



Partial Differential Equations/Functional Analysis

Hardy inequality and Pohozaev identity for operators with boundary singularities: Some applications

Inégalité de Hardy et identité de Pohozaev pour des opérateurs à singularités sur la frontière : Quelques applications

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ABSTRACT

We consider the Schrödinger operator $A_\lambda := -\Delta - \lambda/|x|^2$, $\lambda \in \mathbb{R}$, when the singularity is located on the boundary of a smooth domain $\Omega \subset \mathbb{R}^N$, $N \geq 1$.

The aim of this Note is two folded. Firstly, we justify the extension of the classical Pohozaev identity for the Laplacian to this case. The problem we address is very much related to Hardy–Poincaré inequalities with boundary singularities. Secondly, the new Pohozaev identity allows us to develop the multiplier method for the wave and the Schrödinger equations. In this way we extend to the case of boundary singularities well known observability and control properties for the classical wave and Schrödinger equations when the singularity is placed in the interior of the domain (Vancostenoble and Zuazua (2009) [16]).

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RÉSUMÉ

On considère l'opérateur de Schrödinger $A_\lambda := -\Delta - \lambda/|x|^2$, $\lambda \in \mathbb{R}$, lorsque l'origine est située sur la frontière d'un domaine borné et régulier $\Omega \subset \mathbb{R}^N$, $N \geq 1$.

Cette Note a deux objectifs. Premièrement, on montre, dans ce cas, l'extension de l'identité classique de Pohozaev pour le laplacien. Le problème abordé est très lié aux inégalités de Hardy–Poincaré avec des singularités sur la frontière. En second lieu, la nouvelle identité de Pohozaev permet de d'obtenir la méthode des multiplicateurs pour l'équation des ondes et pour l'équation de Schrödinger. De cette façon on étend au cas de la singularité frontière les propriétés d'observabilité et contrôle pour l'équation des ondes classique et pour l'équation de Schrödinger bien connues dans le cas d'une singularité intérieure (Vancostenoble et Zuazua (2009) [16]).

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Dans cette Note nous nous intéressons à l'opérateur $A_\lambda := -\Delta - \lambda/|x|^2$, $\lambda \in \mathbb{R}$, lorsque l'origine est située sur la frontière d'un domaine régulier de $\Omega \subset \mathbb{R}^N$, $N \geq 1$. Il est connu que la valeur $\lambda(N) := N^2/4$ est la meilleure constante dans

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les inégalités de Hardy ci-après, elle est critique lorsqu'on étudie les propriétés qualitatives de A_λ . Dans la première partie de cette Note nous montrons que pour tout $\lambda \leq \lambda(N)$, l'identité de Pohozaev (voir Théorème 2) est vérifiée dans le domaine de A_λ , défini par $D(A_\lambda) := \{u \in H_\lambda \mid A_\lambda u \in L^2(\Omega)\}$. Nous définirons plus loin l'espace H_λ et nous donnerons quelques unes de ses propriétés. Formellement, le Théorème 2 peut être obtenu par intégration directe. Cependant, la singularité $x = 0$ entraîne une perte de régularité de l'opérateur $(A_\lambda, D(A_\lambda))$ et les intégrations par parties ne sont plus justifiées rigoureusement. De plus, la régularité L^2 de la dérivée normale n'est plus vraie car les estimations elliptiques standards ne s'appliquent plus car la singularité est localisée sur le bord. Néanmoins, la trace d'un élément de $D(A_\lambda)$ existe dans un espace L^2 à poids, dont le poids est dégénéré à l'origine, comme on le montre dans le Théorème 1. Dans la deuxième partie de cette Note, nous donnons plusieurs applications des Théorèmes 1, 2. D'abord des solutions non triviales d'une EDP singulière sont traitées dans le Théorème 3. Ensuite, nous utilisons des techniques de multiplicateurs pour démontrer «la régularité cachée» de la dérivée normale dans le cas de l'équation des ondes et de l'équation de Schrödinger, correspondant à A_λ (voir le Théorème 4). En particulier, nous donnons la réponse à la question concernant la contrôlabilité des systèmes conservatifs. Le résultat principal est donné par le Théorème 5 et est résulte de l'identité des multiplicateurs (10), en combinaison avec une inégalité forte de Hardy, formulée dans le Théorème 6. Pour plus de clarté dans la présentation, nous traitons notamment le cas C1 de la Fig. 1. Cependant, les mêmes résultats peuvent être étendus aux cas C2, C3, C4 dans un cadre fonctionnel plus faible, associé à l'inégalité de Hardy plus faible (2).

