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Mathematical Analysis

On m-symmetric d-orthogonal polynomials

Sur les polynômes d-orthogonaux m-symétriques

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ABSTRACT

In this Note, we prove that all the components of a d-symmetric classical d-orthogonal are classical and in the case where the sequence is m-symmetric and d-orthogonal, we prove that the first component of an m-symmetric classical d-orthogonal is classical. That generalized the Douak and Maroni (1992) [8] results for the case $m = d$. Then we discuss, as far as we know, a new symmetric classical 3-PS.

RÉSUMÉ

Dans cette Note, on montre que les composantes d'une suite d-symétrique d-orthogonale et classique sont aussi classiques. Dans le cas où la suite est d-orthogonale classique et m-symétrique, on montre que la première composante est d-orthogonale classique. On généralise ainsi les résultats de Douak et Maroni (1992) [8]. On donne à la fin de cette note un exemple d'une nouvelle suite 3-orthogonale symétrique classique.

1. Introduction

Let $\mathcal{P}$ be the vector space of polynomials with coefficients in $\mathbb{C}$ and let $\mathcal{P}'$ be its algebraic dual. A polynomial sequence $\{P_n\}_{n\geqslant 0}$ in $\mathcal{P}$ is called a polynomial set (PS, for shorter) if $\deg P_n = n$ for all integer n. We denote by $\langle u, f\rangle$ the effect of the linear functional $u \in \mathcal{P}'$ on the polynomial $f \in \mathcal{P}$. A natural extension of the notion of orthogonality was introduced by Van Iseghem [14] and Maroni [9] as follows:

Definition 1.1. Let d be a positive integer and let $\{P_n\}_{n\geqslant 0}$ be a PS in $\mathcal{P}$. $\{P_n\}_{n\geqslant 0}$ is called a d-orthogonal polynomial set (d-OPS, for shorter) with respect to the d-dimensional functional vector $\Gamma = {}^t(\Gamma_0, \Gamma_1, \dots, \Gamma_{d-1})$ if it satisfies the following conditions:

$$\begin{cases} \langle \Gamma_k, P_m P_n\rangle = 0, & m > nd+k,\ n \geqslant 0,\ k = 0,\dots,d-1, \\ \langle \Gamma_k, P_n P_{nd+k}\rangle \neq 0, & n \geqslant 0. \end{cases}$$

For the particular case $d = 1$, we meet the well known notion of orthogonality [7].

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Definition 1.2. Let m be a nonnegative integer. A PS $\{P_n\}_{n\geqslant 0}$ is called m-symmetric if $P_n(wx)=w^n P_n(x)$ for all n, where $w=e^{\frac{2i\pi}{m+1}}$ an $(m+1)$-root of the unity.

For the particular case: $m=1$, we meet the well-known notion of symmetric PS [7]. A characteristic property of m-symmetric PS is given by the following:

Proposition 1.3. *A PS $\{P_n\}_{n\geqslant 0}$ is m-symmetric if and only if there exist $(m+1)$ PSs $\{P_n^k\}_{n\geqslant 0}$, $k=0,\dots,m$, such that $P_{(m+1)n+k}(x)=x^k P_n^k(x^{m+1})$, $n\geqslant 0$.*

The PSs $\{P_n^k\}_{n\geqslant 0}$, $k=0,\dots,m$, are called *the components of* the m-symmetric PS $\{P_n\}_{n\geqslant 0}$.

There exist in the literature many works dealing with m-symmetric d-orthogonal polynomials for particular couples (m,d). One of the main questions related to this notion asks to find properties satisfied by the components and corresponding to fixed ones satisfied by the involved m-symmetric d-OPS.

The case $(m,d)=(1,1)$ is widely known (see, for instance, Chihara [7]). The case $(m,d)=(m,1)$, $m>1$, corresponds to the orthogonality on certain sets in the complex domain and having some symmetrical properties. This case was investigated by Ben Cheikh [1] where the author unified some previous works written by Carlitz [6], Milovanovič [11], Marcellàn and Sansigre [10] and Ricci [12]. The case $(m,d)=(d,d)$, $d>1$, was initiated by Douak and Maroni [8] where the authors characterized the d-symmetric d-OPSs by means of a lacunary $(d+1)$-order recurrence relation and showed the d-orthogonality of the components. Other results for these polynomials were derived by Ben Cheikh and Douak [2] and Ben Cheikh and Gaied [5]. In [4], the authors gave some characteristic properties for the d-symmetric classical d-orthogonal polynomials related to generating functions and recuro-differential equation. The aim of this Note is to generalize some results obtained by Douak and Maroni [8] to the case (m,d) where $d>1$ and $m\leqslant d$. Without loosing the generality, in which follows we assume that the polynomials P_n, $n\geqslant 0$, are monic.

