C. R. Acad. Sci. Paris, Ser. I 349 (2011) 1255–1259

Contents lists available at SciVerse ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Partial Differential Equations/Numerical Analysis

A robust two-level domain decomposition preconditioner for systems of PDEs

Une méthode de décomposition de domaine à deux niveaux robuste pour les systèmes d'EDPs

Nicole Spillane [a,b], Victorita Dolean [c], Patrice Hauret [b], Frédéric Nataf [a], Clemens Pechstein [d], Robert Scheichl [e]

[a] *Laboratoire J.L. Lions, UMR 7598, UPMC université Paris 6, 75252 Paris cedex 05, France*
[b] *Michelin Technology Center, place des Carmes-Déchaux, 63000 Clermont-Ferrand, France*
[c] *Laboratoire J.-A. Dieudonné, UMR 6621, université de Nice-Sophia Antipolis, 06108 Nice cedex 02, France*
[d] *Institute of Computational Mathematics, Johannes Kepler Universität, Altenberger Str. 69, A-4040 Linz, Austria*
[e] *Department of Mathematical Sciences, University of Bath, Bath BA2 7AY, UK*

ARTICLE INFO

Article history:
Received 15 July 2011
Accepted 20 October 2011
Available online 9 November 2011

Presented by Olivier Pironneau

ABSTRACT

Coarse spaces are instrumental in obtaining scalability for domain decomposition methods. However, it is known that most popular choices of coarse spaces perform rather weakly in presence of heterogeneities in the coefficients in the partial differential equations, especially for systems. Here, we introduce in a variational setting a new coarse space that is robust even when there are such heterogeneities. We achieve this by solving local generalized eigenvalue problems which isolate the terms responsible for slow convergence. We give a general theoretical result and then some numerical examples on a heterogeneous elasticity problem.

RÉSUMÉ

Un moyen efficace pour obtenir des méthodes de décomposition de domaine extensibles (« scalable » en anglais) est l'utilisation d'une grille grossière. Cependant, lorsque les coefficients des équations présentent de grandes hétérogénéités, les méthodes usuelles tombent en défaut, surtout dans le cas des systèmes. Nous introduisons ici, au niveau variationnel, une grille grossière robuste même en présence de telles discontinuités. Pour cela, nous résolvons des problèmes aux valeurs propres généralisés locaux qui isolent les composantes de la solution nuisant à la convergence. Nous présentons un résultat théorique général puis quelques résultats numériques pour un problème d'élasticité à coefficients discontinus.

E-mail addresses: spillane@ann.jussieu.fr (N. Spillane), dolean@unice.fr (V. Dolean), patrice.hauret@fr.michelin.com (P. Hauret), nataf@ann.jussieu.fr (F. Nataf), clemens.pechstein@numa.uni-linz.ac.at (C. Pechstein), r.scheichl@bath.ac.uk (R. Scheichl).

1631-073X/$ – see front matter
doi:10.1016/j.crma.2011.10.021

Version française abrégée

Ce travail s'intéresse à la résolution d'un système linéaire (3) issu de la discrétisation par éléments finis (2) d'un problème aux limites elliptique donné sous forme variationnelle (1) où les coefficients peuvent être discontinus. Afin d'obtenir des méthodes de décomposition de domaine extensibles (robustes vis à vis du nombre de sous-domaines), nous considérons des méthodes à deux niveaux [7]. Ces méthodes sont étroitement liées aux méthodes multigrilles et de déflation. Elles sont définies par deux ingrédients : une grille grossière V_H composée de m vecteurs avec m petit devant la taille du problème initial et une formulation algébrique de la correction qui consistera ici en la méthode de Schwarz à deux niveaux (4) que l'on utilise comme préconditionneur pour un solveur de type gradient conjugué. Ce choix nous permet d'appliquer des résultats connus qui ramènent l'étude de la convergence de l'algorithme à celle du conditionnement de l'opérateur préconditionné. La contribution clé de ce travail consiste en une définition systématique de la grille grossière fondée sur les plus basses fréquences de problèmes spectraux généralisés locaux (Définitions 2.3 et 2.4), voir aussi [2] et les références citées. Par rapport à [2], notre grille grossière présente l'avantage d'être construite à partir de la matrice avant assemblage sans calculs supplémentaires de contributions élémentaires. De plus, l'estimation ne dépend pas d'une hypothèse de stabilité d'un interpolant éléments finis. Enfin, on fournit un critère de sélection optimal du nombre de vecteurs propres. Cette approche étend au cas de systèmes et dans un cadre variationnel la méthode analysée et validée par [1] et [5] dans le cas scalaire. En particulier, elle permet l'obtention d'une convergence efficace de l'algorithme de Schwarz indépendamment des paramètres du problème, y compris des hétérogénéités. Le résultat théorique (Théorème 2.5) démontre en effet l'existence d'une décomposition stable, au sens de la Définition 2.1, de toute fonction discrète sur la grille grossière et les sous-espaces locaux. La section 3 constitue une première illustration de la méthode introduite sur un cas élémentaire mais fortement hétérogène d'élasticité bidimensionnelle.