1. Introduction

Let us consider Ω to be a smooth subset of \mathbb{R}^N , $N \geq 1$, with the origin $x = 0$ placed on its boundary Γ . Without losing the generality we distinguish the following geometrical configurations for Ω as in Fig. 1:

- C1 Ω is a subset of $\mathbb{R}_+^N := \{x = (x_1, \dots, x_N) = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} \mid x_N > 0\}$ (Fig. 1, top, left).
- C2 Close to $x = 0$, the points $x \in \Gamma$ satisfy $x \cdot \nu \geq 0$. Nevertheless, Ω crosses the hyperplane $x_N = 0$ far from origin (Fig. 1, top, right).
- C3 Close to $x = 0$, the points $x \in \Gamma$ verify $x \cdot \nu \leq 0$ (Fig. 1, bottom, left).
- C4 For $x \in \Gamma$ the sign of $x \cdot \nu$ changes at origin (Fig. 1, bottom, right).

The following Hardy inequalities are well known: if Ω verifies the case C1 in Fig. 1, then (e.g. [4]) for any $u \in C_0^\infty(\Omega)$ it holds that

$$\int_\Omega |\nabla u|^2 \, dx \geq \frac{N^2}{4} \int_\Omega \frac{u^2}{|x|^2} \, dx + \frac{1}{4} \int_\Omega \frac{u^2}{|x|^2 \log^2(R_\Omega/|x|)} \, dx, \tag{1}$$

where $R_\Omega = \sup_{x \in \bar{\Omega}} |x|$. If Ω satisfies the cases C2, C3, C4 as in Fig. 1 then (e.g. [8,9]) there exist $C_2 = C_2(\Omega) \in \mathbb{R}$ and $C_3 = C_3(\Omega, N) > 0$ such that any $u \in C_0^\infty(\Omega)$ satisfies

$$C_2 \int_\Omega u^2 \, dx + \int_\Omega |\nabla u|^2 \, dx \geq \frac{N^2}{4} \int_\Omega \frac{u^2}{|x|^2} \, dx + C_3 \int_\Omega \frac{u^2}{|x|^2 \log^2(R_\Omega/|x|)} \, dx. \tag{2}$$

In both situations above, the constant $\lambda(N) = N^2/4$ is optimal. For the sake of clarity and because similar results can be obtained in the other cases in an other functional framework, next we analyze the situation when Ω verifies the case C1 in Fig. 1. In the sequel let us consider the operator $(A_\lambda, D(A_\lambda))$, $\lambda \leq \lambda(N)$, acting on such Ω .

Firstly, this Note is aimed to justify the Pohozaev identity (5) in the functional setting $(A_\lambda, D(A_\lambda))$ in which Theorem 1 plays a crucial role. Pohozaev-type identities (e.g. p. 515, [7]) have been widely used to show non-existence results to nonlinear elliptic equations. In particular, we point out that this issue has been also studied for nonlinear equations with singular potentials (see e.g. [5,10]). In those cases, due to the regularizing effect of the nonlinearity, the solutions become regular enough to obtain the corresponding Pohozaev identity by direct computations. This is not precisely our case. To the best of our knowledge, the regularity of the operator $(A_\lambda, D(A_\lambda))$ in terms of the Sobolev spaces is yet unknown and this affects the direct justification of (5). To do this we proceed by approximations arguments near the singularity. We analyze two situations when discussing the values of λ : the subcritical case $\lambda < \lambda(N)$ respectively the critical case $\lambda = \lambda(N)$. However, the main novelty appears for the critical value $\lambda = \lambda(N)$ in which case H_λ is strictly larger than $H_0^1(\Omega)$. This case requires a better understanding of the norm of $H_{\lambda(N)}$ as discussed in Section 2.