2. *m*-Symmetric *d*-OPSs

2.1. Characterizations of *m*-symmetric *d*-OPSs

Let d be a positive integer and m be a nonnegative integer satisfying $m\leqslant d$. Next, we give a necessary condition on m and d to have an m-symmetric d-OPS and two characterizations of m-symmetric d-OPSs. We denote by $\tilde{X}^k$ the multiplication operator by x^k in $\mathcal{P}$.

Theorem 2.1. *Let $\{P_n\}_{n\geqslant 0}$ be a d-OPS. Then the following properties are equivalent*:

(i) *The PS $\{P_n\}_{n\geqslant 0}$ is m-symmetric.*
(ii) *$d+1$ is a multiple of $m+1$, say $d+1=p(m+1)$, and the PS $\{P_n\}_{n\geqslant 0}$ satisfies a $(d+1)$-order recurrence relation of type*

$$\tilde{X}P_n=P_{n+1}+\sum_{j=1}^{p}\gamma_{n,j}P_{n-j(m+1)+1}, \tag{1}$$

with $\gamma_{n,p}\neq 0$ and the convention $P_{-n}=0$ for all $n\geqslant 1$.

Proof. (i) ⇒ (ii) Since $\{P_n\}_{n\geqslant 0}$ is a d-OPS, it satisfies a $(d+1)$-order recurrence relation of type (cf. [9]):

$$\tilde{X}P_n=P_{n+1}+\sum_{k=0}^{d}\alpha_{k,n-d+k}P_{n-d+k},\quad \alpha_{0,n-d}\neq 0. \tag{2}$$

Take the polynomials in (2) at wx, and use the fact that the PS $\{P_n\}_{n\geqslant 0}$ is m-symmetric, we obtain

$$\tilde{X}P_n=P_{n+1}+\sum_{k=0}^{d}\alpha_{k,n-d+k}w^{k-d-1}P_{n-d+k}. \tag{3}$$

Comparing the coefficients of P_{n-d} in (2) and (3) we deduce that $w^{d+1}=1$ since $\alpha_{0,n-d}\neq 0$. It follows then $d+1$ is a multiple of $m+1$. If we compare the coefficients of P_{n-d-k} in (2) and (3) we deduce that

$$\tilde{X}P_n=P_{n+1}+\sum_{j=0}^{p-1}\alpha_{j(m+1),n-d+j(m+1)}P_{n-d+j(m+1)}=P_{n+1}+\sum_{j=1}^{p}\gamma_{n,j}P_{n-j(m+1)+1},\quad \text{with } \gamma_{n,p}\neq 0.$$

(ii) ⇒ (i) From (1) we get $P_j(x)=x^j$ for $0\leqslant j\leqslant m$. The result is obtained by induction. □

2.2. Properties of the components

As an analogue of the Hahn's characterization for classical polynomials when $d=1$, Douak and Maroni [8] introduced the concept of classical d-orthogonal polynomials as follows:

Definition 2.2. A PS $\{P_n\}_{n\geqslant 0}$ is called classical d-orthogonal if and only if both $\{P_n\}_{n\geqslant 0}$ and $\{(1/(n+1))P'_{n+1}\}_{n\geqslant 0}$ are d-orthogonal.

They showed that if $\{P_n\}_{n\geqslant 0}$ is a d-symmetric d-OPS, all the components $\{P_n^k\}_{n\geqslant 0}$, $k=0,\ldots,d$, are d-orthogonal and if moreover $\{P_n\}_{n\geqslant 0}$ is classical, the first component P_n^0 is classical. In this subsection, we generalize these two results by proving that they remain true for m-symmetric d-OPS and we improve the second one by proving that all the $d+1$ components are classical.

Theorem 2.3. *Let $\{P_n\}_{n\geqslant 0}$ be an m-symmetric d-OPS. Then its components $\{P_n^k\}_{n\geqslant 0}$, $k=0,\ldots,m$, are d-orthogonal.*

Proof. Since $\{P_n\}_{n\geqslant 0}$ is an m-symmetric d-OPS, it verifies the following recurrence relation:

$$\tilde{X}P_n = P_{n+1} + \sum_{j=1}^{p} \alpha_{j,n} P_{n+1-j(m+1)}, \quad \alpha_{p,n-d} \neq 0. \tag{4}$$