1. Introduction

This work is the extension to systems of PDEs of the coarse space introduced and validated in [1] and [5] for the Darcy equation (scalar PDE). In that specific case, in order to define the coarse space, generalized eigenvalue problems defined on the interfaces between subdomains were solved. The theoretical proof used weighted Poincaré inequalities [6] to write inequalities between quantities defined on the interfaces and ones defined in the whole overlapped regions. In order to be more general and avoid the need for weighted Poincaré type inequalities, this time we will define the generalized eigenproblems directly in the overlapped regions. We also bypass any stability assumption for the finite element interpolation. First we define the framework in which we will introduce the new coarse space.

Given a Hilbert space V_0, a symmetric and coercive bilinear form $a : V_0 \times V_0 \to \mathbb{R}$ and an element f in the dual space V_0', we consider the abstract variational problem: Find $u \in V_0$ such that

$$a(u, v) = \langle f, v\rangle \quad \text{for all } v \in V_0. \tag{1}$$

This variational problem is associated with an elliptic boundary value problem on a given polygonal (polyhedral) domain $\Omega \subset \mathbb{R}^d$ ($d = 2$ or 3) with suitable boundary conditions (possibly homogeneous Dirichlet on part of the boundary), posed in a suitable space of functions V_0 on Ω.

We consider a discretization of the variational problem (1) with finite elements (FE). To define the FE spaces and the approximate solution, we assume that we have a mesh $\mathcal{T}_h$ of Ω: $\overline{\Omega} = \bigcup_{\tau\in\mathcal{T}_h} \tau$.

The corresponding space of finite element functions w.r.t. $\mathcal{T}_h$ is then denoted by V_h, and the subspace of functions from V_h that fulfill the homogeneous Dirichlet boundary conditions by $V_{h,0} = V_h \cap V_0$. In the case where a is a bilinear form derived from a system of PDEs, V_h is a space of vector functions. In our analysis, we will also need restrictions of FE functions into subdomains $D \subset \Omega$ that are resolved by $\mathcal{T}_h$. The space of restrictions of the functions in V_h to D is denoted by $V_h(D)$. Similarly, the space of restrictions of functions from $V_{h,0}$ which vanish on $\Omega \setminus D$ is denoted by $V_{h,0}(D)$.

The finite element discretization of (1) is: Find $u_h \in V_{h,0}$ such that

$$a(u_h, v_h) = \langle f, v_h\rangle \quad \text{for all } v_h \in V_{h,0}. \tag{2}$$

Let $\{\phi_i\}_{i=1}^n$ be a basis for $V_{h,0}$ with $n := \dim(V_{h,0})$, then (2) can be compactly written as

$$A\mathbf{u} = \mathbf{f}, \tag{3}$$

where $A_{i,j} := a(\phi_j, \phi_i)$, $f_i = \langle f, \phi_i\rangle$, $i, j = 1, \ldots, n$ and $\mathbf{u}$ is the vector of coefficients corresponding to the unknown FE function u_h in (2).