The controllability properties and stabilization of the heat and wave equation corresponding to A_λ have been analyzed in [15,6,16] in the case of interior singularity. Roughly speaking, they showed that the parameter $\lambda_* = (N - 2)^2/4$, which is the optimal constant in the Hardy inequality with interior singularity, is critical when asking the well-posedness and control properties of such systems. In the second part of this Note we address the question of controllability for the wave and Schrödinger equations corresponding to A_λ , in the case of boundary singularity. Our main result asserts that we can increase the range of values λ (from λ_* to $\lambda(N)$) for which the exact controllability holds.

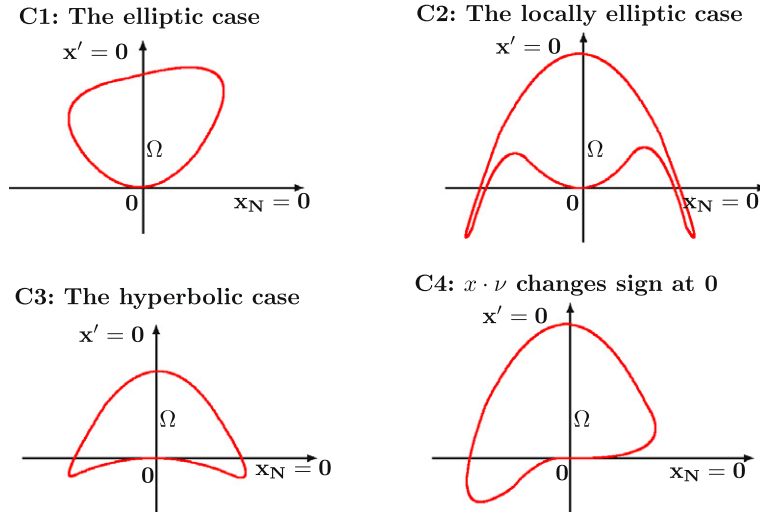


Fig. 1. Geometry of Ω (in red), ν denotes the outward normal vector. (For interpretation of the references to color in this figure, the reader is referred to the web version of this article.)

2. The space H_λ , proper norm and main elliptic results

Following the idea in [18], thanks to inequality (1) we consider the Hardy functional $B_\lambda[u] = \int_\Omega [|\nabla u|^2 - \lambda u^2/|x|^2] dx$, which is positive and finite for all $u \in C_0^\infty(\Omega)$. We define the Hilbert space H_λ to be the completion of $C_0^\infty(\Omega)$ functions in the norm induced by $B_\lambda[u]$. If $\lambda < \lambda(N)$, it holds that $H_0^1(\Omega) = H_\lambda$ due to Hardy inequality which ensures the equivalence of the norms. Similar to the case of interior singularity emphasized in [17], an interesting phenomena appears in the critical space $H_{\lambda(N)}$. Assume $\Omega = \{x \in \mathbb{R}^N \mid |x| \leq 1, x_N > 0\}$ and let $e_1 := x_N|x|^{-N/2} J(z_{0,1}|x|)$ where $z_{0,1}$ is the first positive zero of the Bessel function J_0 . Then there exists $\lim_{\epsilon \rightarrow 0} \int_{x \in \Omega, |x| \geq \epsilon} [|\nabla e_1|^2 - \lambda(N)e_1^2/|x|^2] dx < \infty$ although $e_1 \notin H_0^1(\Omega)$. Surprisingly, the meaning of $B_{\lambda(N)}[\cdot]$ is not well-defined in the sense of principle value. Indeed, if it were one can check that $B_{\lambda(N)}[e_1 - \phi] \geq C_0 > 0$, for all $\phi \in C_0^\infty(\Omega)$ for some universal constant $C_0 > 0$. This is in contradiction with the definition of $H_{\lambda(N)}$. The remedy for this is to consider the functional

