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Lie Algebras

On the Kostant conjecture for Clifford algebras

Sur la conjecture de Kostant pour les algèbres de Clifford

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ABSTRACT

Let $\mathfrak{g}$ be a complex simple Lie algebra, and $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra. In the end of 1990s, B. Kostant defined two filtrations on $\mathfrak{h}$, one using the Clifford algebras and the odd analogue of the Harish-Chandra projection $\mathrm{hc}_{\mathrm{odd}} : \mathrm{Cl}(\mathfrak{g}) \to \mathrm{Cl}(\mathfrak{h})$, and the other one using the canonical isomorphism $\check{\mathfrak{h}} = \mathfrak{h}^*$ (here $\check{\mathfrak{h}}$ is the Cartan subalgebra in the simple Lie algebra $\check{\mathfrak{g}}$ corresponding to the dual root system) and the adjoint action of the principal $\mathfrak{sl}_2$-triple. Kostant conjectured that the two filtrations coincide.
Recently, A. Joseph proved that the second Kostant filtration coincides with the filtration on $\mathfrak{h}$ induced by the generalized Harish-Chandra projection $(U\mathfrak{g} \otimes \mathfrak{g})^{\mathfrak{g}} \to S\mathfrak{h} \otimes \mathfrak{h}$ and the evaluation at $\rho \in \mathfrak{h}^*$. In this Note, we prove that Joseph's result is equivalent to the Kostant Conjecture. We also show that the standard Harish-Chandra projection $U\mathfrak{g} \to S\mathfrak{h}$ composed with evaluation at ρ induces the same filtration on $\mathfrak{h}$.

RÉSUMÉ

Soient $\mathfrak{g}$ une algèbre de Lie simple complexe, et $\mathfrak{h} \subset \mathfrak{g}$ une sous-algèbre de Cartan. Vers la fin des années 1990, B. Kostant définit deux filtrations sur $\mathfrak{h}$; la première utilise les algèbres de Clifford et l'analogue impair de la projection de Harish-Chandra $\mathrm{hc}_{\mathrm{odd}} : \mathrm{Cl}(\mathfrak{g}) \to \mathrm{Cl}(\mathfrak{h})$, la seconde l'isomorphisme canonique $\check{\mathfrak{h}} = \mathfrak{h}^*$ (ici, $\check{\mathfrak{h}}$ est la sous-algèbre de Cartan dans l'algèbre de Lie simple $\check{\mathfrak{g}}$ correspondant au système de racines dual) et l'action adjointe du $\mathfrak{sl}_2$-triplet principal. Kostant conjectura que ces deux filtrations coïncident.
Récemment, A. Joseph a démontré que la seconde filtration de Kostant coïncidait avec la filtration sur $\mathfrak{h}$ induite par la projection de Harish-Chandra généralisée $(U\mathfrak{g} \otimes \mathfrak{g})^{\mathfrak{g}} \to S\mathfrak{h} \otimes \mathfrak{h}$ et l'évaluation au point $\rho \in \mathfrak{h}^*$. Dans cette Note, nous montrons que le résultat de Joseph est équivalent à la conjecture de Kostant. Nous obtenons de plus que la projection de Harish-Chandra standard $U\mathfrak{g} \to S\mathfrak{h}$ composée avec l'évaluation au point ρ induit la même filtration sur $\mathfrak{h}$.

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Le point de départ de ce travail est l'article fondamental de Kostant [11] sur la structure des algèbres de Clifford engendrées par une algèbre de Lie simple complexe. Soit $\mathfrak{g}$ une algèbre de Lie simple complexe de rang r. D'après le théorème de Hopf–Koszul–Samelson, $(\wedge\mathfrak{g}^*)^{\mathfrak{g}}$ est isomorphe à l'algèbre extérieure $\wedge P$ où P est un espace vectoriel gradué dont les générateurs sont de degrés $2m_i + 1$ définis par les exposants de $\mathfrak{g}$, $m_1, \ldots, m_r$. Le produit scalaire invariant, $B_{\mathfrak{g}}$, sur $\mathfrak{g}$ (unique à un multiple près) définit une structure d'algèbre de Clifford $\mathrm{Cl}(\mathfrak{g}) = T\mathfrak{g}/\langle x\otimes x - B_{\mathfrak{g}}(x,x)\rangle$ où $T\mathfrak{g}$ est l'algèbre tensorielle de $\mathfrak{g}$. L'un des résultats principaux de [11] stipule que la partie invariante $\mathrm{Cl}(\mathfrak{g})^{\mathfrak{g}}$ de l'algèbre de Clifford sous l'action adjointe est isomorphe à l'algèbre de Clifford $\mathrm{Cl}(P)$ sur P (avec un produit scalaire B_P induit par $B_{\mathfrak{g}}$).

Soient $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ une décomposition triangulaire de $\mathfrak{g}$, et $\theta : \mathfrak{g} \to \mathrm{Cl}(\mathfrak{g})$ l'injection canonique de $\mathfrak{g}$ comme générateur de $\mathrm{Cl}(\mathfrak{g})$. Considérons la décomposition en somme directe (1) de l'algèbre de Clifford et la *projection de Harish-Chandra impaire* $\mathrm{hc}_{\mathrm{odd}} : \mathrm{Cl}(\mathfrak{g}) \to \mathrm{Cl}(\mathfrak{h})$. Un résultat non trivial de Bazlov [5, Proposition 4.5] et Kostant (communications privées) assure que $\mathrm{hc}_{\mathrm{odd}} \circ q(P) = \theta(\mathfrak{h}) \subset \mathrm{Cl}(\mathfrak{h})$ où $q : \wedge\mathfrak{g} \to \mathrm{Cl}(\mathfrak{g})$ est l'anti-symétrisation. L'espace P possède une filtration naturelle, $P^{(k)}$, induite par le degré et $\mathrm{hc}_{\mathrm{odd}}(q(P^{(k)}))$ définit alors une filtration sur la sous-algèbre de Cartan $\mathfrak{h}$.

