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Complex Analysis

An avoidance criterion for normal functions ☆

Un critère d'évitement pour des familles normales

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ABSTRACT

Let f be a meromorphic function in the unit disc Δ, φ_1, φ_2 and φ_3 be three functions meromorphic in Δ and continuous on closure of Δ such that $\varphi_i(z) \neq \varphi_j(z)$ $(1 \leqslant i < j \leqslant 3)$ on the unit circle $|z| = 1$. If $f(z) \neq \varphi_i(z)$ $(i = 1, 2, 3)$ in Δ, then f is normal.

RÉSUMÉ

Soit f une fonction méromorphe dans le disque unité Δ, soient φ_1, φ_2 et φ_3 trois fonctions méromorphes dans Δ et continues sur l'adhérence de Δ et dont les restrictions au cercle unité sont deux à deux distinctes. Alors, si la fonction f est distincte des $\varphi_i(z)$ $(i = 1, 2, 3)$, elle est normale.

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1. Introduction

Let D be a domain in the complex plane. The family $\mathcal{F}$ is said to be normal in D, in the sense of Montel, if for any sequence $\{f_n\}$ in $\mathcal{F}$ there exists a subsequence $\{f_{n_j}\}$, such that $\{f_{n_j}\}$ converges spherically locally uniformly in D to a meromorphic function or ∞ (see [4,5]).

A function f meromorphic in the unit disc $\Delta = \{z\colon\ |z| < 1\}$ is called a normal function if and only if the family $\{f(S(z))\}$, where $z' = S(z)$ denotes an arbitrary one–one conformal mapping of Δ onto itself, is normal. The notion was introduced by Lehto and Virtanen [2].

A well-known result about normal functions is the following:

Theorem A. *Let f be a meromorphic function in the unit disc Δ. If f omits at least three distinct values in Δ, then f is normal.*

We say the functions f and g avoid each other uniformly if there exists a $\delta > 0$ such that, for each point z in their common domain, the spherical distance between $f(z)$ and $g(z)$ is at least δ. In [1], Lappan extended three distinct values in Theorem A to three continuous functions that avoid each other uniformly.

Theorem B. *Let g_1, g_2 and g_3 be three continuous functions that avoid each other uniformly in the unit disc Δ. Further, for each $j = 1, 2, 3$, let the family $\{g_j \circ \phi\colon\ \phi \in \Phi\}$ be normal in Δ, where $\Phi = \{\phi : \Delta \to \Delta,\ \phi$ is conformal mapping$\}$. Let f be a function meromorphic in Δ such that $f(z) \neq g_i(z)$ for $i = 1, 2, 3$. Then f is a normal function.*

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Lappan [1] also pointed out that the hypothesis that the functions g_1, g_2 and g_3 avoid each other *uniformly* is necessary. However, when the functions g_1, g_2, and g_3 are all meromorphic functions in the unit disc Δ and continuous on the closure of Δ, we need these functions to avoid each other only at each point of the unit circle, but not necessarily on Δ.

Theorem 1. *Let f be a meromorphic function in the unit disc Δ, φ_1, φ_2 and φ_3 be three functions meromorphic in Δ and continuous on closure of Δ such that $\varphi_i(z) \neq \varphi_j(z)$ $(1 \leqslant i < j \leqslant 3)$ on the unit circle $|z| = 1$. If $f(z) \neq \varphi_i(z)$ $(i = 1, 2, 3)$ in Δ, then f is normal.*

Clearly, Theorem 1 extends Theorem A.

2. Proof of Theorem 1

To prove our result, we need the following result due to Lohwater and Pommerenke:

Lohwater–Pommerenke Theorem. *(See [3].) A function f meromorphic in the unit disc Δ is a normal function if and only if there do not exist sequences $\{z_n\}$ and $\{\rho_n\}$ with $z_n \in \Delta$, and $\rho_n > 0$, $\rho_n \to 0$ such that $\{g_n(z) = f(z_n + \rho_n z)\}$ converges uniformly on each compact subset of the complex plane to a function $g(z)$, where $g(z)$ is a non-constant meromorphic function.*

Proof of Theorem 1. Assume that f, φ_1, φ_2, and φ_3 satisfy the hypotheses of the theorem and that f is not a normal function. Then, by Lohwater–Pommerenke Theorem, there exist sequences $\{z_n\}$ and $\{\rho_n\}$, with $z_n \in \Delta$ and $\rho_n > 0$, $\rho_n \to 0$ such that the sequence $\{g_n(z) = f(z_n + \rho_n z)\}$ converges uniformly on each compact subset of the complex plane to a function $g(z)$, where $g(z)$ is a non-constant meromorphic function. By taking a subsequence, if necessary, we may assume that $z_n \to z_0 \in \bar{\Delta}$, the closure of Δ. If $z_0 \in \Delta$, then $z_n + \rho_n z \to z_0$ for each complex number z, and $g_n(z) = f(z_n + \rho_n z) \to f(z_0)$, which would mean $g(z) \equiv f(z_0)$, violating the assumption that g is a non-constant function. Thus, we must have that $|z_0| = 1$.

Fix i, $1 \leqslant i \leqslant 3$, assume that $\varphi_i(z_0) \neq \infty$, and let

$$h_n(z) = f(z_n + \rho_n z) - \varphi_i(z_n + \rho_n z).$$

Then $h_n(z) \to g(z) - \varphi_i(z_0)$ uniformly on each compact subset of the plane. Since $f(z) - \varphi_i(z)$ is assumed to be never zero, it follows from a well-known theorem of Hurwitz that either $g(z) - \varphi_i(z_0) \equiv 0$ or $g(z) - \varphi_i(z_0)$ is never zero. But $g(z) - \varphi_i(z_0) \equiv 0$ means that $g(z)$ is a constant function, violating the assumption that it is not. Thus, it follows that $g(z)$ never assumes the value $\varphi_i(z_0)$.

If $\varphi_i(z_0) = \infty$, then we can take

$$h_n^*(z) = \frac{1}{f(z_n + \rho_n z)} - \frac{1}{\varphi_i(z_n + \rho_n z)}$$

and use the same argument (using h^* in place of h) to conclude that $1/g(z)$ does not assume the value 0, which means that $g(z)$ does not assume the value $\infty = \varphi_i(z_0)$.

This same argument can be applied to each i, $1 \leqslant i \leqslant 3$, so we know that the non-constant meromorphic function g avoids the three distinct values $\varphi_1(z_0)$, $\varphi_2(z_0)$ and $\varphi_3(z_0)$. But this violates Picard's theorem. Thus the assumption that f is not a normal function is untenable, and we conclude that f is a normal function. □

3. A remark

Using the argument above and Nevanlinna's second fundamental theorems (see [5]), we obtain the following more general result:

Theorem 2. *Let f be a meromorphic function in the unit disc Δ, φ_1, φ_2 and φ_3 be three functions meromorphic in Δ and continuous on closure of Δ such that $\varphi_i(z) \neq \varphi_j(z)$ $(1 \leqslant i < j \leqslant 3)$ on the unit circle $|z| = 1$, and let l_1, l_2 and l_3 be positive integers or ∞ with $1/l_1 + 1/l_2 + 1/l_3 < 1$. If all zeros of $f(z) - \varphi_i(z)$ have multiplicity at least l_i for $i = 1, 2, 3$ in Δ, then f is normal.*

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