$$B_{\lambda,1}[u] = \int_\Omega \left| \nabla u + \frac{N}{2} \frac{x}{|x|^2} u - \frac{e_N}{x_N} u \right|^2 dx + (\lambda(N) - \lambda) \int_\Omega \frac{u^2}{|x|^2} dx, \tag{3}$$

where e_N denotes the N -th canonical vector of \mathbb{R}^N . We observe that $B_\lambda[u] = B_{\lambda,1}[u]$, for any Ω as in case C1, and all $u \in C_0^\infty(\Omega)$. Therefore, $B_{\lambda,1}[u]$ induces a new norm in the space H_λ which is well understood in the sense of principal value. In the sequel, we denote by $\|\cdot\|_{H_\lambda}$ the norm induced by $B_{\lambda,1}$.

Notations. For any $\epsilon > 0$, θ_ϵ is a smooth cut-off function which satisfies $\theta_\epsilon = 0$ for $|x| \leq \epsilon$ respectively $\theta_\epsilon = 1$ for $|x| \geq 2\epsilon$. Besides, $\vec{q} \in (C^2(\overline{\Omega}))^N$ denotes a vector field such that $\vec{q} = \nu$ on Γ . H'_λ denotes the dual space of H_λ . Next we state our main elliptic results.

Theorem 1 (Trace regularity). Assume $\Omega \subset \mathbb{R}^N$, $N \geq 1$, verifies the case C1. Let us consider $\lambda \leq \lambda(N)$ and $u \in D(A_\lambda)$. Then $(\partial u / \partial \nu)|_{x| \in L^2(\Gamma)}$ and there exists $C = C(\Omega) > 0$ such that

$$\int_\Gamma \left(\frac{\partial u}{\partial \nu} \right)^2 |x|^2 d\sigma \leq C (\|u\|_{H_\lambda}^2 + \|A_\lambda u\|_{L^2(\Omega)}^2). \tag{4}$$

Sketch of the proof. In order to avoid the singularity, we multiply $A_\lambda u$ by $|x|^2(\vec{q} \cdot \nabla u)\theta_\epsilon$ and we integrate by parts. Then we obtain an identity which, combined with Cauchy-Schwartz inequality, allows to get uniform upper bounds for the boundary term. Then, by Fatou Lemma we can pass to the limit as ϵ tends to zero to end up the proof. \square

Theorem 2 (Pohozaev identity). Assume $\Omega \subset \mathbb{R}^N$, $N \geq 1$, verifies the case C1 and let $\lambda \leq \lambda(N)$. Then for all $u \in D(A_\lambda)$ it holds that

$$\frac{1}{2} \int_\Gamma (x \cdot \nu) \left(\frac{\partial u}{\partial \nu} \right)^2 d\sigma = - \int_\Omega A_\lambda u (x \cdot \nabla u) dx - \frac{N-2}{2} \|u\|_{H_\lambda}^2. \tag{5}$$

Sketch of the proof. Note firstly that all terms in (5) are finite. Indeed, thanks to the fact that $x \cdot \nu = O(|x|^2)$ and Theorem 1 we obtain the integrability of the boundary term. Moreover, $x \cdot \nabla u \in L^2(\Omega)$ as shown in Theorem 6. We proceed by multiplier technique. If $\lambda < \lambda(N)$ the multiplier $(x \cdot \nabla u)\theta_\epsilon$ is used to obtain (5) in the limit process. If $\lambda = \lambda(N)$ we apply the subcritical result for $(\lambda(N) - \delta)$ and we argue we can pass to the limit as δ tends to zero. Indeed, following approximation lemma which is proved in generality in [1], for a given $u \in D(A_\lambda)$, the solution u_δ of $A_{\lambda(N)-\delta}u_\delta = A_{\lambda(N)}u$, converges strongly to u in $H_{\lambda(N)}$. Secondly, by comparison arguments we show the asymptotic behavior of $u_\delta, \nabla u_\delta$ when $A_\lambda u$ is smooth. This is done by constructing super solutions and using rescaling arguments in [3] to get rid of the singularity. A density argument together with Theorem 1 concludes the proof. \square