We apply the operator $\tilde{X}$ on both sides of (4) and we replace $\tilde{X}P_q$ by a relation of type (4). We obtain a relation of type $\tilde{X}^2 P_n = P_{n+2} + \sum_{j=1}^{2p} \alpha_{2,j,n} P_{n+2-j(m+1)}$, with $\alpha_{2,2p,n} \neq 0$. By iteration, we deduce that for all $r \in \mathbb{N}$ and $n \geqslant rd$

$$\tilde{X}^r P_n = P_{n+r} + \sum_{j=1}^{rp} \alpha_{r,j,n} P_{n+r-j(m+1)} \tag{5}$$

with $\alpha_{r,rp,n} \neq 0$. Thus $\tilde{X}^{m+1} P_{n(m+1)+k} = P_{(n+1)(m+1)+k} + \sum_{j=1}^{p(m+1)=d+1} \alpha_{m+1,j,n(m+1)+k} P_{(n-j+1)(m+1)+k}$, which is equivalent to $\tilde{X}^{m+1} P_{n(m+1)+k} = P_{(n+1)(m+1)+k} + \sum_{j=0}^{d} \alpha_{j,n-d+j} P_{(n-d+j)(m+1)+k}$, with $\alpha_{0,n-d} \neq 0$. It results that $x^{m+1} P_n^k(x^{m+1}) = P_{n+1}^k(x^{m+1}) + \sum_{j=0}^{d} \alpha_{j,n-d+j} P_{n-d+j}^k(x^{m+1})$, with $\alpha_{0,n-d} \neq 0$. In other words $\tilde{X} P_n^k = P_{n+1}^k + \sum_{j=0}^{d} \alpha_{j,n-d+j} P_{n-d+j}^k$, $\alpha_{0,n-d} \neq 0$. Then $\{P_n^k\}_{n\geqslant 0}$, $k=0,\ldots,m$, is a d-OPS. □

2.2.1. Classical d-OPSs

Douak and Maroni showed in [8] that if $\{P_n\}_{n\geqslant 0}$ is a d-symmetric classical d-OPS, then the first component $\{P_n^0\}_{n\geqslant 0}$ is classical. Next, we prove that all the components $\{P_n^k\}_{n\geqslant 0}$, $k=0,\ldots,d$, are classical and if $\{P_n\}_{n\geqslant 0}$ is m-symmetric and classical, we prove that the first component is classical. We state the following:

Theorem 2.4. *If $\{P_n\}_{n\geqslant 0}$ is a d-symmetric classical d-OPS, then its components $\{P_n^k\}_{n\geqslant 0}$, $k=0,\ldots,d$, are classical d-orthogonal.*

Proof. Since $\{P_n\}_{n\geqslant 0}$ and $\{A_n = (1/(n+1))P'_{n+1}\}_{n\geqslant 0}$ are d-symmetric d-orthogonal, it results from Theorem 2.3 that the families $\{P_n^k\}_{n\geqslant 0}$ and $\{A_n^k\}_{n\geqslant 0}$ are d-orthogonal, $k=0,\ldots,d$. Our goal here is to prove that the family $\{P_n^k\}_{n\geqslant 0}$ is classical. It is enough to prove that the PS $\{K_n^k = (1/(n+1))(P_{n+1}^k)'\}_{n\geqslant 0}$ verifies a recurrence relation of type (2).

Case $k=0$. We recall that $P_{n+1}^0(x^{d+1}) = P_{(n+1)(d+1)}(x)$. Then taking derivatives in both sides of this relation, we obtain:

$$x^d K_n^0\big(x^{d+1}\big) = A_{(n+1)(d+1)-1}(x). \tag{6}$$

Since $\{A_n\}_{n\geqslant 0}$ is d-symmetric and d-orthogonal, we replace in (5) r by $d+1$ and n by $(n+1)(d+1)-1$, we have:

$$\begin{aligned}\tilde{X}^{d+1} A_{(n+1)(d+1)-1} &= A_{(n+2)(d+1)-1} + \sum_{j=1}^{d} \beta_{j,(n+2-j)(d+1)} A_{(n+2-j)(d+1)-1} + \gamma_{n-d} A_{(n+1-d)(d+1)-1} \\ &= A_{(n+2)(d+1)-1} + \sum_{j=1}^{d} \beta_{d+1-j,(n-d+j+1)(d+1)} A_{(n-d+j+1)(d+1)-1},\end{aligned}$$

with $\beta_{d+1,(n-d+1)(d+1)} \neq 0$. Then $x^{d+1} K_n^0(x^{d+1}) = K_{n+1}^0(x^{d+1}) + \sum_{j=0}^{d} \beta_{d+1-j,(n-d+j+1)(d+1)} K_{n-d+j}^0(x^{d+1})$. Thus $\tilde{X} K_n^0 = K_{n+1}^0 + \sum_{j=0}^{d} C_{n-d+j} K_{n-d+j}^0$, $C_{n-d} \neq 0$, which means that $\{P_n^0\}_{n\geqslant 0}$ is classical and d-orthogonal.