Soient $\check{\mathfrak{g}}$ l'algèbre de Lie duale définie par le système de racines dual, et $\check{\mathfrak{h}} \subset \check{\mathfrak{g}}$ la sous-algèbre de Cartan correspondante. Kostant observe que $\rho \in \mathfrak{h}^*$, la demi-somme des racines positives, vu comme un élément de $\check{\mathfrak{h}}$ coïncide avec l'élément de Cartan $\check{h}$ du $\mathfrak{sl}_2$-triplet principal $(\check{e}, \check{h}, \check{f}) \subset \check{\mathfrak{g}}$. Comme $\check{\mathfrak{h}}^*$ et $\mathfrak{h}$ sont canoniquement isomorphes, l'action coadjointe de $\check{e}$ induit une filtration, $\check{\mathcal{F}}^{(m)}\mathfrak{h}$, sur $\mathfrak{h}$ donnée par l'égalité (2). Il résulte de la relation $[\check{e}, [\check{e}, \check{h}]] = 0$ que $\rho = \check{h} \in \mathcal{F}^{(1)}\check{\mathfrak{h}}$. Kostant conjecture alors que $h(q(P^{(2m+1)})) = \theta(\check{\mathcal{F}}^{(m)}\mathfrak{h})$ (Conjecture 1.1).

Nous établissons un lien entre les deux filtrations impliquées dans la conjecture de Kostant en deux temps. La première étape, cruciale, a récemment été franchie par Joseph dans [6]. Ce dernier montre que la filtration $\check{\mathcal{F}}^{(m)}\mathfrak{h}$ apparaît dans le cadre des projections de Harish-Chandra généralisées, introduites dans [8] ; voir le Théorème 2.1 pour un énoncé précis.

Dans cette Note, nous montrons (Théorème 2.2) l'équivalence entre la conjecture de Kostant et le théorème de Joseph. Nous obtenons également un second résultat (Théorème 3.1) qui est une variante du précédent sur le même thème. La démonstration du Théorème 2.2 repose sur trois propositions clés que nous présentons brièvement ici.

Soient $\tau : U\mathfrak{g} \to \mathrm{Cl}(\mathfrak{g})$ l'unique homomorphisme d'algèbres défini par les propriétés $[\tau(x), \theta(y)] = \theta([x,y]_{\mathfrak{g}})$ et $\deg(\tau(x)) = 2$, pour $x, y \in \mathfrak{g}$, $\mathrm{m}_{\mathrm{Cl}} : \mathrm{Cl}(\mathfrak{g}) \otimes \mathfrak{g} \to \mathrm{Cl}(\mathfrak{g})$, $a \otimes b \to a\theta(b)$, l'application produit dans l'algèbre de Clifford, et $\mu = \mathrm{m}_{\mathrm{Cl}} \circ (\tau \otimes 1) : U\mathfrak{g} \otimes \mathfrak{g} \to \mathrm{Cl}(\mathfrak{g})$. Puisque τ est de degré deux, il vient $\mu(U^{(m)}\mathfrak{g} \otimes \mathfrak{g}) \subset \mathrm{Cl}^{(2m+1)}(\mathfrak{g})$ où $U^{(m)}\mathfrak{g}$ est la filtration naturelle de $U\mathfrak{g}$. L'espace $U\mathfrak{g} \otimes \mathfrak{g}$ possède deux actions ρ_L et ρ_R de $\mathfrak{g}$ données par les formules (4) qui commutent entre elles. On note $\mathrm{hc} : U\mathfrak{g} \otimes \mathfrak{g} \to S\mathfrak{h} \otimes \mathfrak{g}$ la projection de Harish-Chandra généralisée relativement à la décomposition (5). Notre première proposition (Proposition 2.1) affirme que les sous-espaces $\mu(\rho_L(\mathfrak{n}_-)(U\mathfrak{g} \otimes \mathfrak{g}))$ et $\mu(\rho_R(\mathfrak{n}_+)(U\mathfrak{g} \otimes \mathfrak{g}))$ sont contenus dans le noyau de $\mathrm{hc}_{\mathrm{odd}}$. Nous montrons ensuite (Proposition 2.3) que les applications $\theta \circ (\mathrm{ev}_\rho \otimes 1) \circ \mathrm{hc}_{\mathfrak{g}}$ et $\mathrm{hc}_{\mathrm{odd}} \circ \mu$, définies sur $(U\mathfrak{g} \otimes \mathfrak{g})^{\mathfrak{g}}$ et à valeurs dans $\mathrm{Cl}(\mathfrak{h})$, coïncident. Enfin nous montrons (Proposition 2.4) que pour tout $m \in \mathbb{N}$, $\mu((U^{(m)}\mathfrak{g} \otimes \mathfrak{g})^{\mathfrak{g}}) = q(P^{(2m+1)})$. Les résultats de [11] sont cruciaux dans cette dernière étape.

Le Théorème 2.2 s'ensuit : ce qui précède montre que $\theta \circ (\mathrm{ev}_\rho \otimes 1) \circ \mathrm{hc}_{\mathfrak{g}}((U^{(m)}\mathfrak{g} \otimes \mathfrak{g})^{\mathfrak{g}}) = \mathrm{hc}_{\mathrm{odd}} \circ \mu((U^{(m)}\mathfrak{g} \otimes \mathfrak{g})^{\mathfrak{g}}) = \mathrm{hc}_{\mathrm{odd}}(q(P^{(2m+1)}))$. L'équivalence souhaitée en découle.