Theorem 3. Assume $\lambda \leq \lambda(N)$ and $\Omega \subset \mathbb{R}^N, N \geq 3$, satisfying the case C1. Let us consider the problem

$$-\Delta u - \frac{\lambda}{|x|^2}u = |u|^{\alpha-1}u, \quad x \in \Omega, \quad u = 0, \quad x \in \Gamma. \tag{6}$$

- (1) If $1 < \alpha < (N + 2)/(N - 2)$, problem (6) has non-trivial solutions in H_λ . Moreover, if $1 < \alpha < N/(N - 2)$ problem (6) has non-trivial solutions in $D(A_\lambda)$.
- (2) Assume Ω is a star-shaped domain (i.e. $x \cdot \nu \geq 0$, for all $x \in \Gamma$). If $\alpha \geq (N + 2)/(N - 2)$, problem (6) does not have non-trivial solutions in $D(A_\lambda)$.

Sketch of the proof. The existence of non-trivial solutions reduces to finding minimizers for the problem $I = \inf_{\|u\|_{L^{\alpha+1}(\Omega)}=1} \|u\|_{H_\lambda}^2$ and is due to the compact embedding $H_\lambda \hookrightarrow L^{\alpha+1}(\Omega)$. The non-existence result is proved by combining Theorem 2 and unique continuation results as in [11]. \square

3. Applications to controllability

Let us set $\Gamma_0 := \{x \in \Gamma \mid x \cdot \nu \geq 0\}$ a non-empty part of the boundary Γ . Next we consider the wave-like process

$$\begin{cases} u_{tt} - \Delta u - \lambda \frac{u}{|x|^2} = 0, & (t, x) \in Q_T, \\ u(t, x) = h(t, x)\chi_{\Gamma_0}, & (t, x) \in \Sigma_T, \\ (u(0, x), u_t(0, x)) = (u_0(x), u_1(x)), & x \in \Omega, \end{cases} \tag{7}$$

where $Q_T = (0, T) \times \Omega, \Sigma_T = (0, T) \times \Gamma$ and χ_{Γ_0} denotes the characteristic function of Γ_0 .

The solution of (7) is defined in weak sense by the transposition method (J.L. Lions [12]). In this section we address the question of exact controllability of system (7), i.e. whether for any initial data $(u_0, u_1) \in L^2(\Omega) \times H'_\lambda$ and any target $(\bar{u}_0, \bar{u}_1) \in L^2(\Omega) \times H'_\lambda$, there exists a finite time $T > 0$ and a control $h \in L^2((0, T) \times \Gamma_0)$ such that the solution of (7) satisfies $(u_t(T, x), u(T, x)) = (\bar{u}_1(x), \bar{u}_0(x))$ for all $x \in \Omega$. In view of the time-reversibility of the equation it is enough to consider the null-controllability problem, i.e. the case where the target $(\bar{u}_0, \bar{u}_1) = (0, 0)$. By now classical *Hilbert Uniqueness Method* (HUM) (see J.L. Lions [12]) the null-controllability of system (7) is characterized through the adjoint system

$$\begin{cases} v_{tt} - \Delta v - \lambda \frac{v}{|x|^2} = 0, & (t, x) \in Q_T, \\ v(t, x) = 0, & (t, x) \in \Sigma_T, \\ (v(0, x), v_t(0, x)) = (v_0(x), v_1(x)), & x \in \Omega. \end{cases} \tag{8}$$

The operator $(\mathcal{A}_\lambda, D(\mathcal{A}_\lambda))$ defined by $\mathcal{A}_\lambda(w_1, w_2) = (w_2, \Delta w_1 + \lambda/|x|^2 w_1)$ for all $(w_1, w_2) \in D(\mathcal{A}_\lambda) := D(A_\lambda) \times H_\lambda$ generates the wave semigroup in $H_\lambda \times L^2(\Omega)$. In view of that, the adjoint system is well-posed.

