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A notion of polyconvex function on a surface suggested by nonlinear shell theory

Une notion de fonction polyconvexe sur une surface suggérée par la théorie des coques non linéaires

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ABSTRACT

Combining the definitions set forth by J. Ball in 1977 and by J. Ball, J.C. Currie, and P.J. Olver in 1981, we propose in this Note a definition of a "polyconvex function on a surface". When the surface is thought of as the middle surface of a nonlinearly elastic shell and the function as its stored energy function, we show that it is possible to assume in addition that this function is coercive for appropriate Sobolev norms and that it satisfies specific growth conditions that prevent the vectors of the covariant bases along the deformed middle surface to become linearly dependent, a condition that is the "surface analogue" of the orientation-preserving condition of J. Ball. We then show that a functional with such a polyconvex integrand is weakly lower semi-continuous, which eventually allows to establish the existence of minimizers.

RÉSUMÉ

Combinant les définitions proposées par J. Ball en 1977 et par J. Ball, J.C. Currie, et P.J. Olver en 1981, nous proposons dans cette Note une définition de « fonction polyconvexe sur une surface ». Quand la surface est vue comme la surface moyenne d'une coque non linéairement élastique et la fonction comme sa densité d'énergie, on montre qu'il est loisible de supposer en plus que cette fonction est coercive pour des normes de Sobolev convenables et qu'elle satisfait des conditions de croissance particulières qui empêchent les vecteurs des bases covariantes le long de la surface déformée de devenir linéairement dépendants, une condition qui est « l'analogue pour une surface » de la condition de préservation de l'orientation de J. Ball. On montre ensuite qu'une fonctionnelle avec un tel intégrande polyconvexe est faiblement semi-continue inférieurement, ce qui finalement permet d'établir l'existence de minimiseurs.

1. Introduction

Let ω be a *domain* in $\mathbb{R}^2$, i.e., a bounded, connected, open subset of $\mathbb{R}^2$ with a Lipschitz-continuous boundary γ, the set ω being locally on the same side of γ, let $\boldsymbol{\theta} \in \mathcal{C}^2(\overline{\omega}; \mathbb{R}^3)$ be an injective immersion, and let $\boldsymbol{a}_3 := \dfrac{\partial_1\boldsymbol{\theta} \wedge \partial_2\boldsymbol{\theta}}{|\partial_1\boldsymbol{\theta} \wedge \partial_2\boldsymbol{\theta}|}$, where

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$|\cdot|$ designates the Euclidean norm in $\mathbb{R}^3$; hence $\boldsymbol{a}_3 : \overline{\omega} \to \mathbb{R}^3$ is a unit normal vector field along the surface $\boldsymbol{\theta}(\overline{\omega})$. A large class of nonlinearly elastic shells with middle surface $\boldsymbol{\theta}(\overline{\omega})$ are modeled by a minimization problem of the following form (for an overview, see, e.g., Ciarlet [4] or Valid [16]): Find two vector fields $\boldsymbol{\varphi} : \overline{\omega} \to \mathbb{R}^3$ and $\boldsymbol{\zeta} : \overline{\omega} \to \mathbb{R}^3$ such that $\boldsymbol{u} = (\boldsymbol{\varphi}, \boldsymbol{\zeta})$ minimizes a functional of the form

$$I(\boldsymbol{v}) = \int_{\omega} W\big(y, \boldsymbol{v}(y), \nabla \boldsymbol{v}(y)\big)\,\mathrm{d}y - L(\boldsymbol{v}), \tag{1}$$

when the fields $\boldsymbol{v} = (\boldsymbol{\psi}, \boldsymbol{\eta})$ vary in a subset $\boldsymbol{U}$ of the Sobolev space $\boldsymbol{W}^{1,r}(\omega) \times \boldsymbol{W}^{1,r}(\omega)$ for some large enough $r > 1$. The function $L : \boldsymbol{U} \to \mathbb{R}$, which takes into account the *applied forces*, will be assumed for simplicity to be a continuous linear form over the space $\boldsymbol{W}^{1,r}(\omega) \times \boldsymbol{W}^{1,r}(\omega)$. The function W, which is the *stored energy function* of the elastic shell (as such, it depends on both the elastic material constituting the shell and on the mapping $\boldsymbol{\theta} : \overline{\omega} \to \mathbb{R}^3$ defining its middle surface $\boldsymbol{\theta}(\overline{\omega})$), is defined over an ad hoc subset of $\overline{\omega} \times \mathbb{R}^6 \times \mathbb{M}^{6\times 2}$ (see Section 2); the notation $\nabla \boldsymbol{v}(y)$ designates the 6×2 gradient matrix of $\boldsymbol{v} = (\boldsymbol{\psi}, \boldsymbol{\eta})$ at each point y of $\overline{\omega}$.

