C. R. Acad. Sci. Paris, Ser. I 350 (2012) 299–302

Contents lists available at SciVerse ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

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Probability Theory/Partial Differential Equations

Random attractors for stochastic porous media equations perturbed by space–time linear multiplicative noise ☆

Attracteurs aléatoires pour des équations aux milieux poreux stochastiques perturbés par un bruit linéaire multiplicatif, distribué dans l'espace et le temps

Benjamin Gess [1]

Faculty of Mathematics, University of Bielefeld, 33615 Bielefeld, Germany

ARTICLE INFO

Article history:
Received 13 August 2011
Accepted after revision 7 February 2012
Available online 24 February 2012

Presented by Alain Bensoussan

ABSTRACT

Unique existence of solutions to porous media equations driven by continuous linear multiplicative space–time rough signals is proven for initial data in $L^1(\mathcal{O})$. The generation of a continuous, order-preserving random dynamical system (RDS) on $L^1(\mathcal{O})$ and the existence of a "small" random attractor for stochastic porous media equations perturbed by linear multiplicative noise in *space and time* is obtained. Uniform L^∞ bounds and uniform space–time continuity of solutions is shown. General noise including fractional Brownian Motion for all Hurst parameters is contained. A pathwise Wong–Zakai result for driving noise given by a continuous semimartingale is obtained.

RÉSUMÉ

L'existence et l'unicité des solutions des équations aux milieux poreux pilotés par des «*rough paths*», continus, linéaire multiplicatifs et distribués dans l'espace et le temps sont démontrées pour des conditions initiales dans $L^1(\mathcal{O})$. On obtient la génération d'un système dynamique aléatoire continu et monotone dans $L^1(\mathcal{O})$ ainsi que l'existence d'un «petit» attracteur aléatoire pour des équations aux milieux poreux stochastiques perturbés par un bruit linéaire multiplicatif, distribué dans *l'espace et le temps*. Des bornes uniformes dans $L^\infty(\mathcal{O})$ et la continuité uniforme des solutions dans l'espace et le temps sont démontrées. Le cas d'un bruit généralisé, y compris le mouvement Brownien fractionnaire pour tous les paramétres de Hurst est contenu. Un résultat trajectoire du type Wong–Zakai pour un bruit mené par une semimartingale continue est obtenu.

1. Porous medium equation driven by rough signals

Let $\mathcal{O} \subseteq \mathbb{R}^d$ be a bounded domain with smooth boundary $\partial\mathcal{O}$ in arbitrary dimension $d \in \mathbb{N}$, $T > 0$ and $\mathcal{O}_T := [0, T] \times \mathcal{O}$. We consider partial differential equation driven by rough signals of the type

☆ A more detailed account of the results presented here can be found in (Gess, 2011 [4]).

E-mail address: bgess@math.uni-bielefeld.de.

[1] Supported by DFG-Internationales Graduiertenkolleg "Stochastics and Real World Models", the SFB-701 and the BiBoS-Research Center. The author would like to thank Michael Röckner for valuable discussions and comments.

1631-073X/$ – see front matter
doi:10.1016/j.crma.2012.02.004

$$\begin{aligned} \mathrm{d}X_t &= \Delta\big(|X_t|^m \operatorname{sgn}(X_t)\big)\,\mathrm{d}t + \sum_{k=1}^{N} f_k X_t \circ \mathrm{d}z_t^{(k)}, \quad \text{on } \mathcal{O}_T \\ X(0) &= X_0, \quad \text{on } \mathcal{O} \end{aligned} \tag{1}$$

with Dirichlet boundary conditions, $m > 1$, driven by signals $z^{(k)} \in C([0,T];\mathbb{R})$ and with $f_k \in C^\infty(\bar{\mathcal{O}})$ (we assume high regularity of f_k for simplicity only). Giving meaning to Eq. (1), in particular to the occurring stochastic integral, is part of the results. For detailed proofs of the results presented here we refer to [4].