In order to automatically construct robust two-level Schwarz type methods for (2) we first partition our domain Ω into a set of non-overlapping subdomains $\{\Omega_j'\}_{j=1}^N$ using for example a graph partitioner such as METIS or SCOTCH. Each subdomain Ω_j' is then extended to a domain Ω_j by adding one or several layers of fine grid elements, thus creating an overlapping decomposition $\{\Omega_j\}_{j=1}^N$ of Ω. Let us suppose that the domains $\{\Omega_j\}_{j=1}^N$ are large enough so that for every k ($1 \leqslant k \leqslant n$) there is a subdomain j ($1 \leqslant j \leqslant N$) such that $\operatorname{supp}\phi_k \subset \bar{\Omega}_j$. Having defined overlapping subdomains we now introduce restriction operators R_j. For $1 \leqslant j \leqslant N$, R_j is a mapping between the dual of $V_{h,0}$ and the dual of $V_{h,0}(\Omega_j)$

– it restricts residuals. The adjoint R_j^T of R_j extends functions from $V_{h,0}(\Omega_j)$ to $V_{h,0}$ by zero. The corresponding matrix (also denoted by R_j^T for simplicity) takes a local vector and makes a global vector by inserting zeros. Finally, let us assume that we have a subspace $V_H \subset V_{h,0}$ (the so called coarse space) and an extension operator R_H^T from V_H to $V_{h,0}$, then the two-level preconditioner that we use is defined as

$$M_{AS,2}^{-1} = R_H^T A_H^{-1} R_H + \sum_{j=1}^{N} R_j^T A_j^{-1} R_j, \quad A_H := R_H A R_H^T \quad \text{and} \quad A_j := R_j A R_j^T. \tag{4}$$

The paper is organized as follows, in Section 2 we finish introducing the method by specifying our choice for the coarse space and give the theoretical bound for the condition number of the preconditioned operator; in Section 3 we give numerical results for the case where the two-dimensional elasticity equations with heterogeneous coefficients are considered.

2. An automatic coarse space construction

Let us first define some notation. Since the bilinear form a originates from a second order elliptic partial differential equation in Ω, it has the following property: there exists a family of bilinear forms $\{a_D\}_D$ indexed by all subsets $D \subset \Omega$ such that for any $D \subset \Omega$, $D' \subset \Omega$ verifying $D \cap D' = \emptyset$, the following holds

$$a_{D\cup D'}(u_{|D\cup D'}, v_{|D\cup D'}) = a_D(u_{|D}, v_{|D}) + a_{D'}(u_{|D'}, v_{|D'}) \quad \forall u, v \in V_{h,0}.$$

In addition for any $D \subset \Omega$ and any $v \in V_h(D)$, let $\|v\|_{a,D}^2 = a_D(v, v)$. This is usually referred to as the energy seminorm on $V_h(D)$, which defines a full norm on $V_{h,0}(D)$.

Since both the preconditioner and the matrix A of the problem that we want to solve are symmetric and positive definite it is well known that the rate of convergence of the preconditioned conjugate gradient method depends only on the condition number of $M_{AS,2}^{-1}A$. In turn, a bound for this condition number relies on the existence of a stable decomposition of the solution onto the coarse space and the local subspaces (see [7] or [4] and the references therein). Next, we give the definition of such a stable decomposition.

Definition 2.1. Given a coarse space $V_H \subset V_{h,0}$, the local subspaces $\{V_{h,0}(\Omega_j)\}_{1\leqslant j\leqslant N}$ and a constant C_0, a C_0-stable decomposition of $u \in V_{h,0}$ is a family of functions $\{z_j\}_{0\leqslant j\leqslant N}$ such that $z_0 \in V_H$, $z_j \in V_{h,0}(\Omega_j)$ for all $1 \leqslant j \leqslant N$ and $u = \sum_{j=0}^N z_j$, which verifies

$$\|z_0\|_a^2 + \sum_{j=1}^{N} \|z_j\|_{a,\Omega_j}^2 \leqslant C_0^2 \|u\|_a^2.$$

What we aim to achieve is to define the coarse space in such a way that there is a constant C_0 for which any $u \in V_{h,0}$ admits a C_0-stable decomposition and C_0 depends only on the ratio $\frac{\operatorname{diam}(\Omega_j)}{\delta_j}$ between the sizes of the subdomains ($\operatorname{diam}(\Omega_j)$) and the width of the overlap (δ_j). In particular C_0 will remain independent of the decomposition into subdomains and the heterogeneities defined by the bilinear form a. First, we introduce a partition of unity operator.