In the sequel, we justify some “hidden regularity” effect for the system (8) which may not be directly deduce from the semigroup regularity but from the equation itself.

Theorem 4 (Hidden regularity). Assume $\lambda \leq \lambda(N)$ and v is the solution of (8) corresponding to the initial data $(v_0, v_1) \in H_\lambda \times L^2(\Omega)$. Then v satisfies

$$\int_0^T \int_\Gamma \left(\frac{\partial v}{\partial \nu} \right)^2 |x|^2 \, d\sigma \, dt \leq C (\|v_0\|_{H_\lambda}^2 + \|v_1\|_{L^2(\Omega)}^2) \tag{9}$$

for some universal constant $C > 0$. Moreover, v verifies the identity

$$\frac{1}{2} \int_0^T \int_\Gamma (x \cdot \nu) \left(\frac{\partial v}{\partial \nu} \right)^2 \, d\sigma \, dt = \frac{T}{2} (\|v_0\|_{H_\lambda}^2 + \|v_1\|_{L^2(\Omega)}^2) + \int_\Omega v_t \left(x \cdot \nabla v + \frac{N-1}{2} v \right) \Big|_0^T \, dx. \tag{10}$$

Sketch of the proof. By density, it suffices to prove Theorem 4, for initial data (v_0, v_1) in $D(\mathcal{A}_\lambda)$. For the proof of (9) we multiply $A_\lambda v$ by $|x|^2(\bar{q} \cdot \nabla v)\theta_\epsilon$ and integrate. The integration in time and the conservation of energy allow to obtain uniform bounds for the boundary term in the energy space. Then by Fatou Lemma we pass to the limit as ϵ tends to zero and the proof finishes.

For the proof of (10) we proceed straightforward from Theorem 2. Indeed, for a fixed time $t \in [0, T]$ we apply Theorem 2 for $A_\lambda v = -v_{tt}$. Then we integrate in time, and due to the equipartition of energy we can finish the proof. \square

Due to Theorem 4 the operator $(v_0, v_1) \mapsto (\int_0^T \int_{\Gamma_0} (x \cdot v)(\partial v / \partial \nu)^2 d\sigma dt)^{1/2}$ is a linear continuous map in $H_\lambda \times L^2(\Omega)$. Let \mathcal{H} be the completion of this norm in $H_\lambda \times L^2(\Omega)$. We consider the functional $J : \mathcal{H} \rightarrow \mathbb{R}$ defined by

$$J(v_0, v_1)(v) := \frac{1}{2} \int_0^T \int_{\Gamma_0} (x \cdot v) \left(\frac{\partial v}{\partial \nu} \right)^2 d\sigma dt - \langle u_1, v_0 \rangle_{H'_\lambda, H_\lambda} + (u_0, v_1)_{L^2(\Omega), L^2(\Omega)}, \tag{11}$$

where v is the solution of (8) corresponding to initial data (v_0, v_1) . Of course, $\langle \cdot, \cdot \rangle_{H'_\lambda, H_\lambda}$ denotes the duality product. The control $h \in L^2((0, T) \times \Gamma_0)$ for (7) could be chosen as $h := (x \cdot v)v_{\min}$ where v_{\min} minimizes the functional J on \mathcal{H} among the solutions v of (8) corresponding to the initial data $(u_0, u_1) \in H'_\lambda \times L^2(\Omega)$. The existence of a minimizer of J is assured by the coercivity of J which is equivalent to the *Observability Inequality* for the adjoint system stated as follows.