We emphasize that by the spatial dependency of the functions f_k the noise acts in space as well as in time. For this type of noise even the generation of a continuous RDS by corresponding stochastic partial differential equations (SPDE) with quasilinear drift has been an open problem and is solved in this paper for the first time. In contrast to the case of additive or real (i.e. non-spatially distributed) multiplicative noise, the standard method of transforming the SPDE into a random PDE becomes highly non-trivial, because the space-dependency of the noise destroys the monotonicity structure of the transformed equation. A construction of stochastic flows and invariant manifolds for semilinear SPDE with linear multiplicative space–time noise can be found in [5].

The construction of solutions to (1) for signals of bounded variation proceeds by first transforming the equation into a PDE and then by construction of solutions to this transformed equation. More precisely, let $\mu_t(\xi) := -\sum_{k=1}^N f_k(\xi) z_t^{(k)}$. Then $Y := e^{\mu} X$ satisfies the transformed equation

$$\partial_t Y_t = e^{\mu_t} \Delta\big(\big(e^{-\mu_t} Y_t\big)^m \operatorname{sgn}\big(e^{-\mu_t} Y_t\big)\big), \quad \text{on } \mathcal{O}_T \tag{2}$$

with Dirichlet boundary conditions and initial condition Y_0. This transformation will be rigorously justified below. Our results extend [1] where under restrictions on the dimension d and the order m unique existence of solutions for (2) with essentially bounded initial conditions has been shown.

Let us define what we mean by a solution to (1) and (2). Defining $B(x)(z) := \sum_{k=1}^N f_k x z^{(k)}$ for $x \in L^1(\mathcal{O})$ and $z \in \mathbb{R}^N$ we can rewrite $B(X_t)\,\mathrm{d}z_t = \sum_{k=1}^N f_k X_t \,\mathrm{d}z_t^{(k)}$. Let $W^{n,p}(\mathcal{O})$ be the Sobolev space of order n in $L^p(\mathcal{O})$, $W_0^{n,p}(\mathcal{O})$ the subspace of functions vanishing on $\partial\mathcal{O}$, $C^{m,n}(\bar{\mathcal{O}}_T) \subseteq C(\bar{\mathcal{O}})$ be the set of all continuous functions on $\mathcal{O}_T$ having m continuous derivatives in time and n continuous derivatives in space and let $C^{1\text{-var}}([0,T];H)$ be the set of functions of bounded variation. Further, let $\Phi(r) := |r|^m \operatorname{sgn}(r)$.

Definition 1.1.

(i) Let $Y_0 \in L^1(\mathcal{O})$. We call $Y \in L^1(\mathcal{O}_T)$ a (very) weak solution to (2) if $\Phi(e^{-\mu} Y) \in L^1([0,T]; W_0^{1,1}(\mathcal{O}))$ ($\in L^1(\mathcal{O}_T)$ resp.) and

$$-\int_{\mathcal{O}_T} Y_r \partial_r \eta \,\mathrm{d}\xi\,\mathrm{d}r - \int_{\mathcal{O}} Y_0 \eta_0 \,\mathrm{d}\xi = \int_{\mathcal{O}_T} \Phi\big(e^{-\mu_r} Y_r\big)\Delta\big(e^{\mu_r}\eta_r\big)\,\mathrm{d}\xi\,\mathrm{d}r, \tag{3}$$

for all $\eta \in C^1(\bar{\mathcal{O}}_T)$ ($\in C^{1,2}(\bar{\mathcal{O}}_T)$ resp.) with $\eta = 0$ on $[0,T] \times \partial\mathcal{O}$ and on $\{T\} \times \mathcal{O}$.

