \dots, d^s$ of d such that $\sum_{p=1}^s d^p = d$ and $\{\sum_{p=1}^b d^p, d\}_Q > 0$ for each $b = 1, \dots, s-1$ to be admissible.

For each $m > 0$ and partitions $\bar{a}, \bar{b}$ of a and b, we define the bipartite quiver $Q^m(\bar{a}, \bar{b})$ as follows. It consists of $S_{\bar{a}}$ source vertices such that for each l, we have a_l vertices v; for our convenience, we say a_l is the label of v and we put $w(v) = l$. It consists of $S_{\bar{b}}$ terminal vertices with labels and $w(\cdot)$ being defined by the same manner. We put $mw(v)w(v')$ arrows from each source vertex v to each terminal vertex v'.

Let us explain Denef's stability conditions in use. For the m-Kronecker quiver K^m, the source vertex $(1,0)$, and the terminal vertex $(0,1)$, the slope function μ satisfies $\mu(1,0) > \mu(0,1)$. For $Q^m(\bar{a}, \bar{b})$ and vertices v and v' with the labels being a_l and $b_{l'}$, central charges $\frac{Z(v)}{w(v)}$ and $\frac{Z(v')}{w(v')}$ coincide with those of the vertices $(1,0)$ and $(0,1)$.

We write $(\bar{a}, \bar{b})$ for the dimension vector which has one on each vertex of the quiver $Q^m(\bar{a}, \bar{b})$. We let $M(Q^m(\bar{a}, \bar{b}))$ to be the moduli space of stable representations of the dimension vector $(\bar{a}, \bar{b})$ of $Q^m(\bar{a}, \bar{b})$. We denote $P(Q^m(\bar{a}, \bar{b}), y)$ to be the Poincaré polynomial and we put $\chi(Q^m(\bar{a}, \bar{b})) = P(Q^m(\bar{a}, \bar{b}), 1)$. For the m-Kronecker quiver K^m, we have the following MPS formula by specializing the Poincaré polynomial formula in [7, Appendix D]:

Theorem 3 *(MPS formula). For each coprime a, b and $m > 0$, we have*

$$\chi\big(K^m(a,b)\big) = \sum_{\bar{a},\bar{b}} \chi\big(Q^m(\bar{a},\bar{b})\big) \cdot \prod_l \frac{1}{\bar{a}_l!} \frac{(-1)^{\bar{a}_l(l-1)}}{l^{2\bar{a}_l}} \cdot \prod_l \frac{1}{\bar{b}_l!} \frac{(-1)^{\bar{b}_l(l-1)}}{l^{2\bar{b}_l}}.$$

Notice that $M(Q^m(\bar{a}, \bar{b}))$ is a non-trivial smooth projective variety, since we have stable representations including ones with invertible maps on every arrow. We have the following:

[1] In [7], they give their formula in terms of Poincaré polynomials for Denef's stabilities on quivers without oriented loops. We use its Euler characteristic version on Kronecker quivers. In [13], their formula has been motivically generalized and, for complete bipartite quivers and Euler characteristics, identified with a degeneration formula of Gromow–Witten theory.

Lemma 2.1.

$$\chi\big(Q^m(\bar{a},\bar{b})\big)=m^{S_{\bar{a}}+S_{\bar{b}}-1}\chi\big(Q^1(\bar{a},\bar{b})\big).$$

Proof. We consider the Poincaré polynomial $P(Q^m(\bar{a},\bar{b}),y)$ with Reineke's formula [10, Corollary 6.8]. For the dimension vector $(\bar{a},\bar{b})$, we take an admissible partition $d^1,\dots,d^s$ and $(-1)^{s-1}y^{2\sum_{k\leqslant l}\sum_{v\to v'}d^l_v d^k_{v'}}$. We notice that $\{\cdot,\cdot\}_{Q^m(\bar{a},\bar{b})}=m\{\cdot,\cdot\}_{Q^1(\bar{a},\bar{b})}$. The set of admissible partitions is invariant under choices of m. For each admissible partition, the power of y above is the m times of that for $P(Q^1(\bar{a},\bar{b}),y)$. We have that $P(Q^1(\bar{a},\bar{b}),y)$ is a non-zero polynomial. Ignoring an overall factor of a power of y and writing y^2 as q for simplicity, for some non-trivial and non-negative integers α_i and β_i, we have $P(Q^1(\bar{a},\bar{b}),q)=(q-1)^{1-S_{\bar{a}}-S_{\bar{b}}}(\sum_{i\geqslant 0}\alpha_i(q-1)^{S_{\bar{a}}+S_{\bar{b}}-1}q^{\beta_i})$. For admissible partitions, the second factor is the sum of terms above. So we have $P(Q^m(\bar{a},\bar{b}),q)=(q-1)^{1-S_{\bar{a}}-S_{\bar{b}}}(\sum_{i\geqslant 0}\alpha_i(q^m-1)^{S_{\bar{a}}+S_{\bar{b}}-1}q^{m\beta_i})$. □

We give a proof of Theorem 1.

Proof. By Lemma 2.1, $\chi(Q^m(a,b))$ carries the highest power of m among $\chi(Q^m(\bar{a},\bar{b}))$ in Theorem 3. □

We give a proof of Corollary 2.

Proof. When $a+b=1$, $M(K^m(a,b))$ is a point. For $a+b\neq 1$ and large enough m so that

$$\left|\frac{\ln(\frac{\chi(Q^1(a,b))}{a!b!})}{(a+b-1)\ln(m)}\right|\ll 1,$$

the first assertion follows. For the second assertion, with a_i,b_i,m_i such that $\frac{b_i}{a_i}\to r$, $\frac{1}{a_i}\to 0$, and $\frac{\ln(\chi(K^{m_i}(a_i,b_i)))}{\ln(m_i)(a_i+b_i-1)}\to 1$ for $i\to\infty$, we use a standard argument. □

Let us compute $\chi(Q^1(a,a+1))$ as in the introduction. From [16], we recall the following:

Theorem 4. *(See [16, Theorem 6.6].)*

$$\chi\big(K^m(a,a+1)\big)=\frac{m}{(a+1)((m-1)a+m)}\binom{(m-1)^2a+(m-1)m}{a}.$$

By Theorem 1, we have the following:

Corollary 5.

$$\chi\big(Q^1(a,a+1)\big)=\lim_{m\to\infty}\frac{\chi(K^m(a,a+1))a!(a+1)!}{m^{2a}}=(a+1)!(a+1)^{-2+a}.$$

Remark 1. With the formula of $\chi(K^m(2,2a+1))$ in [10], Manschot has proved

$$\chi\big(Q^1(2,2a+1)\big)=\frac{(2a+1)!}{a!^2}.$$

This sequence and the one in Corollary 5 coincide with A002457 and A066319 at oeis.org.

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