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Cauchy–Fantappiè transformation and mutual dualities between $A^{-\infty}(\Omega)$ and $A^{\infty}(\widetilde{\Omega})$ for lineally convex domains

La transformation de Cauchy–Fantappiè et les dualités mutuelles entre $A^{-\infty}(\Omega)$ et $A^{\infty}(\widetilde{\Omega})$ pour des domaines linéellement convexes

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ABSTRACT

In this Note we present the following results: (i) a description, via the Cauchy–Fantappiè transformation of analytic functionals, of the mutual dualities between the (*DFS*)-space $A^{-\infty}(\Omega)$ of holomorphic functions in a bounded lineally convex domain Ω of $\mathbb{C}^n$ $(n \geqslant 2)$ with polynomial growth near the boundary $\partial\Omega$, and the (*FS*)-space $A^{\infty}(\widetilde{\Omega})$ of holomorphic functions in the interior of the conjugate set $\widetilde{\Omega}$ that are in $C^{\infty}(\widetilde{\Omega})$; (ii) the existence of countable sufficient sets in $A^{-\infty}(\Omega)$ and $A^{\infty}(\widetilde{\Omega})$; (iii) a possibility (respectively, the failure) of representing functions from $A^{\infty}(\widetilde{\Omega})$ (respectively, $A^{-\infty}(\Omega)$) in the form of series of partial fractions.

RÉSUMÉ

Dans cette Note nous présentons les résultats suivants: (i) la description par la transformation de Cauchy–Fantappiè des fonctionnelles analytiques, des dualités mutuelles entre le (*DFS*)-espace $A^{-\infty}(\Omega)$ des fonctions holomorphes dans le domaine linéellement convexe Ω de $\mathbb{C}^n$ $(n \geqslant 2)$ avec croissance polynomiale près de la frontière de $\partial\Omega$, et le (*FS*)-espace $A^{\infty}(\widetilde{\Omega})$ des fonctions holomorphes dans l'intérieur de l'ensemble conjugué $\widetilde{\Omega}$ qui sont dans $C^{\infty}(\widetilde{\Omega})$; (ii) l'existence d'ensembles suffisants dénombrables dans $A^{-\infty}(\Omega)$ et $A^{\infty}(\widetilde{\Omega})$; (iii) la possibilité (respectivement, l'impossibilité) de représentation des fonctions de $A^{\infty}(\widetilde{\Omega})$ (respectivement, $A^{-\infty}(\Omega)$) sous la forme de séries de fractions partielles.

1. Introduction

1.1. Basic notation and definitions

$\mathcal{O}(\Omega)$ (Ω being a domain in $\mathbb{C}^n$) denotes the space of functions that are holomorphic in Ω, with the topology of uniform convergence on compact subsets of Ω.

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For a set $E \subset \mathbb{C}^n$ $(0 \in E)$ denote $\widetilde{E} := \{\zeta \in \mathbb{C}^n\colon \langle z, \zeta\rangle \neq 1$ for any $z \in E\}$, the conjugate set of E. Here and below, $\langle z, \zeta\rangle := z_1\zeta_1 + \cdots + z_n\zeta_n$, $z, \zeta \in \mathbb{C}^n$. In the case when E is open, $\widetilde{E}$ is a compact set and plays the role of "the exterior" in the duality of A. Martineau and L. Aĭzenberg [3].

Let Ω be a bounded domain in $\mathbb{C}^n$ $(n \geqslant 2)$. W.l.o.g. we can assume that $0 \in \Omega$. Put $d(z) := \inf_{w\in\partial\Omega} |z - w|$, $z \in \Omega$. The space $A^{-\infty}(\Omega)$ of holomorphic functions in Ω having a polynomial growth near $\partial\Omega$ is defined as follows:

$$A^{-\infty}(\Omega) := \Big\{ f \in \mathcal{O}(\Omega)\colon \|f\|_k = \sup_{z\in\Omega} \big|f(z)\big| \big[d(z)\big]^k < +\infty, \text{ for some } k \in \mathbb{N} \Big\},$$

and equipped with its natural inductive limit topology. Hence, $A^{-\infty}(\Omega)$ becomes a dual Fréchet–Schwartz space (briefly, (*DFS*)-space).

Throughout this Note a domain Ω is supposed to be C^2-smooth. In the sequels we often use the following notation: $\delta_\Omega := \min_{z\in\partial\Omega} |z|$, $\Delta_\Omega := \max_{z\in\partial\Omega} |z|$.

