C. R. Acad. Sci. Paris, Ser. I 350 (2012) 91–96

Contents lists available at SciVerse ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

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Probability Theory

Baum–Katz type theorems for martingale arrays

Théorèmes de type Baum–Katz pour un tableau de martingales

Shunli Hao [a,b], Quansheng Liu [a,b]

[a] *Université de Bretagne-Sud, UMR 6205, Laboratoire de Mathématiques de Bretagne Atlantique, campus de Tohannic, BP 573, 56017 Vannes, France*
[b] *Université Européenne de Bretagne, France*

ARTICLE INFO

Article history:
Received 14 November 2011
Accepted after revision 8 December 2011
Available online 27 December 2011

Presented by the Editorial Board

ABSTRACT

We show convergence rates in the law of large numbers for martingale arrays. The results extend the classical theorems of Baum and Katz (1965) [2] for sums of independent and identically distributed (i.i.d.) random variables. They improve a result of Ghosal and Chandra (1998) [6] for martingale arrays, and generalize a result of Alsmeyer (1990) [1] for a single martingale. As an application, we obtain a new theorem about the convergence rate of Cesàro summation of identically distributed random variables.

RÉSUMÉ

Nous montrons la vitesse de convergence dans la loi des grand nombres pour un tableau de martingales. Les résultats étendent les théorèmes classiques de Baum et Katz (1965) [2] pour les sommes de variables aléatoires indépendantes et identiquement distribuées (i.i.d.). Ils améliorent un résultat de Ghosal et Chandra (1998) [6] pour des tableaux de martingales, et généralisent un résultat d'Alsmeyer (1990) [1] pour une seule martingale. Comme application, nous obtenons un théorème nouveau concernant la vitesse de convergence pour des sommes de Cesàro de variables aléatoires identiquement distribuées.

Version française abrégée

On suppose que pour chaque $n \geqslant 1$, $\{(X_{nj}, \mathcal{F}_{nj})\}_{j\geqslant 1}$ est une suite de différences de martingales, c'est-à-dire que X_{nj} sont $\mathcal{F}_{nj}$ mesurables avec $\mathbb{E}[X_{nj}|\mathcal{F}_{n,j-1}] = 0$ presque sûrement. On pose pour $n \geqslant 1$,

$$S_{nk} = \sum_{1\leqslant j\leqslant k} X_{nj} \quad \text{pour } k \geqslant 1, \qquad S_{n,\infty} = \sum_{j\geqslant 1} X_{nj} \quad \text{si la série converge.} \tag{1}$$

Sous une condition simple, nous montrons que $S_{n,\infty} \to 0$ en probabilité et nous étudions la vitesse de convergence de $P\{|S_{n,\infty}| > \varepsilon\}$ lorsque $n \to \infty$, où $\varepsilon > 0$. Remarquons que dans le cas d'une seule martingale en posant $X_{nj} = \frac{1}{n} X_j$ si $1 \leqslant j \leqslant n$, et $X_{nj} = 0$ si $j > n$, nous avons $S_{n,\infty} = \frac{1}{n}\sum_{j=1}^n X_j$; dans ce cas il s'agit de l'étude de la vitesse de convergence de $P\{\frac{1}{n}|\sum_{j=1}^n X_j| > \varepsilon\}$. Nos résultats principaux sont les deux théorèmes suivants qui généralisent les résultats classique de Baum et Katz sur la vitesse de convergence dans la loi des grands nombres. Notons, pour $n \geqslant 1$,

E-mail addresses: shunli.hao@univ-ubs.fr (S. Hao), quansheng.liu@univ-ubs.fr (Q. Liu).