Alors que nous finalisions cette note, A. Joseph nous a informé avoir récemment démontré l'égalité entre filtrations (3) (cf. [7, §1.4 et Theorem 2.4]). Cette égalité jointe au Théorème 3.1 donne une nouvelle démonstration de la conjecture de Kostant. Précisons que l'énoncé de la conjecture de Kostant selon [7, §1.4] diffère du nôtre. Une égalité similaire à (3), avec $U\mathfrak{g}$ en place de $S\mathfrak{g}$ et la projection de $S\mathfrak{g}$ sur $S\mathfrak{h}$ en place de la projection de Harish-Chandra standard, fut établie par Rohr [13]. Dans le cas de l'algèbre enveloppante, la démonstration exige toutefois des outils nettement plus sophistiqués comme les opérateurs de Zhelobenko et Bernstein–Gelfand–Gelfand.

1. Introduction

Our starting point in this work is the fundamental paper by Kostant [11] on the structure of Clifford algebras over complex simple Lie algebras. Let G be a complex simple Lie group of rank r and $\mathfrak{g}$ be the corresponding Lie algebra. The cohomology ring of G (over $\mathbb{C}$) is isomorphic to the ring of bi-invariant differential forms, $H(G) \cong (\wedge\mathfrak{g}^*)^{\mathfrak{g}}$. By the Hopf–Koszul–Samelson theorem, $(\wedge\mathfrak{g}^*)^{\mathfrak{g}}$ is isomorphic to the exterior algebra $\wedge P$, where P is a graded vector space with generators in degrees $2m_i + 1$ defined by the exponents of $\mathfrak{g}$, $m_1, \ldots, m_r$. For all $\mathfrak{g}$ simple, $m_1 = 1$ and the corresponding bi-invariant differential form is the Cartan 3-form

$$\eta(x, y, z) = B_{\mathfrak{g}}\big(x, [y, z]\big),$$

where $x, y, z \in \mathfrak{g}$ and $B_{\mathfrak{g}}$ is the unique (up to multiple) invariant scalar product on $\mathfrak{g}$.

The scalar product $B_{\mathfrak{g}}$ defines a structure of a Clifford algebra $\mathrm{Cl}(\mathfrak{g}) = T\mathfrak{g}/\langle x \otimes x - B_{\mathfrak{g}}(x,x)\rangle$, where $T\mathfrak{g}$ is the tensor algebra of $\mathfrak{g}$. The Clifford algebra carries a Lie superalgebra structure with Lie bracket $[a,b] = ab - (-1)^{|a||b|}ba$. To avoid confusion, we will denote by $[,]_{\mathfrak{g}}$ the usual Lie bracket on $\mathfrak{g}$. One of the main results of [11] is the theorem stating that the adjoint invariant part of the Clifford algebra $\mathrm{Cl}(\mathfrak{g})^{\mathfrak{g}} \cong \mathrm{Cl}(P)$ is isomorphic to the Clifford algebra over P (with a scalar product B_P induced by $B_{\mathfrak{g}}$). Under this isomorphism, the Cartan 3-form defines a canonical cubic element $\hat{\eta} \in \mathrm{Cl}(\mathfrak{g})$. This element plays an important role in the theory of Kostant's cubic Dirac operator (see [12] and [1]), in the theory of group valued moment maps [2], and recently in the Chern–Simons theory in dimension one [3].

Let $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ be a triangular decomposition of $\mathfrak{g}$, and let $\theta : \mathfrak{g} \to \mathrm{Cl}(\mathfrak{g})$ be the canonical injection of $\mathfrak{g}$ as the generating set of $\mathrm{Cl}(\mathfrak{g})$. Consider the direct sum decomposition of the Clifford algebra

$$\mathrm{Cl}(\mathfrak{g}) = \mathrm{Cl}(\mathfrak{h}) \oplus \big(\theta(\mathfrak{n}_-)\mathrm{Cl}(\mathfrak{g}) + \mathrm{Cl}(\mathfrak{g})\theta(\mathfrak{n}_+)\big), \tag{1}$$

where $\mathrm{Cl}(\mathfrak{h}) \subset \mathrm{Cl}(\mathfrak{g})$ is the subalgebra spanned by $\theta(\mathfrak{h})$, and the *odd Harish-Chandra projection* $\mathrm{hc}_{\mathrm{odd}} : \mathrm{Cl}(\mathfrak{g}) \to \mathrm{Cl}(\mathfrak{h})$. The map $\mathrm{hc}_{\mathrm{odd}}$ is important in various applications (*e.g.* in the localization formulas of [2]). In particular, the image of the canonical cubic element $\hat{\eta}$ is given by formula,