The unknown $\boldsymbol{\varphi} : \overline{\omega} \to \mathbb{R}^3$ represents the *position vector field* of the unknown deformed middle surface $\boldsymbol{\varphi}(\overline{\omega})$ of the shell and is assumed to satisfy a boundary condition of the form $\boldsymbol{\varphi} = \boldsymbol{\theta}$ on a subset γ_0 of γ with $\partial\gamma$-meas $\gamma_0 > 0$. In addition, the unknown $\boldsymbol{\varphi}$ is subjected to the constraint

$$\partial_1 \boldsymbol{\varphi} \wedge \partial_2 \boldsymbol{\varphi} \neq \boldsymbol{0} \quad \text{in } \overline{\omega}, \tag{2}$$

so as to insure that the tangent plane is well defined at each point of the deformed surface.

It should be emphasized that both unknowns $\boldsymbol{\varphi}$ and $\boldsymbol{\zeta}$ are identified here by means of their *Cartesian coordinates* over a given Euclidean frame of $\mathbb{R}^3$, and thus *not* by their covariant components over a moving basis, such as, e.g., the basis formed at each point of S by two contravariant vectors in the tangent plane and a unit normal vector.

In the well-known *Koiter model* [10], the unknown $\boldsymbol{\zeta}$ represents the unit normal vector field to the deformed middle surface $\boldsymbol{\varphi}(\overline{\omega})$ and is accordingly subjected to the *constraint* (which makes sense because of (2))

$$\boldsymbol{\zeta} = \frac{\partial_1 \boldsymbol{\varphi} \wedge \partial_2 \boldsymbol{\varphi}}{|\partial_1 \boldsymbol{\varphi} \wedge \partial_2 \boldsymbol{\varphi}|} \quad \text{in } \overline{\omega}. \tag{3}$$

Finally, $\boldsymbol{\zeta}$ is subjected to the boundary condition $\boldsymbol{\zeta} = \boldsymbol{a}_3$ on γ_0, which, together with $\boldsymbol{\varphi} = \boldsymbol{\theta}$ on γ_0, means that the shell is assumed to be *clamped on* $\boldsymbol{\theta}(\gamma_0)$.

Remark. In this "two-vector field" approach, the covariant components $b_{\alpha\beta}$ and $b_{\alpha\beta}(\boldsymbol{\varphi})$ of the second fundamental forms of the surfaces $\boldsymbol{\theta}(\overline{\omega})$ and $\boldsymbol{\varphi}(\overline{\omega})$ (the difference of which are the covariant components of the change of curvature tensor) are respectively written as $b_{\alpha\beta} = -\partial_\alpha \boldsymbol{\theta} \cdot \partial_\beta \boldsymbol{a}_3$ and $b_{\alpha\beta}(\boldsymbol{\varphi}) = -\partial_\alpha \boldsymbol{\varphi} \cdot \partial_\beta \boldsymbol{\zeta}$.

In the equally well-known Naghdi model [13], the unknown $\boldsymbol{\zeta}$ represents the rotated normal vector field along the deformed middle surface $\boldsymbol{\varphi}(\overline{\omega})$ and is assumed to have a strictly positive normal component at each point of $\boldsymbol{\varphi}(\overline{\omega})$; accordingly, $\boldsymbol{\zeta}$ is subjected to the *constraint* (which makes sense again because of (2))

$$(\partial_1 \boldsymbol{\varphi} \wedge \partial_2 \boldsymbol{\varphi}) \cdot \boldsymbol{\zeta} > 0 \quad \text{in } \overline{\omega}, \tag{4}$$

in addition to the constraint $|\boldsymbol{\zeta}| = 1$ in $\overline{\omega}$. Finally, $\boldsymbol{\zeta}$ is subjected to the boundary condition $\boldsymbol{\zeta} = \boldsymbol{a}_3$ on γ_0, which, together with $\boldsymbol{\varphi} = \boldsymbol{\theta}$ on γ_0, again means that the shell is assumed to be clamped on $\boldsymbol{\theta}(\gamma_0)$.