1631-073X/$ – see front matter
doi:10.1016/j.crma.2011.12.006

$$m_n(\gamma) = \sum_{j \geqslant 1} \mathbb{E}\big[|X_{nj}|^\gamma | \mathcal{F}_{n,j-1}\big], \quad \gamma \in]1,2]. \tag{2}$$

Théorème 0.1. *Soit $\phi : \mathbb{N} \to [0, \infty[$ une application positive. Supposons que pour certains $\gamma \in]1, 2]$, $q \in [1, \infty]$ et $\lambda \in]0, q[$, lorsque $n \to \infty$,*

$$\big\|m_n(\gamma)\big\|_q \to 0 \quad \text{et} \quad \phi(n)\big\|m_n(\gamma)\big\|_q^\lambda = O(1), \tag{C1}$$

où $\|\cdot\|_q$ désigne la norme dans L^q. Alors les assertions suivantes sont équivalentes :

$$\phi(n) \sum_{j \geqslant 1} P\big\{|X_{nj}| > \varepsilon\big\} = o(1) \quad \big(\text{resp. } O(1)\big)\ \forall \varepsilon > 0, \tag{3}$$

$$\phi(n) P\big\{|S_{n,\infty}| > \varepsilon\big\} = o(1) \quad \big(\text{resp. } O(1)\big)\ \forall \varepsilon > 0, \tag{4}$$

$$\phi(n) P\Big\{\sup_{j \geqslant 1} |S_{nj}| > \varepsilon\Big\} = o(1) \quad \big(\text{resp. } O(1)\big)\ \forall \varepsilon > 0. \tag{5}$$

Notons que la condition (C1) est satisfaite si pour certains $\gamma \in]1, 2]$, $r \in \mathbb{R}$ et $\varepsilon_1 > 0$, lorsque $n \to \infty$,

$$\phi(n) = O\big(n^r\big) \quad \text{et} \quad \big\|m_n(\gamma)\big\|_\infty = O\big(n^{-\varepsilon_1}\big). \tag{C1$'$}$$

Théorème 0.2. *Soit $\phi : \mathbb{N} \to [0, \infty[$ une application positive. Supposons que pour certains $\gamma \in]1, 2]$, $q \in [1, \infty]$ et $\lambda \in]0, q[$,*

$$\big\|m_n(\gamma)\big\|_q \to 0 \quad \text{et} \quad \sum_{n \geqslant 1} \phi(n)\big\|m_n(\gamma)\big\|_q^\lambda < \infty. \tag{C2}$$

Alors les assertions suivantes sont équivalentes :

$$\sum_{n \geqslant 1} \phi(n) \sum_{j \geqslant 1} P\big\{|X_{nj}| > \varepsilon\big\} < \infty \quad \forall \varepsilon > 0, \tag{6}$$

$$\sum_{n \geqslant 1} \phi(n) P\big\{|S_{n,\infty}| > \varepsilon\big\} < \infty \quad \forall \varepsilon > 0, \tag{7}$$

$$\sum_{n \geqslant 1} \phi(n) P\Big\{\sup_{j \geqslant 1} |S_{nj}| > \varepsilon\Big\} < \infty \quad \forall \varepsilon > 0. \tag{8}$$

Dans le cas où (C2) est satisfaite avec $q = \infty$ et $\gamma = 2$, Ghosal et Chandra [6] ont prouvé que (6) implique (8). Ici nous montrons l'équivalence sous une condition plus faible. Dans le cas d'une seule martingale où $X_{nj} = \frac{1}{n^\alpha} X_j$ avec $\alpha \in]1/2, 1]$ si $1 \leqslant j \leqslant n$, et $X_{nj} = 0$ si $j > n$, et lorsque $\phi(n) = n^{b-1}$ ($b > 0$), l'équivalence de (6)–(8) a été prouvée par Alsmeyer [1] sous la condition suivante plus forte :

$$\sup_{n \geqslant 1} \big\|\underline{m}_n(\gamma)\big\|_q < \infty \quad \text{avec } \underline{m}_n(\gamma) = \frac{1}{n} \sum_{1 \leqslant j \leqslant n} \mathbb{E}\big[|X_j|^\gamma | \mathcal{F}_{j-1}\big] \tag{C3}$$

pour certains $\gamma \in]1/\alpha, 2]$ et $q \in [1, \infty]$ avec $q > b/(\gamma\alpha - 1)$. Dans le cas où les X_j sont i.i.d., les equivalences des Théorèmes 0.1 et 0.2 deviennent les théorèmes classiques de Baum et Katz [2].