(ii) Let $z \in C^{1\text{-var}}([0,T];\mathbb{R}^N)$ and $X_0 \in L^1(\mathcal{O})$. A function $X \in L^1(\mathcal{O}_T)$ such that $t \mapsto (\int_{\mathcal{O}} B(X_t)\eta_t \,\mathrm{d}\xi)$ is continuous is said to be a (very) weak solution to (1) if $\Phi(X) \in L^1([0,T]; W_0^{1,1}(\mathcal{O}))$ ($\in L^1(\mathcal{O}_T)$ resp.) and

$$-\int_{\mathcal{O}_T} X_r\, \partial_r \eta \,\mathrm{d}\xi\,\mathrm{d}r - \int_{\mathcal{O}} X_0 \eta_0 \,\mathrm{d}\xi = \int_{\mathcal{O}_T} \Phi(X_r)\Delta\eta_r \,\mathrm{d}\xi\,\mathrm{d}r + \int_0^T \bigg(\int_{\mathcal{O}} B(X_r)\eta_r \,\mathrm{d}\xi\bigg)\mathrm{d}z_r,$$

for all $\eta \in C^1(\bar{\mathcal{O}}_T)$ ($\in C^{1,2}(\bar{\mathcal{O}}_T)$ resp.) with $\eta = 0$ on $[0,T] \times \partial\mathcal{O}$ and on $\{T\} \times \mathcal{O}$.

A rigorous formulation for the transformation of (1) into (2) can be given as following: Let $X_0 \in L^1(\mathcal{O})$, $z \in C^{1\text{-var}}([0,T];\mathbb{R}^N)$ and $X \in L^1(\mathcal{O}_T)$ with $X \in C([0,T];H)$ or $X \in C([0,T];L^1(\mathcal{O}))$. Then X is a very weak solution to (1) iff $Y := e^{\mu} X$ is a very weak solution to (2). We prove the following:

Theorem 1.2. *Essentially bounded very weak solutions to* (2) *are unique.*

We define $H_0^1(\mathcal{O}) := W_0^{1,2}(\mathcal{O})$ and denote its dual by H.

Theorem 1.3. *Let $Y_0 \in L^\infty(\mathcal{O})$ and $z \in C([0,T];\mathbb{R}^N)$. There exists a unique weak solution $Y \in C([0,T];H) \cap L^\infty(\mathcal{O}_T)$ to* (2) *satisfying $\Phi(e^{-\mu} Y) \in L^2([0,T]; H_0^1(\mathcal{O}))$. There is a function $U : [0,T] \to \bar{\mathbb{R}}$ (taking the value ∞ at $t = 0$) which is piecewise smooth on $(0,T]$ such that for all $Y_0 \in L^\infty(\mathcal{O})$*

$$Y_t \leqslant U_t, \quad \text{a.e. in } \mathcal{O},\ \forall t \in [0,T].$$

If $z \in C^{1\text{-var}}([0,T];\mathbb{R}^N)$ then this yields the existence of a weak solution to (1) given by $X = e^{-\mu}Y$. A key point of Theorem 1.3 is that the upper bound U_t does not depend on the initial condition Y_0. Solutions to (1) for continuous signals are constructed by an approximation of the driving signal.

Definition 1.4. Let $z \in C([0,T];\mathbb{R}^N)$. We call $X \in C([0,T];H)$ a rough weak solution to (1) if $X(0) = X_0$ and for all approximations $z^{(\varepsilon)} \in C^{1\text{-var}}([0,T];\mathbb{R}^N)$ of the driving signal z with $z^{(\varepsilon)} \to z$ in $C([0,T];\mathbb{R}^N)$ and corresponding weak solutions $X^{(\varepsilon)}$ to (1) driven by $z^{(\varepsilon)}$ we have $X_t^{(\varepsilon)} \to X_t$ in H for all $t \in [0,T]$.

Theorem 1.5. *Let $X_0 \in L^\infty(\mathcal{O})$ and $z \in C([0,T];\mathbb{R}^N)$. Then there exists a unique rough weak solution X to* (1) *given by $X = e^{-\mu}Y$, where Y is the corresponding weak solution to* (2)*. X satisfies $X_t \leqslant U_t$ a.e. in $\mathcal{O}$ for all $t \in [0,T]$, where U is as in Theorem* 1.3*.*

Proving Lipschitz continuity in the initial condition with respect to the $L^1(\mathcal{O})$ norm we obtain existence of solutions to (1) for initial conditions in $L^1(\mathcal{O})$ in a limiting sense. Let $(\cdot)^+ := \max(0,\cdot)$ and $C^w([0,T];H)$ be the space of weakly continuous functions in H.