Definition 2.2.

(i) For each k, $1 \leqslant k \leqslant n$ define $\mathcal{N}_k = \{j; 1 \leqslant j \leqslant N \text{ and } \operatorname{supp}\phi_k \subset \bar{\Omega}_j\}$.
(ii) For each j, $1 \leqslant j \leqslant N$ define $\mathcal{M}_j = \{k; 1 \leqslant k \leqslant n \text{ and } \operatorname{supp}\phi_k \subset \bar{\Omega}_j\}$.
(iii) Then, for each j, $1 \leqslant j \leqslant N$, let $\Xi_j : V_h \to V_h$ be defined by: $\Xi_j(u) = \sum_{k\in\mathcal{M}_j} \frac{1}{\#(\mathcal{N}_k)} u_k \phi_k(\boldsymbol{x})$, where ϕ_k are the vector shape functions and $u(\boldsymbol{x}) = \sum_{k=1}^n u_k \phi_k(\boldsymbol{x})$.

Notice that $\sum_{j=1}^N \Xi_j(u) = u$ and $\operatorname{supp}(\Xi_j u) \subset \bar{\Omega}_j$. Now, let $\Omega_j^\circ = \{x \in \Omega_j; \exists j' \neq j \text{ such that } x \in \Omega_{j'}\}$ denote the boundary layer of Ω_j that is overlapped by neighboring domains, and let δ_j denote the width of Ω_j° at the narrowest place. It can be shown that $(\Xi_j(u))_{|\Omega_j\setminus\Omega_j^\circ} = u_{|\Omega_j\setminus\Omega_j^\circ}$.

Then the following eigenproblems enable us to ensure the stability of the decomposition by isolating the terms which slow down convergence:

Definition 2.3. For any given j, $1 \leqslant j \leqslant N$, let $(\lambda_k^j, p_k^j)_{1\leqslant k\leqslant \dim(V_h(\Omega_j))}$ be defined as the solutions of: Find the eigenpairs $(\lambda_k^j, p_k^j) \in (\mathbb{R}^+ \times V_h(\Omega_j))$ of the generalized eigenproblem

$$\lambda_k^j a_{\Omega_j^\circ}\big(\Xi_j\big(p_k^j\big), \Xi_j(v)\big) = a_{\Omega_j}\big(p_k^j, v\big) \quad \forall v \in V_h(\Omega_j), \tag{5}$$

normalize them and order them in increasing eigenvalue order.

Fig. 1. Decomposition into $N = 8$ subdomains: regular (left), with METIS (middle) and coefficient distribution (right).

Fig. 1. Décomposition en $N = 8$ sous-domaines : régulière (gauche), avec METIS (milieu) et choix des coefficients (droite).

These eigenproblems have a clear relationship with but are different from those introduced in [2]. Here we no longer need to make any assumption on a stability result for the finite element interpolant, this simplifies matters greatly: all assumptions hold also in a discrete setting so our theory actually covers the implemented method. Moreover, implementing (5) does not require any additional elementary matrix computations. In addition, we provide here a practical criterion to optimally select the number of vectors to be included in the coarse space.

Definition 2.4. For a given positive real number K_j, define $m_j(K_j) = \min\{k \mid \lambda_k^j \geqslant \frac{1}{K_j}\} - 1$.

After choosing K_j, we prove the following result for the new preconditioner.