$$\mathrm{hc}_{\mathrm{odd}}(\hat{\eta}) = B_{\mathfrak{g}}^{\sharp}(\rho),$$

where $\rho \in \mathfrak{h}^*$ is the half-sum of positive roots, and $B_{\mathfrak{g}}^{\sharp} : \mathfrak{h}^* \to \mathfrak{h}$ is the isomorphism induced by the scalar product $B_{\mathfrak{g}}$. In this context, the natural question is to evaluate the images of higher degree generators of P under the map $\mathrm{hc}_{\mathrm{odd}}$. It is convenient to state this question in terms of the natural filtration, $P^{(k)}$, of P by degree (in contrast to grading, the filtration survives the passage to the Clifford algebra). One can introduce the grading with $P^{(k)} = \bigoplus_{l \leqslant k} P_l$ and $P_k = P^{(k)} \cap (P^{(k-1)})^{\perp}$. The graded components P_k are nonvanishing for $k = 2m_i + 1$, $i = 1, \ldots, r$. A non-trivial result of Bazlov [5, Proposition 4.5] and Kostant (private communications) is that $\mathrm{hc}_{\mathrm{odd}} \circ q(P) = \theta(\mathfrak{h})$, where $q : \wedge\mathfrak{g} \to \mathrm{Cl}(\mathfrak{g})$ is the $\mathrm{ad}_{\mathfrak{g}}$-equivariant isomorphism given by the anti-symmetrization map. Hence, $\mathrm{hc}_{\mathrm{odd}}(q(P^{(k)}))$ defines a filtration on the Cartan subalgebra $\mathfrak{h}$.

Let $\check{\mathfrak{g}}$ be the Lie algebra defined by the dual root system, and $\check{\mathfrak{h}} \subset \check{\mathfrak{g}}$ the corresponding Cartan subalgebra. The key observation of Kostant is that $\rho \in \mathfrak{h}^*$ viewed as an element of $\check{\mathfrak{h}}$ coincides with the Cartan element $\check{h}$ of the principal $\mathfrak{sl}_2$-triple $(\check{e}, \check{h}, \check{f}) \subset \check{\mathfrak{g}}$. Since $\check{\mathfrak{h}}^*$ and $\mathfrak{h}$ are canonically isomorphic, the co-adjoint action of $\check{e}$ induces a filtration on $\mathfrak{h}$,

$$\check{\mathcal{F}}^{(m)}\mathfrak{h} = \big\{x \in \mathfrak{h},\ \big(\mathrm{ad}^*_{\check{e}}\big)^{m+1}x = 0\big\}. \tag{2}$$

The dimension of the vector space $\check{\mathcal{F}}^{(m)}\mathfrak{h}$ jumps at the values $m = m_1, \ldots, m_r$. This follows from the Kostant's theorem [10]. In most cases, the exponents $m_1, \ldots, m_r$ are all distinct. The exception is the case of the D_n series, for even n and $n \geqslant 4$, when there are two coincident exponents (equal to $n-1$). The filtration $\check{\mathcal{F}}^{(m)}\mathfrak{h}$ is induced by a grading. In more detail, observe that the orthogonal complements $\check{\mathcal{F}}^{(m)}\mathfrak{h}^{\perp}$ form a new filtration of $\mathfrak{h}$. Define $\check{\mathcal{F}}_m\mathfrak{h} = \check{\mathcal{F}}^{(m)}\mathfrak{h} \cap \check{\mathcal{F}}^{(m-1)}\mathfrak{h}^{\perp}$. Note that $\check{\mathcal{F}}_m\mathfrak{h}$ are non-empty only for $m = m_i$ (and in most cases these are complex lines). Then, $\check{\mathcal{F}}^{(m)}\mathfrak{h} = \bigoplus_{k \leqslant m} \check{\mathcal{F}}_k\mathfrak{h}$. Since $[\check{e}, [\check{e}, \check{h}]] = 0$, we have $\rho = \check{h} \in \mathcal{F}^{(1)}\check{\mathfrak{h}}$. Kostant suggested the following conjecture (see also [5, §5.6]):

Conjecture 1.1 *(Kostant Conjecture). For any $m \in \mathbb{N}$, we have*: $\mathrm{hc}_{\mathrm{odd}}(q(P^{(2m+1)})) = \theta(\check{\mathcal{F}}^{(m)}\mathfrak{h})$.

The two filtrations of the Kostant Conjecture arise in very different contexts, and comparing them proved to be a difficult task. In [4], Bazlov settled the Kostant conjecture for $\mathfrak{g}$ of type A_n using explicit expressions for higher generators of P. The text [5] claims to prove the conjecture for all simple Lie algebras $\mathfrak{g}$, but there is a gap in the argument (to be more precise, in Lemma 4.4).

The link between the two filtrations in the Kostant Conjecture can be established in two steps. The decisive step has been recently made by Joseph in [6]. He proved that filtration $\check{\mathcal{F}}^{(m)}\mathfrak{h}$ arises in the context of generalized Harish-Chandra projections recently introduced in [8]. We refer the reader to Theorem 2.1 for the precise statement.

Our first result in this Note (Theorem 2.2) is the equivalence of the Kostant Conjecture and Joseph's theorem. Our second result is a variation of the same theme (see Theorem 3.1). While this paper was in preparation, A. Joseph informed us that he proved the following equality of filtrations [7, §1.4 and Theorem 2.4]:

$$(\mathrm{ev}_{\rho} \circ \mathrm{hc} \otimes 1)\big(\big(U^{(m)}\mathfrak{g} \otimes \mathfrak{g}\big)^{\mathfrak{g}}\big) = \check{\mathcal{F}}^{(m)}\mathfrak{h}, \tag{3}$$

where $\mathrm{hc} : U\mathfrak{g} \to S\mathfrak{h}$ is the standard Harish-Chandra projection. Together with our Theorem 3.1, it gives an alternative proof of the Kostant Conjecture. Note that the statement of the Kostant's conjecture in [7, §1.4] differs from ours. A result similar to (3) for $U\mathfrak{g}$ replaced by $S\mathfrak{g}$, and hc by the projection map from $S\mathfrak{g}$ onto $S\mathfrak{h}$, was proved by Rohr [13]. However, in the case of enveloping algebras, the proof is much more involved, it uses the technique of Zhelobenko and Bernstein–Gelfand–Gelfand operators.