1. Introduction

Our work is motivated by the study of convergence rates in the law of large numbers for martingales.

Let (X_j) be a sequence of i.i.d. random variables with $\mathbb{E}X_i = 0$ and set $S_n = \sum_{j=1}^n X_j$. By the law of large numbers, $P\{|S_n| > \varepsilon n\} \to 0$ for $\varepsilon > 0$. For the rate of convergence, Hsu and Robbins [8] showed that $\mathbb{E}X_1^2 < \infty$ implies $\sum_{n=1}^\infty P(|S_n| > \varepsilon n) < \infty$ for all $\varepsilon > 0$; Erdös [5] proved that the converse also holds. Spitzer [12] found that $\sum_{n=1}^\infty n^{-1} P(|S_n| > \varepsilon n) < \infty$ for all $\varepsilon > 0$ whenever $\mathbb{E}X_1 = 0$. Baum and Katz [2] proved that, for $p = \frac{1}{\alpha}$, $\alpha \geqslant \frac{1}{2}$ or for $p > \frac{1}{\alpha}$, $\alpha > \frac{1}{2}$, $\sum_{n=1}^\infty n^{p\alpha-2} P\{|S_n| > \varepsilon n^\alpha\} < \infty$ for all $\varepsilon > 0$ if and only if $\mathbb{E}|X_1|^p < \infty$.

Let (X_j) be a sequence of real-valued martingale differences defined on a probability space $(\Omega, \mathcal{F}, P)$, adapted to a filtration $(\mathcal{F}_j)$, with $\mathcal{F}_0 = \{\emptyset, \Omega\}$. Is the pre-mentioned theorem of Baum and Katz [2] still valid for martingale differences (X_j)? Lesigne and Volný [10] proved that for $p \geqslant 2$, $\sup_{j \geqslant 1} \mathbb{E}|X_j|^p < \infty$ implies $P(|S_n| > \varepsilon n) = O(n^{-p/2})$, and that the exponent $p/2$ is the best possible, even for strictly stationary and ergodic sequences of martingale differences. Therefore the theorem of Baum and Katz does not hold for martingale differences without additional conditions. (Stoica [13] claimed that the theorem of Baum and Katz still holds for $p > 2$ in the case of martingale differences without additional assumption;

but his claim is a contradiction with the conclusion of Lesigne and Volný [10], and his proof contains an error: when $p > 2$, we cannot choose α satisfying (6) of [13].) Alsmeyer [1] proved that the theorem of Baum and Katz of order $p > 1$ still holds for martingale differences (X_j) if for some $\gamma \in (1, 2]$ and $q \in [1, \infty]$ with $q > (p-1)/(\gamma-1)$, (C3) holds, where $\|\cdot\|_q$ denotes the L^q norm. This is a nice result, nevertheless: (a) it does not apply to “non-homogeneous” cases, such as martingales of the form $S_n = \sum_{j=1}^n j^a Y_j$, where $a > 0$ and Y_j are i.i.d., as in this case the condition (C3) (with $X_j = j^a Y_j$) is never satisfied; (b) in applications instead of a single martingale we often need to consider martingale arrays: for example when we use the decomposition of a random sequence S_n into martingale differences (such as in the study of directed polymers in a random environment), the summands usually depend on n: $S_n = \sum_{j=1}^n X_{nj}$, $X_{nj} = \mathbb{E}[S_n|\mathcal{F}_j] - \mathbb{E}[S_n|\mathcal{F}_{j-1}]$, where $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_i = \sigma(S_1, \ldots, S_i)$ for $i \geqslant 1$.