We also consider the following Fréchet–Schwartz space (briefly, (*FS*)-space) $A^\infty(\widetilde{\Omega})$ of all holomorphic functions in $\operatorname{int}\widetilde{\Omega}$, the interior of $\widetilde{\Omega}$, which are in $C^\infty(\widetilde{\Omega})$, endowed with the topology given by the system of norms

$$|f|_k = \sup_{\|\alpha\|\leqslant k} \sup_{\zeta\in\widetilde{\Omega}} \big|D^\alpha f(\zeta)\big|, \quad k = 0, 1, 2, \ldots.$$

1.2. The main goal and results

The space $A^{-\infty}(\Omega)$, for Ω being either the unit disk $\mathbb{D}$ in the complex plane $\mathbb{C}$ or the unit ball $\mathbb{B}$ in $\mathbb{C}^n$, was considered by many authors (see, e.g., [6,10,11] and references therein). Therein, the problems concerning the sets of uniqueness, (weakly) sufficient, sampling and interpolating sets, as well as different relationships between them, have been treated.

Notice also that the duality problem for the space $A^{-\infty}(\Omega)$ has been studied by several authors including S. Bell, D. Barrett, C. Kiselman, E. Straube and others (see, e.g., [4,12] and references therein), and by different methods. They mainly studied, for C^∞-smooth domains Ω, the duality (in fact, mainly its failure) between $A^{-\infty}(\Omega)$ and the space $A^\infty(\overline{\Omega})$, and so Bell's condition R played an important role.

In our recent papers [1,2], for a bounded convex domain Ω of $\mathbb{C}^n$, we established, via the Laplace transformation, the mutual dualities between $A^{-\infty}(\Omega)$ and the space $A_\Omega^{-\infty}$ of entire functions in $\mathbb{C}^n$ with a certain growth condition. We also gave an explicit construction of a countable sufficient set for the dual space and then, applying the so-called "dual relationship" we showed that any function from either $A^{-\infty}(\Omega)$ or $A_\Omega^{-\infty}$ can always be represented in the form of a Dirichlet series.

The aim of this Note is to consider more general case, namely lineally convex [3] domains in $\mathbb{C}^n$ $(n \geqslant 2)$. We establish, via the Cauchy–Fantappiè transformation, the mutual dualities between $A^{-\infty}(\Omega)$ and $A^\infty(\widetilde{\Omega})$, provided Ω is strictly starlike with respect to the origin.

As it will be seen below, the duality theorem guarantees an opportunity to represent each function from $A^\infty(\widetilde{\Omega})$ in the form of series of partial fractions, while a surprising result, in comparison with [1,2], is that it is impossible to have a similar representation for functions from $A^{-\infty}(\Omega)$, despite the existence of sufficient sets in its dual space.

2. Mutual dualities via Cauchy–Fantappiè transformation

The Cauchy–Fantappiè kernel for a domain Ω is defined as $k(z, \zeta) := (1 - \langle z, \zeta\rangle)^{-n}$, $z \in \Omega$, $\zeta \in \widetilde{\Omega}$.

Lemma 2.1. *The following estimates hold*:

(1) $\frac{d(z)}{\Delta_\Omega} \leqslant \inf_{\zeta\in\partial\widetilde{\Omega}} |1 - \langle z, \zeta\rangle| \leqslant \frac{d(z)}{\delta_\Omega}$, $z \in \Omega$.
(2) $\|k(\cdot, \zeta)\|_m \leqslant [\Delta_\Omega]^m$, $\zeta \in \widetilde{\Omega}, m \geqslant n$.
(3) $|k(z, \cdot)|_m \leqslant \frac{A_m}{[d(z)]^{m+n}}$, $z \in \Omega, m \in \mathbb{N}$ $(A_m := \frac{(n+m-1)!}{(n-1)!} [\Delta_\Omega]^{n+m} [\max\{1, \Delta_\Omega\}]^m)$.

From Lemma 2.1 it follows that $k(\cdot, \zeta) \in A^{-\infty}(\Omega)$, for all $\zeta \in \widetilde{\Omega}$, as well as $k(z, \cdot) \in A^\infty(\widetilde{\Omega})$, for all $z \in \Omega$. These results lead to the following definition.