It should be noted that the stored energy function of Koiter's model depends on $\nabla \boldsymbol{v}$, but *not* on $\boldsymbol{v}$ itself. The reason that $\boldsymbol{v}$ appears in (1) is that it will be needed, by means of relation (2) or (4), so as to prevent the covariant basis vectors $\partial_1 \boldsymbol{\varphi}$ and $\partial_2 \boldsymbol{\varphi}$ of the tangent plane to the deformed middle surface to become linearly dependent. In this sense, *relation* (4) *is the "surface analog" of the orientation preserving condition of* Ball [1].

As is well known, there is to this date no available existence theory for the nonlinear Koiter or Naghdi models. As a way to remedy this situation, the purpose of this Note is to give sufficient conditions, bearing on the integrand W (appearing in integrals of the form (1)), but more general than those of Koiter's or Naghdi's models, guaranteeing the *existence of minimizers* over appropriate sets $\boldsymbol{U}$ of *admissible fields* which, in addition to the standard boundary conditions, will incorporate constraints such as (2) and (3), or (4). To this end, an essential use will be made of a notion of *polyconvexity on a surface*, which extends that proposed by Ball [1] for three-dimensional elasticity problems.

Remark. By contrast with Koiter's and Naghdi's models, which incorporate both bending and flexural effects, the existence of solutions to the nonlinear "membrane" and "flexural" shell models, respectively obtained by Le Dret and Raoult [11] and Friesecke, James, Mora, and Müller [9] as Γ-limits of three-dimensional nonlinear elasticity, is *ipso facto* guaranteed (it can be also established directly for the flexural model; cf. Ciarlet and Coutand [5]).

Complete proofs and examples of stored energy functions with the above properties will appear elsewhere; cf. [7].

2. A definition of polyconvexity on a surface

Let $\mathbb{M}^3$ designate the space of 3×3 matrices and let $\mathbb{M}^3_+ := \{\boldsymbol{F} \in \mathbb{M}^3;\ \det \boldsymbol{F} > 0\}$. In a landmark paper [1], John Ball has introduced the notion of *polyconvex stored energy functions* for three-dimensional nonlinearly elastic materials with a domain $\Omega \subset \mathbb{R}^3$ as their reference configuration: A stored energy function $W : \overline{\Omega} \times \mathbb{M}^3_+ \to \mathbb{R}$ is *polyconvex* if, for almost all $x \in \Omega$, there exists a convex function $\mathbb{W}(x, \cdot) : \mathbb{M}^3 \times \mathbb{M}^3 \times]0, \infty[\to \mathbb{R}$ such that

$$W(x, \boldsymbol{F}) = \mathbb{W}(x, \boldsymbol{F}, \mathbf{Cof}\,\boldsymbol{F}, \det \boldsymbol{F}) \quad \text{for all } \boldsymbol{F} \in \mathbb{M}^3_+,$$

where the notation $\mathbf{Cof}\,\boldsymbol{F}$ designates the cofactor matrix of $\boldsymbol{F}$. The restriction that $W(x, \cdot)$ be only defined on $\mathbb{M}^3_+$ is, together with the assumption that $W(x, \boldsymbol{F}) \to \infty$ as $\det \boldsymbol{F} \to 0^+$, intended to take care of the *orientation-preserving condition* that any physically realistic deformation of a three-dimensional body should satisfy.

Observing that the components of $\boldsymbol{F}$, $\mathbf{Cof}\,\boldsymbol{F}$, and $\det \boldsymbol{F}$ are nothing but all the distinct determinants that can be extracted from a matrix $\boldsymbol{F} \in \mathbb{M}^3$, Ball, Currie, and Olver [2] proposed later on a related definition, directly adapted to minimization problems for functionals of the form $\int_\Omega W(x, \nabla^k \boldsymbol{v}(x))\,dx$, where, for some $k \geqslant 1$, $\nabla^k \boldsymbol{v}$ denotes the set of all partial derivatives of $\boldsymbol{v}$ of order k, Ω is a domain of $\mathbb{R}^p$, and the vector fields $\boldsymbol{v}$ map Ω into $\mathbb{R}^q$ for some $p \geqslant 1$ and $q \geqslant 1$; note that, in [2], *there is no longer any restriction* (such as an orientation-preserving condition) on the *admissible vector fields* $\boldsymbol{v}$: In this case, the integrand is said to be *polyconvex* if there exists a *convex* function $\mathbb{W}(x, \cdot)$ such that for almost all $x \in \Omega$,

$$W\big(x, \nabla^k \boldsymbol{v}(x)\big) = \mathbb{W}\big(x, \boldsymbol{J}\big(\nabla^k \boldsymbol{v}(x)\big)\big) \quad \text{for all } \nabla^k \boldsymbol{v},$$

where $\boldsymbol{J}(\nabla^k \boldsymbol{v})$ denote the set of all determinants (in the sense of Eq. (1.3) in [2]) that can be extracted from $\nabla^k \boldsymbol{v}$.