Theorem 2.5. *Assume that each point in Ω belongs to at most k_0 subdomains and the coarse space is constructed as follows: for each $1 \leqslant j \leqslant N$, solve the generalized eigenproblem given in Definition* 2.3*, then select the m_j first eigenvectors according to the strategy given in Definition* 2.4 *for $K_j = \frac{\operatorname{diam}(\Omega_j)}{\delta_j}$, and set V_H to*

$$V_H = \underset{1 \leqslant j \leqslant N, 1 \leqslant k \leqslant m_j}{span} \left\{ \Xi_j\left(p_k^j\right)\right\}. \tag{6}$$

Then for any $u \in V_{h,0}$ there exists a C_0-stable decomposition with

$$C_0^2 = \left[2 + 4k_0(2k_0 + 1)\left(1 + \max_{1 \leqslant j \leqslant N} \frac{\operatorname{diam}(\Omega_j)}{\delta_j}\right)\right]. \tag{7}$$

We insist on the fact that, thanks to the choice of K_j, C_0 depends only on k_0 and on the proportion of overlap in each subdomain $\frac{\operatorname{diam}(\Omega_j)}{\delta_j}$. From this, a bound on the condition number of the preconditioned operator and thus on the convergence rate of the preconditioned conjugate gradient method can be derived immediately using standard domain decomposition theorems (again, see [7] or [4]). These bounds will depend on the same quantities (k_0 and $\frac{\operatorname{diam}(\Omega_j)}{\delta_j}$) leading to a method that should be scalable and robust with regard to high heterogeneities in the coefficients. The particular choice $K_j = \frac{\operatorname{diam}(\Omega_j)}{\delta_j}$ is made to match the well known estimate in the constant coefficient case.

3. Numerical results

The purpose of this section is to illustrate the behavior of our new preconditioner on the two-dimensional linear elasticity equations with heterogeneities. We have used FreeFem++ [3] to define the test cases and build all the finite element data and Matlab for the actual solver. The equations are the following:

$$-\operatorname{div}\big(\sigma(\mathbf{u})\big) = \mathbf{f}, \quad \text{where } \mathbf{u} = (u_1, u_2)^T,$$

and the stress tensor $\sigma(\mathbf{u})$, the Lamé coefficients λ and μ and the right-hand side are

$$\begin{cases} \sigma_{ij}(\mathbf{u}) = 2\mu\varepsilon_{ij}(\mathbf{u}) + \lambda\delta_{ij}\operatorname{div}(\mathbf{u}), \quad \varepsilon_{ij}(\mathbf{u}) = \dfrac{1}{2}\left(\dfrac{\partial u_i}{\partial x_j} + \dfrac{\partial u_j}{\partial x_i}\right), \qquad \mathbf{f} = (0, g)^T = (0, 10)^T, \\ \mu = \dfrac{E}{2(1+\nu)}, \qquad \lambda = \dfrac{E\nu}{(1+\nu)(1-2\nu)}. \end{cases} \tag{8}$$

Here E and ν denote respectively Young's modulus and Poisson's ratio and are discontinuous. The test case is as follows (see Fig. 1): we take a bar of height 1 and length N made up of $21 \times (N \times 20 + 1)$ nodes and divided into N subdomains which we extend by two nodes to form the overlapping partition. The coefficient distribution consists of four horizontal layers. We impose Dirichlet conditions $\mathbf{u} = 0$ on the left-hand side boundary denoted by $\partial\Omega_D$, and Neumann conditions $\sigma(\mathbf{u}) \cdot \mathbf{n} = 0$ on the remaining boundaries. The spaces are: $V_0 = \{v \in H^1(\Omega)^2;\, v_{|\partial\Omega_D} = 0\}$, $V_h = (\mathbb{P}_1(\Omega))^2$ (piecewise linear with respect to τ_h). Throughout this section we compare three methods. The first one is the one-level additive Schwarz method (referred to as AS), defined by the preconditioner $M_{AS,1}^{-1} = \sum_{j=1}^N R_j^T A_j^{-1} R_j$. The second one is the standard two-level method (referred to as $\mathcal{RBM}$), given by (4) with the coarse space which consists of all rigid body modes (so three in 2D) simply weighted by partition of unity functions. The third one is the new two-level method (referred to as NEW). As a stopping criterion we use $\frac{\|u-\bar{u}\|_\infty}{\|\bar{u}\|_\infty} < 10^{-7}$ where $\bar{u}$ is the solution of (2) obtained *via* a direct solver on the global problem.