2. Joseph's theorem is equivalent to the Kostant Conjecture

Recall that $U\mathfrak{g}$ admits a decomposition, $U\mathfrak{g} = S\mathfrak{h} \oplus (\mathfrak{n}_- U\mathfrak{g} + U\mathfrak{g}\mathfrak{n}_+)$. It induces the Harish-Chandra projection $\mathrm{hc} : U\mathfrak{g} \to S\mathfrak{h}$. Recently, generalized Harish-Chandra projections were introduced in [8]. Consider the space $U\mathfrak{g} \otimes \mathfrak{g}$ which carries two commuting $\mathfrak{g}$-actions

$$\rho_L(x) : a \otimes b \to xa \otimes b, \qquad \rho_R(x) : a \otimes b \to -ax \otimes b + a \otimes \mathrm{ad}(x)b. \tag{4}$$

Using these two actions, one defines a direct sum decomposition,

$$U\mathfrak{g}\otimes\mathfrak{g} = S\mathfrak{h}\otimes\mathfrak{g}\oplus\big(\rho_L(\mathfrak{n}_-)(U\mathfrak{g}\otimes\mathfrak{g})+\rho_R(\mathfrak{n}_+)(U\mathfrak{g}\otimes\mathfrak{g})\big), \tag{5}$$

see the equality (4) in [8] which refers to [9, Proposition 3.3]. This decomposition defines the generalized Harish-Chandra projection, $\mathrm{hc}_\mathfrak{g}: U\mathfrak{g}\otimes\mathfrak{g}\to S\mathfrak{h}\otimes\mathfrak{g}$. When restricted to the invariant subspace $(U\mathfrak{g}\otimes\mathfrak{g})^\mathfrak{g}$ for the diagonal action $\rho(x)=\rho_L(x)+\rho_R(x)$, it induces an injection

$$\mathrm{hc}_\mathfrak{g}:(U\mathfrak{g}\otimes\mathfrak{g})^\mathfrak{g}\hookrightarrow S\mathfrak{h}\otimes\mathfrak{h}.$$

For every element $\lambda\in\mathfrak{h}^*$, one can introduce an evaluation map, $\mathrm{ev}_\lambda: S\mathfrak{h}\to\mathbb{C}$, associating to a polynomial $p\in S\mathfrak{h}\cong\mathbb{C}[\mathfrak{h}^*]$ its value at λ, $\mathrm{ev}_\lambda(p)=p(\lambda)$. In particular, we will be interested in the evaluation at ρ, the half-sum of positive roots. Recently, Joseph [6] proved the following theorem:

Theorem 2.1 *(Joseph). For any $m\in\mathbb{N}$, we have*: $(\mathrm{ev}_\rho\otimes 1)\circ\mathrm{hc}_\mathfrak{g}((U^{(m)}\mathfrak{g}\otimes\mathfrak{g})^\mathfrak{g})=\check{\mathcal{F}}^{(m)}\mathfrak{h}$.

Here $U^{(m)}\mathfrak{g}$ is the natural filtration of $U\mathfrak{g}$ induced by assigning degree one to all elements of $\mathfrak{g}\subset U\mathfrak{g}$. In this Note, we show:

Theorem 2.2. *Joseph's Theorem is equivalent to the Kostant conjecture.*

The rest of this section is devoted to the proof of Theorem 2.2. To advance, we formulate several key propositions. Let $\tau: U\mathfrak{g}\to\mathrm{Cl}(\mathfrak{g})$ be the unique algebra homomorphism defined by the properties $[\tau(x),\theta(y)]=\theta([x,y]_\mathfrak{g})$ and $\deg(\tau(x))=2$, for $x,y\in\mathfrak{g}$. Let $\mathrm{m}_{\mathrm{Cl}}:\mathrm{Cl}(\mathfrak{g})\otimes\mathfrak{g}\to\mathrm{Cl}(\mathfrak{g}),\ a\otimes b\to a\theta(b)$ be the product map in the Clifford algebra, and

$$\mu=\mathrm{m}_{\mathrm{Cl}}\circ(\tau\otimes 1): U\mathfrak{g}\otimes\mathfrak{g}\to\mathrm{Cl}(\mathfrak{g}).$$

Since τ is of degree two, μ maps $U^{(m)}\mathfrak{g}\otimes\mathfrak{g}$ to $\mathrm{Cl}^{(2m+1)}(\mathfrak{g})$. Our first proposition is the following:

Proposition 2.1. *The subspaces $\mu(\rho_L(\mathfrak{n}_-)(U\mathfrak{g}\otimes\mathfrak{g}))$ and $\mu(\rho_R(\mathfrak{n}_+)(U\mathfrak{g}\otimes\mathfrak{g}))$ are contained in the kernel of the odd Harish-Chandra projection* $\mathrm{hc}_{\mathrm{odd}}$.

Proof. To start with, observe that $\tau(\mathfrak{n}_-)\subset\theta(\mathfrak{n}_-)\mathrm{Cl}(\mathfrak{g})$ and $\tau(\mathfrak{n}_+)\subset\mathrm{Cl}(\mathfrak{g})\theta(\mathfrak{n}_+)$. Indeed, elements of $\tau(\mathfrak{n}_-)$ (resp. $\tau(\mathfrak{n}_+)$) are of negative (resp. positive) weight under the adjoint $\mathfrak{h}$-action.