We now combine these two notions of polyconvexity and adapt them to the particular functionals (cf. (1)) and constraints (cf. (2) to (4)) introduced in Section 1.

Let $\mathbb{F}$ denote the vector space (of dimension 12) of all matrices $\boldsymbol{F} = (f_{k\alpha})$, $1 \leqslant k \leqslant 6$, $1 \leqslant \alpha \leqslant 2$ (k is the row index), let $\mathbb{G}$ denote the vector space (of dimension 15) of all *distinct* 2×2 determinants

$$J_{k\ell}(\boldsymbol{F}) := (f_{k1} f_{\ell 2} - f_{k2} f_{\ell 1}), \quad 1 \leqslant k, \ell \leqslant 6,$$

that can be extracted from a matrix $\boldsymbol{F} \in \mathbb{F}$ and let the mapping $\boldsymbol{J} : \mathbb{F} \to \mathbb{G}$ be defined by

$$\boldsymbol{J}(\boldsymbol{F}) := \big(J_{kl}(\boldsymbol{F})\big)_{1 \leqslant k < \ell \leqslant 6} \in \mathbb{G} \quad \text{for each } \boldsymbol{F} \in \mathbb{F}.$$

Finally, for each *nonzero* vector $\boldsymbol{q} = (q_i) \in \mathbb{R}^3$, let

$$\mathbb{F}_+(\boldsymbol{q}) := \big\{\boldsymbol{F} \in \mathbb{F};\ J_{23}(\boldsymbol{F}) q_1 + J_{31}(\boldsymbol{F}) q_2 + J_{12}(\boldsymbol{F}) q_3 > 0\big\} \quad \text{and} \quad \mathbb{G}_+(\boldsymbol{q}) := \boldsymbol{J}\big(\mathbb{F}_+(\boldsymbol{q})\big). \tag{5}$$

Note that $\mathbb{G}_+(\boldsymbol{q})$ is a *convex* subset of $\mathbb{G}$.

Given any open subset ω of $\mathbb{R}^2$, we then say that a function

$$W : \omega \times \big(\mathbb{R}^3 \times \mathbb{R}^3\big) \times \mathbb{F} \to \mathbb{R} \cup \{+\infty\}$$

is *polyconvex* if, for almost all $y \in \omega$, for each $\boldsymbol{p} \in \mathbb{R}^3$, and for each *nonzero* $\boldsymbol{q} \in \mathbb{R}^3$, there exist a real-valued *convex* function

$$\mathbb{W}\big(y, (\boldsymbol{p}, \boldsymbol{q}), \cdot, \cdot\big) : (\boldsymbol{F}, \boldsymbol{G}) \in \mathbb{F} \times \mathbb{G}_+(\boldsymbol{q}) \to \mathbb{W}\big(y, (\boldsymbol{p}, \boldsymbol{q}), \boldsymbol{F}, \boldsymbol{G}\big) \in \mathbb{R}$$

such that

$$W\big(y, (\boldsymbol{p}, \boldsymbol{q}), \boldsymbol{F}\big) = \mathbb{W}\big(y, (\boldsymbol{p}, \boldsymbol{q}), \boldsymbol{F}, \boldsymbol{J}(\boldsymbol{F})\big) \quad \text{for all } \boldsymbol{F} \in \mathbb{F}_+(\boldsymbol{q}).$$

The above definition is indeed that of a *polyconvexity on a surface*: As will be shown elsewhere (see [7]), if a stored energy function is polyconvex for one parametrization $\boldsymbol{\theta}$ of a surface S (recall that the stored energy function depends on $\boldsymbol{\theta}$), then it remains so for any other parametrization of S. Hence this notion of polyconvexity is indeed *intrinsic*.