Definition 2.2. For an analytic functional T from the dual space $(A^{-\infty}(\Omega))'$ (respectively, $(A^\infty(\widetilde{\Omega}))'$), the function $\mathcal{F}_T(\zeta) := T(k(\cdot, \zeta))$, $\zeta \in \widetilde{\Omega}$ (respectively, $\mathcal{F}_T(z) := T(k(z, \cdot))$, $z \in \Omega$), is called the *Cauchy–Fantappiè transformation* of T.

Proposition 2.3. *Suppose that Ω is starlike with respect to the origin. Then the Cauchy–Fantappiè transformation $T \to \mathcal{F}_T$ is surjective both from $(A^{-\infty}(\Omega))'$ onto $A^\infty(\widetilde{\Omega})$ and from $(A^\infty(\widetilde{\Omega}))'$ onto $A^{-\infty}(\Omega)$.*

The proof follows those of [1,2], in which the C^2-smoothness of Ω plays a key role.

In view of Proposition 2.3 and the open mapping theorem, the Cauchy–Fantappiè transformation establishes the mutual duality between $A^{-\infty}(\Omega)$ and $A^{\infty}(\widetilde{\Omega})$ for a starlike domain Ω if and only if it is injective. This is the case if, for instance, Ω is strictly starlike with respect to some point. Recall that a set E is said to be *strictly starlike with respect to* $a \in \Omega$ if every real line passing through a intersects the boundary ∂E in at most two points. W.l.o.g. we may assume that a coincides with the origin. Note that Ω and $\widetilde{\Omega}$ have the property of strict starlikeness with respect to the origin simultaneously. Thus, we have the following result.

Theorem 2.4. *Let Ω be a bounded lineally convex, strictly starlike, C^2-smooth domain in $\mathbb{C}^n$ ($n \geqslant 2$). The Cauchy–Fantappiè transformation establishes a topological isomorphism between the strong dual $(A^{-\infty}(\Omega))'_b$ of $A^{-\infty}(\Omega)$ and the space $A^{\infty}(\widetilde{\Omega})$, as well as the strong dual $(A^{\infty}(\widetilde{\Omega}))'_b$ of $A^{\infty}(\widetilde{\Omega})$ and the space $A^{-\infty}(\Omega)$.*

Remark 2.5. The property of strict starlikeness is intermediate between starlikeness and strong starlikeness (E is said to be *strongly starlike* with respect to a if it is starlike with respect to every point in some neighborhood of a). In connection with this, note that in [8], where a duality theorem for an (FS)-space of holomorphic functions with prescribed behavior near the boundary was proved, a domain Ω was supposed to be strongly starlike. Theorem 2.4 contains a similar result for (DFS)-space and, more than that, we also obtain the "converse" duality, for Ω, in both cases, being only strictly starlike.

3. Sufficient sets and representation of functions by series of partial fractions

Let Ω be a bounded domain in $\mathbb{C}^n$. Denote by τ the natural internal inductive limit topology in $A^{-\infty}(\Omega)$. Let S be a subset of Ω. Define $A^{-k,S}(\Omega) := \{f \in A^{-\infty}(\Omega)\colon \|f\|_{k,S} := \sup_{z\in S} |f(z)|[d(z)]^k < +\infty\}$. In this case, one can endow $A^{-\infty}(\Omega)$ with another (weaker, and in fact strictly weaker in many cases) internal inductive limit topology, denoted by τ_S, of semi-normed spaces $(A^{-k,S}(\Omega), \|\cdot\|_{k,S})$. A subset S is called *weakly sufficient* [13] for $A^{-\infty}(\Omega)$ if the two topologies τ and τ_S are equivalent. Note that in $A^{-\infty}(\Omega)$ each weakly sufficient set is sufficient one [5] (the converse is true in general).

Obviously, the domain Ω itself is weakly sufficient for the space $A^{-\infty}(\Omega)$. By [9] there always exists a sequence (z_k) in Ω that has no limit points in Ω and forms a weakly sufficient set for $A^{-\infty}(\Omega)$.

It should be noted that these arguments work for any domain in $\mathbb{C}^n$. It however guarantees only a "theoretical existence" of such a discrete set. In [7] it was presented an explicit method for construction of a countable weakly sufficient set for any bounded domain with C^1 smooth boundary.