Theorem 5 (Observability). For all $\lambda \leq \lambda(N)$, there exists a positive constant $D_1 = D_1(\Omega, \lambda, T) > 0$ such that for all $T \geq 2R_\Omega$, and any initial data $(v_0, v_1) \in H_\lambda \times L^2(\Omega)$ the solution of (8) verifies

$$\|v_0\|_{H_\lambda}^2 + \|v_1\|_{L^2(\Omega)}^2 \leq D_1 \int_0^T \int_{\Gamma_0} (x \cdot v) \left(\frac{\partial v}{\partial \nu} \right)^2 d\sigma dt. \tag{12}$$

Sketch of the proof. The proof of Theorem 5 relies mainly on Theorem 4, combining compactness uniqueness argument (cf. [13]) and the sharp Hardy inequality stated in Theorem 6. \square

Theorem 6. Assume Ω satisfies one of the cases C1–C4. Then, there exists a constant $C = C(\Omega) \in \mathbb{R}$ such that

$$\int_\Omega |x|^2 |\nabla v|^2 dx \leq R_\Omega^2 \left[\int_\Omega |\nabla v|^2 dx - \frac{N^2}{4} \int_\Omega \frac{v^2}{|x|^2} dx \right] + C \int_\Omega v^2 dx \quad \forall v \in C_0^\infty(\Omega). \tag{13}$$

Remark 1. The proof of Theorem 6 is quite technical and we omit it here. The constant R_Ω^2 which appears in inequality (13), helps to obtain the control time $T > T_0 = 2R_\Omega$, which is sharp from the Geometric Control Condition considerations, see [2].

The results above guarantee the exact boundary controllability of (7). More precisely, we obtain

Theorem 7 (Controllability). Assume that Ω satisfies C1 and $\lambda \leq \lambda(N)$. For any time $T > 2R_\Omega$, $(u_0, u_1) \in L^2(\Omega) \times H'_\lambda$ and $(\bar{u}_0, \bar{u}_1) \in L^2(\Omega) \times H'_\lambda$ there exists $h \in L^2((0, T) \times \Gamma_0)$ such that the solution of (7) satisfies $(u_t(T, x), u(T, x)) = (\bar{u}_1(x), \bar{u}_0(x))$ for all $x \in \Omega$.

Schrödinger equation. In the above geometrical settings, we consider the Schrödinger equation

$$\begin{cases} iu_t - \Delta u - \lambda \frac{u}{|x|^2} = 0, & (t, x) \in Q_T, \\ u(t, x) = h(t, x)\chi_{\Gamma_0}, & (t, x) \in \Sigma_T, \\ u(0, x) = u_0(x), & x \in \Omega. \end{cases} \tag{14}$$

For all $\lambda \leq \lambda(N)$, we define the Hilbert space $H_\lambda(\Omega; \mathbb{C})$ as the completion of $H_0^1(\Omega; \mathbb{C})$ with respect to the norm induced by the inner product $\langle u, v \rangle_{H_\lambda(\Omega; \mathbb{C})} := \text{Re} \int_\Omega (\nabla u(x) \cdot \nabla \bar{v}(x) - \lambda u(x)\bar{v}(x)/|x|^2) dx$. Then

Theorem 8 (Controllability). For any $\lambda \leq \lambda(N)$, $u_0 \in H'_\lambda$, $\bar{u}_0 \in H'_\lambda$ and any time $T > 0$ there exists $h \in L^2((0, T) \times \Gamma_0)$ such that the solution of (14) satisfies $u(T, x) = \bar{u}_0(x)$ for all $x \in \Omega$.

This result holds true due to the result valid for the wave equation. Indeed, the general theory presented in an abstract form in [14], assures the observability of systems like $\dot{z} = iA_0 z$ using results available for systems of the form $\ddot{z} = -A_0 z$.

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