3. Existence theorems

The next existence theorem is adapted to the minimization problems corresponding to *Naghdi-like shells* whose middle surface is defined as $\boldsymbol{\theta}(\overline{\omega})$ for some immersion $\boldsymbol{\theta} \in \mathcal{C}^2(\overline{\omega}; \mathbb{R}^3)$ (cf. Section 1). The notations $\mathbb{F}, \mathbb{G}, \mathbb{F}_+(\boldsymbol{q}), \mathbb{G}_+(\boldsymbol{q}), \boldsymbol{J}, J_{k\ell}(\boldsymbol{F})$ have been defined in Section 2; the notation $\|\cdot\|$ designates any given norm in a finite-dimensional space.

Theorem 1. *Let ω be a domain in $\mathbb{R}^2$ and let $W : \omega \times (\mathbb{R}^3 \times \mathbb{R}^3) \times \mathbb{F} \to \mathbb{R} \cup \{+\infty\}$ be a stored energy function with the following properties*:

(a) **Polyconvexity**: *For almost all $y \in \omega$, for each $\boldsymbol{p} \in \mathbb{R}^3$, and for each nonzero $\boldsymbol{q} \in \mathbb{R}^3$, there exists a convex function*

$$\mathbb{W} : \big(y, (\boldsymbol{p}, \boldsymbol{q}), \cdot, \cdot\big) : (\boldsymbol{F}, \boldsymbol{G}) \in \mathbb{F} \times \mathbb{G}_+(\boldsymbol{q}) \to \mathbb{W}\big(y, (\boldsymbol{p}, \boldsymbol{q}), \boldsymbol{F}, \boldsymbol{G}\big) \in \mathbb{R}$$

such that

$$W\big(y, (\boldsymbol{p}, \boldsymbol{q}), \boldsymbol{F}\big) = \mathbb{W}\big(y, (\boldsymbol{p}, \boldsymbol{q}), \boldsymbol{F}, \boldsymbol{J}(\boldsymbol{F})\big) \quad \textit{for all } \boldsymbol{F} \in \mathbb{F}_+(\boldsymbol{q}). \tag{6}$$

(b) **Measurability and continuity**: *For all $((\boldsymbol{p}, \boldsymbol{q}), \boldsymbol{F}, \boldsymbol{G}) \in (\mathbb{R}^3 \times \mathbb{R}^3) \times \mathbb{F} \times \mathbb{G}$, the function $\tilde{\mathbb{W}}(\cdot, (\boldsymbol{p}, \boldsymbol{q}), \boldsymbol{F}, \boldsymbol{G}) : \omega \to \mathbb{R} \cup \{+\infty\}$ is measurable. Besides, for almost all $y \in \omega$, the function $\tilde{\mathbb{W}}(y, (\cdot,\cdot), \cdot, \cdot) : ((\boldsymbol{p}, \boldsymbol{q}), \boldsymbol{F}, \boldsymbol{G}) \in (\mathbb{R}^3 \times \mathbb{R}^3) \times \mathbb{F} \times \mathbb{G} \to \tilde{\mathbb{W}}(y, (\boldsymbol{p}, \boldsymbol{q}), \boldsymbol{F}, \boldsymbol{G}) \in \mathbb{R} \cup \{+\infty\}$ defined by*

$$\tilde{\mathbb{W}}\big(y, (\boldsymbol{p}, \boldsymbol{q}), \boldsymbol{F}, \boldsymbol{G}\big) = \mathbb{W}\big(y, (\boldsymbol{p}, \boldsymbol{q}), \boldsymbol{F}, \boldsymbol{G}\big) \quad \textit{for all } \big((\boldsymbol{p}, \boldsymbol{q}), \boldsymbol{F}, \boldsymbol{G}\big) \in \big(\mathbb{R}^3 \times \big(\mathbb{R}^3 \setminus \{\boldsymbol{0}\}\big)\big) \times \mathbb{F} \times \mathbb{G}_+(\boldsymbol{q}),$$

and by

$$\tilde{\mathbb{W}}\big(y, (\boldsymbol{p}, \boldsymbol{q}), \boldsymbol{F}, \boldsymbol{G}\big) = +\infty \quad \textit{otherwise},$$

is continuous.