For the first subspace, we have $\mu(\rho_L(\mathfrak{n}_-)(U\mathfrak{g}\otimes\mathfrak{g}))\subset\mathrm{m}_{\mathrm{Cl}}(\tau(\mathfrak{n}_-)\tau(U\mathfrak{g})\otimes\mathfrak{g})\subset\tau(\mathfrak{n}_-)\mathrm{Cl}(\mathfrak{g})\subset\theta(\mathfrak{n}_-)\mathrm{Cl}(\mathfrak{g})$. For the second subspace, a more detailed analysis is needed: For $a\in U\mathfrak{g}$ and $x,b\in\mathfrak{g}$, one has,

$$\mu\big(\rho_R(x)(a\otimes b)\big)=\mathrm{m}_{\mathrm{Cl}}\circ(\tau\otimes 1)\big(-ax\otimes b+a\otimes[x,b]_\mathfrak{g}\big)=-\tau(a)\tau(x)\theta(b)+\tau(a)\theta\big([x,b]_\mathfrak{g}\big)=-\tau(a)\theta(b)\tau(x).$$

Hence, $\mu(\rho_R(\mathfrak{n}_+)(U\mathfrak{g}\otimes\mathfrak{g}))\subset\mathrm{Cl}(\mathfrak{g})\tau(\mathfrak{n}_+)\subset\mathrm{Cl}(\mathfrak{g})\theta(\mathfrak{n}_+)$. By definition, both $\theta(\mathfrak{n}_-)\mathrm{Cl}(\mathfrak{g})$ and $\mathrm{Cl}(\mathfrak{g})\theta(\mathfrak{n}_+)$ are contained in the kernel of $\mathrm{hc}_{\mathrm{odd}}$ and the proposition follows. □

The next fact is proved for instance in Lemma 4.2 of [5].

Proposition 2.2. *For any $p\in S\mathfrak{h}$, one has* $\mathrm{hc}_{\mathrm{odd}}(\tau(p))=\mathrm{ev}_\rho(p)$. *In particular,* $\mathrm{hc}_{\mathrm{odd}}\circ\tau$ *maps $S\mathfrak{h}$ to $\mathbb{C}$.*

Define two maps, $\mu_i:(U\mathfrak{g}\otimes\mathfrak{g})^\mathfrak{g}\to\mathrm{Cl}(\mathfrak{h})$ for $i=1,2$, as follows. The first map, μ_1, is the composition $\theta\circ(\mathrm{ev}_\rho\otimes 1)\circ\mathrm{hc}_\mathfrak{g}$. The second map, μ_2, is the composition of $\mu: U\mathfrak{g}\otimes\mathfrak{g}\to\mathrm{Cl}(\mathfrak{g})$ and of the odd Harish-Chandra projection $\mathrm{hc}_{\mathrm{odd}}:\mathrm{Cl}(\mathfrak{g})^\mathfrak{g}\to\mathrm{Cl}(\mathfrak{h})$.

Proposition 2.3. *The maps μ_1 and μ_2 are equal to each other.*

Proof. By Proposition 2.1, $\mu_2(\alpha)=\mu_2(\mathrm{hc}_\mathfrak{g}(\alpha))$ for $\alpha\in(U\mathfrak{g}\otimes\mathfrak{g})^\mathfrak{g}$ (here we view $S\mathfrak{h}\otimes\mathfrak{h}$ as a subspace of $U\mathfrak{g}\otimes\mathfrak{g}$). Furthermore, writing $\mathrm{hc}_\mathfrak{g}(\alpha)=\sum_k a_k\otimes x_k$ with $a_k\in S\mathfrak{h}, x_k\in\mathfrak{h}$, one has

$$\mu_2(\alpha)=\sum_k\mu_2(a_k\otimes x_k)=\sum_k\mathrm{hc}_{\mathrm{odd}}\circ\mathrm{m}_{\mathrm{Cl}}\circ(\tau\otimes 1)(a_k\otimes x_k)=\sum_k\mathrm{hc}_{\mathrm{odd}}\big(\tau(a_k)\theta(x_k)\big).$$

The map $\mathrm{hc}_{\mathrm{odd}}$ is an algebra homomorphism on $\mathrm{ad}_\mathfrak{h}$-invariant elements [5, Lemma 2.4]. Hence, by Proposition 2.2, we get:

$$\mu_2(\alpha)=\sum_k\mathrm{hc}_{\mathrm{odd}}\big(\tau(a_k)\big)\mathrm{hc}_{\mathrm{odd}}\big(\theta(x_k)\big)=\sum_k\mathrm{ev}_\rho(a_k)\theta(x_k)=(\mathrm{ev}_\rho\otimes\theta)\big(\mathrm{hc}_\mathfrak{g}(\alpha)\big)=\mu_1(\alpha).\qquad\square$$

Proposition 2.4. *For any $m \in \mathbb{N}$, we have* $\mu((U^{(m)}\mathfrak{g} \otimes \mathfrak{g})^{\mathfrak{g}}) = q(P^{(2m+1)})$.

Proof. By Theorem D of [11], $\mathrm{Cl}(\mathfrak{g})$ is a free module over the subalgebra $\mathrm{Cl}(\mathfrak{g})^{\mathfrak{g}}$. Namely, $\mathrm{Cl}(\mathfrak{g}) = \tau(U\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{g})^{\mathfrak{g}}$. Corollary 80 and Theorem 89 of [11] give an explicit expression for the elements in $q(P^{(2m_i+1)})$, $i = 1, \dots, r$: For any $p \in q(P^{(2m_i+1)})$, we have $p = \sum_k \tau(a_k)\theta(b_k)$, where $\sum_k a_k \otimes b_k$ is in $(U^{(m_i)}\mathfrak{g} \otimes \mathfrak{g})^{\mathfrak{g}}$.