Table 1
Iteration count vs. number of subdomains.

Tableau 1
Nombre d'itérations en fonction du nombre de sous-domaines.

	Regular			METIS		
	AS	$\mathcal{RBM}$	NEW (V_H)	AS	$\mathcal{RBM}$	NEW (V_H)
4 sub	51	55	28 (22/1008)	56	59	24 (29/1040)
8 sub	108	115	35 (46/2352)	121	124	31 (66/2410)
16 sub	282	207	53 (94/5040)	260	239	36 (131/5146)
32 sub	747	442	66 (190/10 416)	>1000	537	41 (268/10 706)

Table 2
Iteration count vs. jump in the coefficients.

Tableau 2
Nombre d'itérations en fonction du saut dans les coefficients.

(E_2, ν_2)	AS	$\mathcal{RBM}$	NEW (V_H)
$(2 \times 10^7, 0.49)$	152	97	36 (60/2352)
$(2 \times 10^8, 0.45)$	97	86	35 (45/2352)
$(2 \times 10^9, 0.4)$	79	68	33 (45/2352)
$(2 \times 10^{10}, 0.35)$	87	42	30 (30/2352)
$(2 \times 10^{11}, 0.3)$	50	33	31 (23/2352)

Every time we give an iteration count for the NEW method we add in brackets the size of the coarse space compared to the total number of d.o.f.s in the overlaps. Table 1 gives the number of iterations that are needed to reach convergence for different decompositions and a coefficient distribution which mimics layers of steel and rubber (without dealing with incompressibility): $(E_1, \nu_1) = (2 \times 10^{11}, 0.3)$ and $(E_2, \nu_2) = (2 \times 10^7, 0.45)$. The NEW method is a lot more efficient and it is also more robust with regard to the number of subdomains. Table 2 gives the number of iterations that are needed to reach convergence for different jumps in the coefficients which are indexed by different values of (E_2, ν_2). We use a regular decomposition into $N = 8$ subdomains and $(E_1, \nu_1) = (2 \times 10^{11}, 0.3)$. When $(E_1, \nu_1) = (E_2, \nu_2)$ (last line) we are looking at a homogeneous steel domain. As expected the method selects the three rigid body modes and only the three rigid body modes in all floating subdomains (plus two in the remaining subdomain).

4. Conclusion

We have constructed a coarse space for a two-level overlapping Schwarz method that is robust with regard to jumps in the coefficients in the equation. This coarse space is computed locally and automatically. It is suitable for parallel implementation. We have a theoretical proof for the bound on the condition number and the numerical tests for two-dimensional elasticity are in agreement with theoretical conclusions. Since the method is constructed at the variational level, numerical experiments could be conducted for any other second order elliptic system of equations.

References

[1] V. Dolean, F. Nataf, R. Scheichl, N. Spillane, Analysis of a two-level Schwarz method with coarse spaces based on local Dirichlet-to-Neumann maps, 2011, submitted for publication, http://hal.archives-ouvertes.fr/hal-00586246/fr/.

[2] Y. Efendiev, J. Galvis, R. Lazarov, J. Willems, Robust domain decomposition preconditioners for abstract symmetric positive definite bilinear forms, RICAM report, 2011.

[3] F. Hecht, FreeFem++, Laboratoire J.L. Lions, CNRS UMR 7598, http://www.freefem.org/ff++/.

[4] P. Le Tallec, Domain decomposition methods in computational mechanics, Comput. Mech. Adv. 1 (2) (1994) 121–220.

[5] F. Nataf, H. Xiang, V. Dolean, N. Spillane, A coarse space construction based on local Dirichlet to Neumann maps, SISC, 2011, http://hal.archives-ouvertes.fr/hal-00491919/fr/.

[6] C. Pechstein, R. Scheichl, Weighted Poincaré inequalities, Tech. Report NuMa-Report 2010-10, Institute of Computational Mathematics, Johannes Kepler University, Linz, December 2010, submitted for publication.

[7] A. Toselli, O. Widlund, Domain Decomposition Methods: Algorithms and Theory, Springer, 2005.