For $A^{\infty}(\widetilde{\Omega})$, we consider a more general situation than $A^{\infty}(\widetilde{\Omega})$, namely, the space $A^{\infty}(\overline{D}) := \mathcal{O}(D)\bigcap C^{\infty}(\overline{D})$, where D is a bounded pseudoconvex domain in $\mathbb{C}^n$ with $\operatorname{int}\overline{D} = D$. Note that the last assumption is natural for spaces of such a kind. We endow $A^{\infty}(\overline{D})$ with the usual topology of a Fréchet space given by the system of norms $|f|_m := \max_{|\alpha|\leqslant m} \max_{\zeta\in\overline{D}} |D^{\alpha} f(\zeta)|$, $f \in A^{\infty}(\overline{D})$, $m \in \mathbb{N}_0$.

For a set $S \subset \overline{D}$, put $|f|_{m,S} := \max_{|\alpha|\leqslant m} \sup_{\zeta\in S} |D^{\alpha} f(\zeta)|$, $f \in A^{\infty}(\overline{D})$, $m \in \mathbb{N}_0$. Then the system of semi-norms $(|f|_{m,S})_{m=0}^{\infty}$ defines another topology in $A^{\infty}(\overline{D})$ that is weaker than the origin one.

Similar to the definition of weakly sufficiency above, the set S is called *sufficient* for $A^{\infty}(\overline{D})$ if the two topologies coincide. It is clear that S is sufficient for $A^{\infty}(\overline{D})$ if and only if

$$\forall m\ \exists \ell\ \exists A_m > 0\colon \quad |f|_m \leqslant A_m |f|_{\ell,S}, \quad \forall f \in A^{\infty}(\overline{D}).$$

Proposition 3.1. *Let D be a bounded pseudoconvex domain in $\mathbb{C}^n$ with $\operatorname{int}\overline{D} = D$ and $S(D)$ the Shilov boundary of D. If a set $S \subset \overline{D}$ is such that $\overline{S} \supset S(D)$, then S is sufficient for $A^{\infty}(\overline{D})$.*

Corollary 3.2. 1) *Let D be as in Proposition* 3.1 *and $S \subset \overline{D}$. If $\overline{S} \supset \partial D$, then S is sufficient for $A^{\infty}(\overline{D})$.*

2) *For each bounded pseudoconvex domain D in $\mathbb{C}^n$ with $\operatorname{int}\overline{D} = D$, there always exists a sequence $(\zeta_k)_{k=1}^{\infty} \subset \overline{D}$ which forms a sufficient set for $A^{\infty}(\overline{D})$.*

From Theorem 2.4 as well as the existence of discrete (weakly) sufficient sets for the spaces $A^{-\infty}(\Omega)$ and $A^{\infty}(\widetilde{\Omega})$, we are "traditionally" expected that functions of both spaces $A^{\infty}(\widetilde{\Omega})$ and $A^{-\infty}(\Omega)$ can be represented in the form of a series of partial fractions, generated by the Cauchy–Fantappiè kernel.

For considering this problem we might use the "dual relationship" between absolutely representing systems of partial fractions in these spaces and (weakly) sufficient sets in their dual spaces, as it has been investigated in the case of Laplace transformation and Dirichlet series presented in our recent papers [1,2].

However, it is surprising that for the space $A^{\infty}(\widetilde{\Omega})$ such a system does exist, while in the space $A^{-\infty}(\Omega)$ there is no absolutely representing system of partial fractions. This also means that the "dual relationship" above, in contrary to the case of Laplace transformation and Dirichlet series presented in [2], does not hold for $A^{-\infty}(\Omega)$.

Namely we have the following two results.

Theorem 3.3. *Let Ω be a bounded lineally convex, strictly starlike, C^2-smooth domain in $\mathbb{C}^n$ $(n \geqslant 2)$. Then there is a discrete sequence $(z_k)_{k=1}^{\infty} \subset \Omega$, such that any function $g \in A^{\infty}(\widetilde{\Omega})$ can be represented in the form of a series of partial fractions*

$$g(\zeta) = \sum_{k=1}^{\infty} \frac{c_k}{(1 - \langle z_k, \zeta \rangle)^n}, \quad \forall \zeta \in \widetilde{\Omega}, \tag{1}$$

that converges absolutely in the space $A^{\infty}(\widetilde{\Omega})$.

Theorem 3.4. *Let Ω be a bounded lineally convex, strictly starlike, C^2-smooth domain in $\mathbb{C}^n$ $(n \geqslant 2)$. For any sequence (z_k) in $\widetilde{\Omega}$ there always exists a function $g \in A^{-\infty}(\Omega)$ such that it cannot be represented in the form of a series (1) that converges absolutely in $A^{-\infty}(\Omega)$.*

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