As a consequence, we obtain inclusions, $q(P) \subset \mu((U\mathfrak{g} \otimes \mathfrak{g})^{\mathfrak{g}})$ and $q(P^{(2m_i+1)}) \subset \mu((U^{(m_i)}\mathfrak{g} \otimes \mathfrak{g})^{\mathfrak{g}})$. The vector space $q(P)$ has dimension r and $(U\mathfrak{g} \otimes \mathfrak{g})^{\mathfrak{g}}$ is a free module of rank r over $Z(U\mathfrak{g})$, the center of $U\mathfrak{g}$. Then $\tau(Z(U\mathfrak{g})) = \mathbb{C}$ (see *e.g.* [11, Corollary 36]) implies $\mu((U\mathfrak{g} \otimes \mathfrak{g})^{\mathfrak{g}}) = q(P)$. Moreover, since $\deg(\tau(x)) = 2$ and $\deg(\theta(x)) = 1$ for $x \in \mathfrak{g}$, we get $\mu((U^{(m)}\mathfrak{g} \otimes \mathfrak{g})^{\mathfrak{g}}) \subset q(P^{(2m+1)})$. Hence, $\mu((U^{(m)}\mathfrak{g} \otimes \mathfrak{g})^{\mathfrak{g}}) = q(P^{(2m+1)})$. □

We are now in the position to prove Theorem 2.2: By Propositions 2.3 and 2.4, we have

$$\mu_1\big((U^{(m)}\mathfrak{g} \otimes \mathfrak{g})^{\mathfrak{g}}\big) = \mu_2\big((U^{(m)}\mathfrak{g} \otimes \mathfrak{g})^{\mathfrak{g}}\big) = \mathrm{hc}_{\mathrm{odd}}\big(q\big(P^{(2m+1)}\big)\big).$$

But $\mu_1((U^{(m)}\mathfrak{g} \otimes \mathfrak{g})^{\mathfrak{g}}) = \theta \circ (\mathrm{ev}_\rho \otimes 1) \circ \mathrm{hc}_{\mathfrak{g}}((U^{(m)}\mathfrak{g} \otimes \mathfrak{g})^{\mathfrak{g}})$. This shows the required equivalence.

Remark 1. In fact, the original formulation of Kostant's conjecture states that $\mathrm{hc}_{\mathrm{odd}}(q(P_{2k+1})) = \theta(\check{\mathcal{F}}_k\mathfrak{h})$. However, as we explain below, it is sufficient to prove the statement about filtrations.

Let $v_1, v_2 \in P$ and $x_1, x_2 \in \mathfrak{h}$ such that $\mathrm{hc}_{\mathrm{odd}}(q(v_1)) = \theta(x_1)$ and $\mathrm{hc}_{\mathrm{odd}}(q(v_2)) = \theta(x_2)$ (see Bazlov–Kostant result). Since the restriction of $\mathrm{hc}_{\mathrm{odd}} \circ q$ to $\mathrm{ad}_{\mathfrak{g}}$-invariant elements is an isomorphism, we get $2B(x_1, x_2) = [\theta(x_1), \theta(x_2)] = [\mathrm{hc}_{\mathrm{odd}}(q(v_1)), \mathrm{hc}_{\mathrm{odd}}(q(v_2))] = \mathrm{hc}_{\mathrm{odd}}([q(v_1), q(v_2)]) = 2B_P(v_1, v_2)$. Hence, $\mathrm{hc}_{\mathrm{odd}}(q(P^{(2k+1)})) = \theta(\check{\mathcal{F}}^{(k)}\mathfrak{h})$ implies $\mathrm{hc}_{\mathrm{odd}}(q(P^{(2k+1)})^{\perp}) = \theta(\check{\mathcal{F}}^{(k)}\mathfrak{h}^{\perp})$. Note that $P^{(2k)} = P^{(2k-1)}$. Indeed, remember that the graded components P_k are non-vanishing for $k = 2m_i + 1$, $i = 1, \dots, r$. So, $\mathrm{hc}_{\mathrm{odd}}(q(P_{2k+1})) = \mathrm{hc}_{\mathrm{odd}}(q(P^{(2k+1)} \cap (P^{(2k)})^{\perp})) = \mathrm{hc}_{\mathrm{odd}}(q(P^{(2k+1)} \cap (P^{(2k-1)})^{\perp})) = \theta(\check{\mathcal{F}}^{(k)}\mathfrak{h} \cap \check{\mathcal{F}}^{(k-1)}\mathfrak{h}^{\perp}) = \theta(\check{\mathcal{F}}_k\mathfrak{h})$.

3. More on Harish-Chandra projections

The standard Harish-Chandra projection $\mathrm{hc} : U\mathfrak{g} \to S\mathfrak{h}$ is equivariant under the adjoint action of $\mathfrak{h}$. Hence, the image of $(U\mathfrak{g} \otimes \mathfrak{g})^{\mathfrak{g}}$ under $\mathrm{hc} \otimes 1$ lands in $S\mathfrak{h} \otimes \mathfrak{h}$. Our second result is the following equality of filtrations on $\mathfrak{h}$:

Theorem 3.1. *For any $m \in \mathbb{N}$, we have*: $(\theta \circ \mathrm{ev}_\rho \circ \mathrm{hc} \otimes 1)((U^{(m)}\mathfrak{g} \otimes \mathfrak{g})^{\mathfrak{g}}) = \mathrm{hc}_{\mathrm{odd}}(q(P^{(2m+1)}))$.