Definition 1.6. Let $X_0 \in L^1(\mathcal{O})$ and $z \in C([0,T];\mathbb{R}^N)$. A function $X \in C^w([0,T];L^1(\mathcal{O}))$ is said to be a limit solution to (1) if $X(0) = X_0$ and for all approximations $X_0^{(\delta)} \in L^\infty(\mathcal{O})$ with $X_0^{(\delta)} \to X_0$ in $L^1(\mathcal{O})$ and corresponding rough weak solutions $X^{(\delta)}$ to (1) we have $X_t^{(\delta)} \to X_t$ in $L^1(\mathcal{O})$ uniformly in time.

Theorem 1.7. *Let $z \in C([0,T];\mathbb{R}^N)$. For each $X_0 \in L^1(\mathcal{O})$ there is a unique limit solution X to* (1) *satisfying $\Phi(X) \in L^1(\mathcal{O}_T)$. For $X_0^{(i)} \in L^1(\mathcal{O})$, $i = 1,2$, the corresponding limit solutions satisfy*

$$\sup_{t\in[0,T]} \big\| \big(X_t^{(1)} - X_t^{(2)}\big)^+ \big\|_{L^1(\mathcal{O})} + \big\| \big(\Phi\big(X^{(1)}\big) - \Phi\big(X^{(2)}\big)\big)^+ \big\|_{L^1(\mathcal{O}_T)} \leqslant C \big\| \big(X_0^{(1)} - X_0^{(2)}\big)^+ \big\|_{L^1(\mathcal{O})}.$$

In addition, $X_t \leqslant U_t$ a.e. in $\mathcal{O}$ for all $t \in [0,T]$, where U_t is as in Theorem 1.3*.*

As a special application we obtain a comparison principle: For $X_0^{(1)}, X_0^{(2)} \in L^1(\mathcal{O})$ with $X_0^{(1)} \leqslant X_0^{(2)}$ almost everywhere we have $X_t^{(1)} \leqslant X_t^{(2)}$, for all $t \in [0,T]$, a.e. in $\mathcal{O}$.

We say that a quantity depends only on the data if it is a function of d, m, T. By proving that the regularity results given in [3] may be applied in our situation we obtain:

Theorem 1.8. *Let $z \in C([0,T];\mathbb{R}^N)$, $X_0 \in L^1(\mathcal{O})$ and X be the corresponding limit solution. Then*

(i) *X is uniformly continuous on every compact set $K \subseteq (0,T] \times \mathcal{O}$, with modulus of continuity depending only on the data and* $\mathrm{dist}(K, \partial\mathcal{O}_T)$.
(ii) *If $X_0 \in L^\infty(\mathcal{O})$ is continuous on a compact set $K \subseteq \mathcal{O}$, then X is uniformly continuous on $[0,T] \times K'$ for every compact set $K' \subseteq \mathring{K}$, with modulus of continuity depending only on the data,* $\mathrm{dist}(K, \partial\mathcal{O})$, $\mathrm{dist}(K', \partial K)$, $\|X_0\|_{L^\infty(\mathcal{O})}$ *and the modulus of continuity of X_0 over K.*
(iii) *Assume*:
($\mathcal{O}1$) *There exist $\theta^* > 0$, $R_0 > 0$ such that $\forall x_0 \in \partial\mathcal{O}$ and $\forall R \leqslant R_0$: $|\mathcal{O} \cap B_R(x_0)| < (1-\theta^*)|B_R(x_0)|$.*
Then for every $\tau > 0$, X is uniformly continuous on $[\tau,T] \times \bar{\mathcal{O}}$ with modulus of continuity depending only on the data, θ^* and τ.