Our first objective is to extend the theorem of Baum and Katz [2] to a large class of martingale arrays. Our results improve a result of Ghosal and Chandra [6] for martingale arrays, and extend Alsmeyer's result (cf. [1]) for martingales. The consideration of a martingale array (rather than a single martingale) makes the results very adapted in the study of weighted sums of identically distributed random variables. As an example, we prove a new theorem about the rate of convergence of Cesàro summation of identically distributed random variables.

Our second objective is to extend another important and closely related theorem of Baum and Katz [2] which states that for i.i.d. random variables X_j with $\mathbb{E}X_j = 0$ and for each $p \geqslant 1$, $P(|X_1| > n) = o(n^{-p})$ if and only if $P(|S_n| > \varepsilon n) = o(n^{-(p-1)})$ for all $\varepsilon > 0$. We prove that a similar result holds for a large class of martingale arrays. The result is new and sharp even for independent but not identically distributed random variables.

2. Convergence rates of martingale arrays

In this section, we show Baum–Katz type theorems for convergence rates of martingale arrays. For every $n \geqslant 1$, let $\{(X_{nj}, \mathcal{F}_{nj})\}_{j \geqslant 1}$ be a sequence of real-valued martingale differences defined on a probability space $(\Omega, \mathcal{F}, P)$: for every $j \geqslant 1$, X_{nj} is $\mathcal{F}_{nj}$ measurable and $\mathbb{E}[X_{nj}|\mathcal{F}_{n,j-1}] = 0$ a.s., where $\{\mathcal{F}_{nj}\colon\ j \geqslant 0\}$ is a sequence of increasing sub-σ-fields of $\mathcal{F}$, with $\mathcal{F}_{n0} = \{\emptyset, \Omega\}$. Set for $n \geqslant 1$,

$$S_{nk} = \sum_{1 \leqslant j \leqslant k} X_{nj} \quad \text{for } k \geqslant 1, \qquad S_{n,\infty} = \sum_{j \geqslant 1} X_{nj} \quad \text{if the series converges.} \tag{9}$$

We first show a preliminary result which is a weak law of large numbers for $S_{n,\infty}$.

Lemma 2.1. *If for some $\gamma \in (1, 2]$, $\sum_{j=1}^\infty \mathbb{E}|X_{nj}|^\gamma \to 0$ as $n \to \infty$, then*

$$\sup_{j \geqslant 1} |S_{nj}| \to 0 \quad \textit{and} \quad S_{n,\infty} \to 0 \quad \textit{in probability}. \tag{10}$$

We are interested in the convergence rate of $P\{|S_{n,\infty}| > \varepsilon\}$ as $n \to \infty$. We describe the rate of convergence by comparing the probability with an auxiliary function $\phi(n)$, and by considering the convergence of the corresponding series.

Theorem 2.2. *Let $\phi : \mathbb{N} \to [0, \infty)$ be a positive function. Suppose that* (C1) *holds for some $\gamma \in (1, 2]$, $q \in [1, \infty]$ and $\lambda \in (0, q)$. Then the assertions* (3)–(5) *are all equivalent.*

We mention that the condition (C1) holds if (C1′) holds for some $\gamma \in (1, 2]$, $r \in \mathbb{R}$ and $\varepsilon_1 > 0$.

Theorem 2.3. *Let $\phi : \mathbb{N} \mapsto [0, \infty)$ be a positive function. Suppose that* (C2) *holds for some $\gamma \in (1, 2]$, $q \in [1, \infty]$ and $\lambda \in (0, q)$. Then the assertions* (6)–(8) *are all equivalent.*

Ghosal and Chandra [6, Theorem 2] proved that if (C2) holds with $q = \infty$ and $\gamma = 2$, then (6) implies (8). Here we show the equivalence of (6) and (8) under a weaker condition.