Case $k \geqslant 1$. The PS $\{P_n\}_{n\geqslant 0}$ is classical d-symmetric d-orthogonal, then

$$\tilde{X} P_n = P_{n+1} + b_{n-d} P_{n-d}, \quad b_{n-d} \neq 0. \tag{7}$$

Moreover since $\{A_n\}_{n \geqslant 0}$ is d-symmetric and d-orthogonal, $\tilde{X} A_n = A_{n+1} + a_{n-d} A_{n-d}$, with $a_{n-d} \neq 0$. Taking derivatives in both sides of (7) and $x^k P_{n+1}^k(x^{d+1}) = P_{(n+1)(d+1)+k}(x)$, we obtain:

$$P_{n+1} = A_{n+1} + \big((n+1-d) b_{n+1-d} - (n+1) a_{n-d}\big) A_{n-d}. \tag{8}$$

Then $k P_{n+1}^k + (d+1)(n+1) \tilde{X} K_n^k = ((n+1)(d+1)+k)(A_{n+1}^k + a_{n(d+1)+k} A_n^k)$.

From (8), we deduce that $P_{n(d+1)+k} = A_{n(d+1)+k} + (((n-1)(d+1)+k) b_{(n-1)(d+1)+k} - (n(d+1)+k) a_{(n-1)(d+1)+k}) \times A_{(n-1)(d+1)+k}$ and then

$$\tilde{X} K_n^k = A_{n+1}^k + \delta_{n,k} A_n^k. \tag{9}$$

We recall that

$$\tilde{X} A_n^k = A_{n+1}^k + \sum_{j=0}^{d} a_{k,n-d+j} A_{n-d+j}^k, \quad \text{with } a_{k,n-d} \neq 0. \tag{10}$$

Replace in this equation n by $n+1$ and A_{j+1}^k by $\tilde{X} K_j^k - \delta_{j,k} A_j^k$, to obtain: $\tilde{X}^2 K_n^k = \tilde{X} K_{n+1}^k + \sum_{j=0}^d b_{k,n-d+j} \tilde{X} K_{n-d+j}^k + c_{k,n-d-1} A_{n-d-1}^k$. If $c_{k,n-d-1} \neq 0$ for a suitable n, then $A_{n-d-1}^k(0) = 0$, and from (9) $A_{n-d}^k(0) = 0$. Then from (10), we deduce that $A_1 = 0$ which is impossible. □

Theorem 2.5. *Let $\{P_n\}_{n \geqslant 0}$ be an m-symmetric classical d-OPS, then its first component $\{P_n^0\}_{n \geqslant 0}$ is a classical d-OPS.*

Proof. Since the PS $\{A_n\}_{n \geqslant 0}$ is m-symmetric d-orthogonal, it fulfills (5). We replace r by $m+1$ and n by $(n+1)(m+1)-1$ in this relation to obtain: $\tilde{X}^{m+1} A_{(n+1)(m+1)-1} = A_{(n+2)(m+1)-1} + \sum_{j=1}^{d+1} \gamma_{n,j} A_{(n+2-j)(m+1)-1}$, with $\gamma_{n,n-d} \neq 0$. Thus

$$\tilde{X}^{m+1} A_{(n+1)(m+1)-1} = A_{(n+2)(m+1)-1} + \sum_{j=0}^{d} \alpha_{n-d+j} A_{(n-d+j+1)(m+1)-1}, \tag{11}$$

with $\alpha_{n-d} \neq 0$. If we take derivatives in both sides of the relation $P_{n+1}^0(x^{m+1}) = P_{(n+1)(m+1)}(x)$, we obtain:

$$x^m K_n^0\big(x^{m+1}\big) = A_{(n+1)(m+1)-1}(x), \tag{12}$$

with $K_n^0 = (1/(n+1))(P_{n+1}^0)'$. From (11) and (12) we deduce that $x^{m+1} x^m K_n^0(x^{m+1}) = x^m K_{n+1}^0(x^{m+1}) + \sum_{j=0}^d \alpha_{n-d+j} x^m \cdot K_{n-d+j}^0(x^{m+1})$, with $\alpha_{n-d} \neq 0$, which is equivalent to $\tilde{X} K_n^0 = K_{n+1}^0 + \sum_{j=0}^d \alpha_{n-d+j} K_{n-d+j}^0$, with $\alpha_{n-d} \neq 0$, and the desired result follows. □

3. Example

We introduce, as far as we know, a new 3-OPS defined by a generating function. Using the identity 2, Problem 7, p. 213 in [13] and Theorem 1 in [3], one can easily prove that the PS $\{P_n\}_{n \geqslant 0}$ generated by: $e^{t^{m+1}} {}_0F_r\left(\begin{matrix} - \\ b_1, \ldots, b_r \end{matrix}\; -xt\right) = \sum_{n=0}^{\infty} P_n(x) t^n$ is m-symmetric classical $((m+1)(r+1)-1)$-orthogonal. Moreover the corresponding $m+1$ components are also classical $((m+1)(r+1)-1)$-orthogonal.

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