Corollary 1.9. *Let $z \in C([0,T];\mathbb{R}^N)$, $X_0 \in L^1(\mathcal{O})$. Then $X \in C([0,T];L^1(\mathcal{O})) \cap C((0,T];L^p(\mathcal{O}))$ for every $p \in [1,\infty)$. If $X_0 \in L^\infty(\mathcal{O})$ then $X \in C([0,T];L^p(\mathcal{O}))$ for every $p \in [1,\infty)$.*

2. Stochastic porous medium equation and RDS

We now pass to the case of stochastically perturbed porous media equations. Let $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ be a filtered probability space, $(z_t)_{t\in\mathbb{R}}$ be an $\mathbb{R}^N$-valued adapted stochastic process and $((\Omega, \mathcal{F}, \mathbb{P}), (\theta_t)_{t\in\mathbb{R}})$ be a metric dynamical system. We assume

($S1$) (Strictly stationary increments) For all $t,s \in \mathbb{R}$, $\omega \in \Omega$: $z_t(\omega) - z_s(\omega) = z_{t-s}(\theta_s\omega)$.
($S2$) (Regularity) z_t has continuous paths.

We have assumed $z_0 = 0$ for notational convenience only. In particular, applications include fractional Brownian Motion with arbitrary Hurst parameter. We then consider the SPDE

$$\mathrm{d}X_t = \Delta\Phi(X_t)\,\mathrm{d}t + \sum_{k=1}^{N} f_k X_t \circ \mathrm{d}z_t^{(k)}, \quad \text{on } \mathcal{O}_T$$
$$X(0) = X_0, \quad \text{on } \mathcal{O}. \tag{4}$$

For $x \in L^1(\mathcal{O})$ and $\omega \in \Omega$ let $X(t,s;\omega)x$ denote the solution to (1) with initial value x at time s driven by the continuous signal $z_\cdot(\omega)$. If the signal z is given by a continuous semimartingale then (4) can be interpreted in the sense of stochastic Stratonovich integration. In this case we show that the limit solution X is a probabilistic solution to (4). Together with the pathwise convergence of the approximants $X^{(\varepsilon)} \to X$ obtained in Theorem 1.5 via approximation by paths of bounded variation this yields a pathwise Wong–Zakai result (cf. e.g. [6]). For the notions of (order-preserving) RDS and random attractors we refer to [2] and references therein.

Theorem 2.1. *The map φ given by $\varphi(t-s,\theta_s\omega)x := X(t,s;\omega)x$ $(t \geqslant s,\ \omega \in \Omega,\ x \in L^1(\mathcal{O}))$ is a continuous RDS and φ is order preserving, i.e. $\varphi(t,\omega)x_1 \leqslant \varphi(t,\omega)x_2$ a.e. in $\mathcal{O}$ if $x_1, x_2 \in L^1(\mathcal{O})$ with $x_1 \leqslant x_2$ a.e. in $\mathcal{O}$.*

Let $\mathcal{D}$ be the system of all random closed sets. The RDS φ satisfies the same regularity and regularizing properties as proved for the pathwise solutions in Theorem 1.8. Using this we prove

Theorem 2.2. *The RDS φ has a $\mathcal{D}$-random attractor A (as an RDS on $L^1(\mathcal{O})$). A is compact in each $L^p(\mathcal{O})$ and attracting in $L^p(\mathcal{O})$-norm, $p \in [1,\infty)$. Moreover, $A(\omega)$ is a bounded set in $L^\infty(\mathcal{O})$ and the functions in $A(\omega)$ restricted to any compact set $K \subseteq \mathcal{O}$ are equicontinuous on K. If $(\mathcal{O}1)$ is satisfied, then $A(\omega)$ is compact in $C(\bar{\mathcal{O}})$ and attracting in $L^\infty(\mathcal{O})$-norm.*

The random attractor A is unique since it is an invariant, random closed set.

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