3. Convergence rates of martingales

In this section, we consider the convergence rate in the law of large numbers for a single sequence of martingale differences. Let $\{(X_j, \mathcal{F}_j)\}_{j \geqslant 1}$ be a sequence of real-valued martingale differences defined on a probability space $(\Omega, \mathcal{F}, P)$. This means that for each $j \geqslant 1$, X_j is $\mathcal{F}_j$-measurable and $\mathbb{E}[X_j|\mathcal{F}_{j-1}] = 0$ a.s., where $\{\mathcal{F}_j\colon\ j \geqslant 0\}$ is a sequence of increasing sub-σ-fields of $\mathcal{F}$, with $\mathcal{F}_0 = \{\emptyset, \Omega\}$. Set for $n \geqslant 1$,

$$S_n = \sum_{1 \leqslant j \leqslant n} X_j, \qquad S_n^* = \max_{1 \leqslant j \leqslant n} |S_j|. \tag{11}$$

Notice that $S_n/n^\alpha \to 0$ a.s. if and only if $P(\sup_{j \geqslant n} j^{-\alpha}|S_j| > \varepsilon) \to 0$ for any $\varepsilon > 0$. So the following theorems describe the a.s. convergence of S_n/n^α. Let $l(\cdot) > 0$ be a function slowly varying at ∞, that is, a positive measurable function defined on

$(0,\infty)$ such that $\lim_{x\to\infty} l(\lambda x)/l(x) = 1$ for each $\lambda > 0$. As usual, we write $a_n = o(1)$ if $\lim_{n\to\infty} a_n = 0$, and $a_n = O(1)$ if (a_n) is bounded.

Theorem 3.1. *Let $\alpha, b > 0$ and $\phi(n) = n^b l(n)$. Suppose that for some $\gamma \in (1,2]$, $q \in [1,\infty]$ and $\lambda \in (0,q)$,*

$$\phi(n)\big\|m_n(\gamma)\big\|_q^\lambda = O(1), \quad \text{where } m_n(\gamma) = n^{-\gamma\alpha} \sum_{1\leqslant j\leqslant n} \mathbb{E}\big[|X_j|^\gamma \,|\, \mathcal{F}_{j-1}\big]. \tag{C4}$$

Then the following assertions are all equivalent:

$$\phi(n) \sum_{1\leqslant j\leqslant n} P\big\{|X_j| > \varepsilon n^\alpha\big\} = o(1) \quad \big(\text{resp. } O(1)\big)\ \forall \varepsilon > 0, \tag{12}$$

$$\phi(n) P\big\{|S_n| > \varepsilon n^\alpha\big\} = o(1) \quad \big(\text{resp. } O(1)\big)\ \forall \varepsilon > 0, \tag{13}$$

$$\phi(n) P\big\{S_n^* > \varepsilon n^\alpha\big\} = o(1) \quad \big(\text{resp. } O(1)\big)\ \forall \varepsilon > 0, \tag{14}$$

$$\phi(n) P\Big\{\sup_{j\geqslant n} j^{-\alpha} S_j^* > \varepsilon\Big\} = o(1) \quad \big(\text{resp. } O(1)\big)\ \forall \varepsilon > 0. \tag{15}$$

Theorem 3.1 is an extension of Theorem 4 of Baum and Katz [2] for i.i.d. random variables.

Theorem 3.2. *Let $\alpha, b > 0$ and $\phi(n) = n^{b-1} l(n)$. Suppose that for some $\gamma \in (1,2]$, $q \in [1,\infty]$ and $\lambda \in (0,q)$,* (C2) *holds with $m_n(\gamma) = n^{-\gamma\alpha} \sum_{j=1}^n \mathbb{E}[|X_j|^\gamma | \mathcal{F}_{j-1}]$. Then the following assertions are all equivalent*:

$$\sum_{n\geqslant 1} \phi(n) \sum_{1\leqslant j\leqslant n} P\big\{|X_j| > \varepsilon n^\alpha\big\} < \infty \quad \forall \varepsilon > 0, \tag{16}$$

$$\sum_{n\geqslant 1} \phi(n) P\big\{|S_n| > \varepsilon n^\alpha\big\} < \infty \quad \forall \varepsilon > 0, \tag{17}$$

$$\sum_{n\geqslant 1} \phi(n) P\big\{S_n^* > \varepsilon n^\alpha\big\} < \infty \quad \forall \varepsilon > 0, \tag{18}$$

$$\sum_{n\geqslant 1} \phi(n) P\Big\{\sup_{j\geqslant n} j^{-\alpha} S_j^* > \varepsilon\Big\} < \infty \quad \forall \varepsilon > 0. \tag{19}$$