We first state an auxiliary statement preparing the proof of Theorem 3.1:

Proposition 3.1. *The image of the map* $\mathrm{hc} - 1 : U\mathfrak{g} \to U\mathfrak{g}$ *is contained in the kernel of* $\mathrm{hc}_{\mathrm{odd}} \circ \tau : U\mathfrak{g} \to \mathrm{Cl}(\mathfrak{h})$.

Proof. The image of the map $\mathrm{hc} - 1$ is the space $\mathfrak{n}_- U\mathfrak{g} + U\mathfrak{g}\mathfrak{n}_+ \subset U\mathfrak{g}$. For the first term, we have $\tau(\mathfrak{n}_- U\mathfrak{g}) = \tau(\mathfrak{n}_-)\tau(U\mathfrak{g}) \subset \theta(\mathfrak{n}_-)\mathrm{Cl}(\mathfrak{g}) \subset \ker(\mathrm{hc}_{\mathrm{odd}})$. Indeed, as already observed, we have $\tau(\mathfrak{n}_-) \subset \theta(\mathfrak{n}_-)\mathrm{Cl}(\mathfrak{g})$ (see the proof of Proposition 2.1). Similarly, for the second term, $\tau(U\mathfrak{g}\mathfrak{n}_+) = \tau(U\mathfrak{g})\tau(\mathfrak{n}_+) \subset \mathrm{Cl}(\mathfrak{g})\theta(\mathfrak{n}_+) \subset \ker(\mathrm{hc}_{\mathrm{odd}})$. □

Now we are ready to present the proof of Theorem 3.1.

Proof of Theorem 3.1. By Proposition 3.1, the following maps from $U\mathfrak{g}$ to $\mathrm{Cl}(\mathfrak{h})$ coincide:

$$\mathrm{hc}_{\mathrm{odd}} \circ \tau = \mathrm{hc}_{\mathrm{odd}} \circ \tau \circ \mathrm{hc}.$$

In turn, by Proposition 2.2, $\mathrm{hc}_{\mathrm{odd}} \circ \tau = \mathrm{ev}_\rho$ on the image of the Harish-Chandra map $\mathrm{hc} : U\mathfrak{g} \to S\mathfrak{h}$. Hence, we obtain an equality of maps:

$$\mathrm{hc}_{\mathrm{odd}} \circ \tau = \mathrm{ev}_\rho \circ \mathrm{hc}.$$

This observation shows that the image of both maps is in fact equal to $\mathbb{C} \subset \mathrm{Cl}(\mathfrak{h})$.

For $x \in \mathfrak{g}$ define a linear form, $\beta_x : \mathfrak{g} \to \mathbb{C}$, given by $\beta_x(y) = B_{\mathfrak{g}}(x, y)$. Then, for all $x \in \mathfrak{h}$ and $\alpha \in (U\mathfrak{g} \otimes \mathfrak{g})^{\mathfrak{g}}$, we have

$$(\mathrm{hc}_{\mathrm{odd}} \circ \tau \otimes \beta_x)(\alpha) = (\mathrm{ev}_\rho \circ \mathrm{hc} \otimes \beta_x)(\alpha).$$

For $x \in \mathfrak{h}$, $v \in q(P)$, consider the expression

$$\big[\theta(x), \mathrm{hc}_{\mathrm{odd}}(v)\big] = \mathrm{hc}_{\mathrm{odd}}\big(\big[\theta(x), v\big]\big).$$

By the result of Bazlov–Kostant, $\mathrm{hc}_{\mathrm{odd}}(v) \in \theta(\mathfrak{h})$ and the above expression is equal to $B_{\mathfrak{g}}(x, y)$, where $\mathrm{hc}_{\mathrm{odd}}(v) = \theta(y)$. Assume that it vanishes for all $v \in q(P^{(2m+1)})$. By definition, it is equivalent to $x \in \mathrm{hc}_{\mathrm{odd}}(q(P^{(2m+1)}))^{\perp}$. By Theorem F (equality (n)) and Corollary 80 (equality (302)) in [11], for any $v \in q(P^{(2m+1)})$ there is an element $\alpha \in (U^{(m)}\mathfrak{g} \otimes \mathfrak{g})^{\mathfrak{g}}$ such that $[\theta(x), v] = (\tau \otimes \beta_x)(\alpha)$. Furthermore, the space $(U^{(m)}\mathfrak{g} \otimes \mathfrak{g})^{\mathfrak{g}}$ surjects on $[\theta(x), q(P^{(2m+1)})]$ under the map $\tau \otimes \beta_x : U\mathfrak{g} \otimes \mathfrak{g} \to \mathrm{Cl}(\mathfrak{g})$ for $m \in \mathbb{N}$. Then,

$$\mathrm{hc}_{\mathrm{odd}}\big(\big[\theta(x), v\big]\big) = (\mathrm{hc}_{\mathrm{odd}} \circ \tau \otimes \beta_x)(\alpha) = (\mathrm{ev}_\rho \circ \mathrm{hc} \otimes \beta_x)(\alpha) = B_{\mathfrak{g}}\big(x, (\mathrm{ev}_\rho \circ \mathrm{hc} \otimes 1)(\alpha)\big).$$

Vanishing of this expression for any $\alpha \in (U^{(m)}\mathfrak{g} \otimes \mathfrak{g})^{\mathfrak{g}}$ is equivalent to $x \in ((\mathrm{ev}_\rho \circ \mathrm{hc} \otimes 1)(U^{(m)}\mathfrak{g} \otimes \mathfrak{g})^{\mathfrak{g}})^{\perp}$.

We have shown that the orthogonal complements of the two filtrations coincide. Hence, so do the filtrations in question. □

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