If additionally X_j have the same law, then (16)–(19) *are all equivalent to $\mathbb{E}|X_1|^{(b+1)/\alpha} l(|X_1|^{1/\alpha}) < \infty$.*

We mention that in the case where $\alpha \in (1/2, 1]$ and (C3) holds for some $\gamma \in (1/\alpha, 2]$ and $q \in [1,\infty]$ with $q > b/(\gamma\alpha - 1)$, the condition (C2) of Theorem 3.2 is satisfied. In this case and when $l = 1$, the equivalence of (16)–(19) was proved by Alsmeyer [1]. In applications, especially when we consider sums of weighted random variables, the condition (C2) is more adapted than (C3) (cf. Section 4).

Theorem 3.2 is an extension of Theorems 1, 2 and 3 of Baum and Katz [2] for i.i.d. random variables.

4. Applications to weighted sums of martingale differences

We can study convergence rates for weighted sums of identically distributed random variables by using Theorem 2.3. In the following we give an example about Cesàro summation for identically distributed martingale differences. Cesàro summation is the most commonly studied method of summation. Let $\{(X_j, \mathcal{F}_j)\}_{j\geqslant 1}$ be martingale differences that are identically distributed. For $a > -1$, set

$$S_n^a = \big(A_n^a\big)^{-1} \sum_{0\leqslant j\leqslant n} A_{n-j}^{a-1} X_j \quad \text{for } n \geqslant 0, \tag{20}$$

where $A_0^a = 1$ and $A_n^a = (a+1)(a+2)\cdots(a+n)/n!$ for $n \geqslant 1$. The coefficients A_n^a satisfy $A_n^a \sim \frac{n^a}{\Gamma(a+1)}$ as $n \to \infty$ and $(A_n^a)^{-1} \sum_{j=0}^n A_{n-j}^{a-1} = 1$ for $n \geqslant 0$. When X_j are centered independent random variables, the a.s. convergence of S_n^a to 0 was shown by Lorentz [11] for $a \in (1/2, 1)$, by Chow and Lai [3] for $a \in (0, 1/2)$ and by Déniel and Derriennic [4] for $a = 1/2$. Here we study the convergence rate for martingale differences.

Theorem 4.1. *Let $b \geqslant 0$, $a > 0$, $\alpha > -a$ and $p, \beta \in \mathbb{R}$. Suppose that* (C3) *holds for some $\gamma \in (1,2]$ and $q \in [1,\infty]$ with $q > \frac{b}{\gamma(\alpha+1)-1}$ and $\gamma(\alpha+1) - 1 > 0$.*

(a) *Let S_n^a be defined by* (20)*. Then*

$$\sum_{n\geqslant 2} n^{b-1}(\log n)^p P\left\{\left|S_n^a\right| > \varepsilon n^\alpha (\log n)^\beta\right\} < \infty \quad \forall \varepsilon > 0 \tag{21}$$

is equivalent to

$$\begin{cases} \mathbb{E}|X_1|^{\frac{b+1}{1+\alpha}}\left(\log^+|X_1|\right)^{-\frac{\beta(b+1)}{1+\alpha}+p} < \infty & \text{if } a > \dfrac{b-\alpha}{b+1}, \\ \mathbb{E}|X_1|^{\frac{1}{1-a}}\left(\log^+|X_1|\right)^{-\frac{\beta}{1-a}+p+1} < \infty & \text{if } a = \dfrac{b-\alpha}{b+1}, \\ \mathbb{E}|X_1|^{\frac{b}{a+\alpha}}\left(\log^+|X_1|\right)^{-\frac{\beta b}{a+\alpha}+p} < \infty & \text{if } a < \dfrac{b-\alpha}{b+1}. \end{cases} \tag{22}$$

(b) *The conclusion of Part* (a) *remains true when S_n^a is defined by $S_n^a = n^{-a}\sum_{j=1}^n j^{a-1}X_j$.*

When X_j are centered i.i.d. random variables, if $p = \beta = 1$, $a \leqslant 1$ and $\alpha = 0$, Part (a) of Theorem 4.1 reduces to Theorem 2.2 of Gut in [7]; Part (b) contains Theorems 2 and 3 of Lanzinger and Stadtmüller [9] (more precisely, our result reduces to Theorems 2 and 3(a)(i) and (iii) of [9] if $p = \beta = 0$ and $a \in (0,1)$, to Theorem 3(a)(ii) of [9] if $p = -1$, $\beta = 0$ and $a \in (0,1)$, and to Theorem 3(b) of [9] if $p < 0$, $p \neq -1$, $\beta = 1$ and $a = 0$). Notice that when $p = \beta = 0$ and $a = 1$, in (22) only the first case occurs, so that the moment condition in Theorem 4.1 coincides with that in Theorem 3.2 (with $l(x) = 1$).

5. Maximal inequalities

The proofs of results are based on the following maximal inequalities. For a sequence of random variables (X_j), let $X_n^* = \max_{1\leqslant j\leqslant n}|X_j|$, and let S_n, S_n^* be defined by (11). For $\gamma \in (1,2]$, let

$$m(\gamma, n) = \sum_{1\leqslant j\leqslant n} \mathbb{E}\left[|X_j|^\gamma | \mathcal{F}_{j-1}\right]. \tag{23}$$

Lemma 5.1 *(Relation between $P\{X_n^* > \varepsilon\}$ and $\sum_{1\leqslant j\leqslant n} P\{|X_j| > \varepsilon\}$). Let $\{(X_j, \mathcal{F}_j)\}_{j=1}^n$ be an adapted sequence of random variables. Then for any $\varepsilon, \gamma > 0$ and $q \geqslant 1$,*

$$P\left\{X_n^* > \varepsilon\right\} \leqslant \sum_{1\leqslant j\leqslant n} P\left\{|X_j| > \varepsilon\right\} \leqslant \left(1 + \varepsilon^{-\gamma}\right) P\left\{X_n^* > \varepsilon\right\} + \varepsilon^{-\gamma}\mathbb{E}m^q(\gamma, n). \tag{24}$$

Lemma 5.2 *(Relation between $P\{X_n^* > \varepsilon\}$ and $P\{S_n^* > \varepsilon\}$). Let $\{(X_j, \mathcal{F}_j)\}_{j=1}^n$ be a sequence of martingale differences. Then for any $\varepsilon > 0$, $\gamma \in (1,2]$, $q \geqslant 1$ and any integer $L \geqslant 0$,*

$$P\left\{X_n^* > 2\varepsilon\right\} \leqslant P\left\{S_n^* > \varepsilon\right\} \leqslant P\left\{X_n^* > \varepsilon/\left(4(L+1)\right)\right\} + \varepsilon^{\frac{-q\gamma(L+1)}{q+L}} C(\gamma, q, L)\left(\mathbb{E}m^q(\gamma, n)\right)^{\frac{1+L}{q+L}}, \tag{25}$$

where $C(\gamma, q, L)$ is a constant depending only on γ, q and L.

Lemma 5.3 *(Relation between $P\{|S_n| > \varepsilon\}$ and $P\{S_n^* > \varepsilon\}$). Let $\{(X_j, \mathcal{F}_j)\}_{j=1}^n$ be a sequence of martingale differences. Then for any $\varepsilon > 0$, $\gamma \in (1,2]$ and $q \geqslant 1$,*

$$P\left\{|S_n| > \varepsilon\right\} \leqslant P\left\{S_n^* > \varepsilon\right\} \leqslant 2P\left\{|S_n| > \varepsilon/2\right\} + \varepsilon^{-q\gamma} C(\gamma, q)\mathbb{E}m^q(\gamma, n), \tag{26}$$

where $C(\gamma, q)$ is a constant depending only on γ and q.

Lemma 5.4 *(Relation between $P\{S_n^* > \varepsilon n^\alpha\}$ and $P\{\sup_{j\geqslant n} j^{-\alpha} S_j^* > \varepsilon\}$). Let $(X_j)_{j\geqslant 1}$ be a sequence of any random variables. Let $\varepsilon, \alpha, b, \delta > 0$ and $\phi(n) = n^b l(n)$. If there exists $n_0 > 0$, such that for all $n \geqslant n_0$,*

$$\phi(n) P\left\{S_n^* > \frac{1}{2}\varepsilon n^\alpha\right\} \leqslant \delta, \tag{27}$$

then there exists $n_0' > 0$ depending only on n_0, b and α, such that for all $n \geqslant n_0'$,

$$\phi(n) P\left\{\sup_{j\geqslant n} j^{-\alpha} S_j^* > 2\varepsilon\right\} \leqslant \delta C(b, \alpha), \tag{28}$$

where $C(b, \alpha)$ is a constant depending only on b and α. Conversely, $P\{S_n^* > \varepsilon n^\alpha\} \leqslant P\{\sup_{j\geqslant n} j^{-\alpha} S_j^* > \varepsilon\}$.

Acknowledgements

We are grateful to the referee for her/his careful reading and useful remarks. The work has been partially supported by the National Natural Science Foundation of China, Grant no. 11101039 and Grant no. 11171044.

References

[1] G. Alsmeyer, Convergence rates in the law of large numbers for martingales, Stochastic Process. Appl. 36 (1990) 181–194.
[2] L.E. Baum, M. Katz, Convergence rates in the law of large numbers, Trans. Amer. Math. Soc. 120 (1965) 108–123.
[3] Y.S. Chow, T.L. Lai, Limiting behavior of weighted sums of independent random variables, Ann. Prob. 1 (1973) 810–824.
[4] Y. Déniel, Y. Derriennic, Sur la convergence prèsque sure, au sens de Cesàro d'ordre α, $0 < \alpha < 1$, de variables aléatoires et indépendantes et identiquement distribuées, Probab. Theory Related Fields 79 (1949) 629–636.
[5] P. Erdös, On a theorem of Hsu and Robbins, Ann. Math. Statist. 20 (1949) 286–291.
[6] S. Ghosal, T.K. Chandra, Complete convergence of martingale arrays, J. Theor. Prob. 3 (1998) 621–631.
[7] A. Gut, Complete convergence and Cesàro summation for i.i.d. random variables, Probab. Theory Related Fields 97 (1993) 169–178.
[8] P.L. Hsu, H. Robbins, Complete convergence and the law of large numbers, Proc. Natl. Acad. Sci. USA 33 (1947) 25–31.
[9] H. Lanzinger, U. Stadtmüller, Baum–Katz laws for certain weighted sums of independent and identically distributed random variables, Bernoulli 9 (2003) 985–1002.
[10] E. Lesigne, D. Volný, Large deviations for martingales, Stochastic Process. Appl. 96 (2001) 143–159.
[11] G.G. Lorentz, Borel and Banach properties of methods of summation, Duke Math. J. 22 (1955) 129–141.
[12] F. Spitzer, A combinatorial lemma and its applications to probability theory, Trans. Amer. Math. Soc. 82 (1956) 323–339.
[13] G. Stoica, Baum–Katz–Nagaev type results for martingales, J. Math. Anal. Appl. 336 (2007) 1489–1492.