(c) **Coerciveness**: *There exist constants $\alpha > 0$, $\beta \in \mathbb{R}$, $r > 4/3$, $s > 1$, and a function $g \in L^1(\omega)$ such that, for almost all $y \in \omega$ and all $((\boldsymbol{p}, \boldsymbol{q}), \boldsymbol{F}) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{F}$,*

$$W\big(y, (\boldsymbol{p}, \boldsymbol{q}), \boldsymbol{F}\big) \geqslant \alpha\big(\|\boldsymbol{F}\|^r + \big\|\boldsymbol{J}(\boldsymbol{F})\big\|^s\big) + \beta\big(|\boldsymbol{p}| + |\boldsymbol{q}|\big) + g(y). \tag{7}$$

Given a $\mathrm{d}\gamma$-measurable subset γ_0 of $\gamma := \partial\omega$ with $\mathrm{d}\gamma$-meas $\gamma_0 > 0$, let

$$\begin{aligned} \boldsymbol{U} := \big\{ &\boldsymbol{v} = (\boldsymbol{\psi}, \boldsymbol{\eta}) \in \boldsymbol{W}^{1,r}(\omega) \times \boldsymbol{W}^{1,r}(\omega);\ \boldsymbol{J}(\nabla\boldsymbol{v}) \in L^s(\omega; \mathbb{G}), \\ &(\partial_1\boldsymbol{\psi} \wedge \partial_2\boldsymbol{\psi}) \cdot \boldsymbol{\eta} > 0 \textit{ and } |\boldsymbol{\eta}| = 1 \textit{ a.e. in } \omega,\ \boldsymbol{\psi} = \boldsymbol{\theta} \textit{ and } \boldsymbol{\eta} = \boldsymbol{a}_3\ \mathrm{d}\gamma\textit{-a.e. on } \gamma_0 \big\}. \end{aligned} \tag{8}$$

Finally, let $L : \boldsymbol{W}^{1,r}(\omega) \times \boldsymbol{W}^{1,r}(\omega) \to \mathbb{R}$ be a continuous linear form.

Assume that

$$\inf_{\boldsymbol{v}\in\boldsymbol{U}} I(\boldsymbol{v}) < +\infty, \quad \textit{where } I(\boldsymbol{v}) := \int_\omega W\big(y, \boldsymbol{v}(y), \nabla\boldsymbol{v}(y)\big)\,\mathrm{d}y - L(\boldsymbol{v}) \quad \textit{for each } \boldsymbol{v} \in \boldsymbol{U}. \tag{9}$$

Then there exists at least one vector field $\boldsymbol{u} = (\boldsymbol{\varphi}, \boldsymbol{\zeta})$ such that

$$\boldsymbol{u} \in \boldsymbol{U} \quad \textit{and} \quad I(\boldsymbol{u}) = \inf_{\boldsymbol{v}\in\boldsymbol{U}} I(\boldsymbol{v}). \tag{10}$$

Sketch of proof. The proof, which is inspired by those of Theorems 7.3 and 7.6 in [1], comprises the following main steps (for details, see [7]).

First, it easily follows from (b) and (c) that the integrals $\int_\omega W(y, \boldsymbol{v}(y), \nabla\boldsymbol{v}(y))\,\mathrm{d}y$ are well defined in $\mathbb{R} \cup \{+\infty\}$ for all $\boldsymbol{v} \in \boldsymbol{U}$. Besides, thanks to the Poincaré inequality and to the coerciveness assumption (c), there exist constants c and d such that

$$c > 0 \quad \text{and} \quad I(\boldsymbol{v}) \geqslant c\big(\|\nabla\boldsymbol{v}\|^r_{0,r,\omega} + \big\|\boldsymbol{J}(\nabla\boldsymbol{v})\big\|^s_{0,s,\omega}\big) + d \quad \text{for all } \boldsymbol{v} \in \boldsymbol{U},$$

where $\|\cdot\|_{0,r,\omega}$ designates the norm of vector-valued or matrix-valued functions with components in $L^r(\omega)$.

Next, let $(\boldsymbol{v}^k)$ be an *infimizing sequence* of the functional I. The sequence $(\boldsymbol{v}^k, \boldsymbol{J}(\nabla\boldsymbol{v}^k))$ being thus bounded in the reflexive Banach space $W^{1,r}(\omega; \mathbb{R}^6) \times L^s(\omega; \mathbb{G})$, there exists a subsequence $(\boldsymbol{v}^\ell, \boldsymbol{J}(\nabla\boldsymbol{v}^\ell))_\ell$ that weakly converges to an element $(\boldsymbol{u}, \boldsymbol{H}) \in W^{1,r}(\omega; \mathbb{R}^6) \times L^s(\omega; \mathbb{G})$; a *compensated compactness argument* (cf. Murat [12] and Tartar [15]) then shows that $\boldsymbol{H} = \boldsymbol{J}(\boldsymbol{u})$.

It remains to show that $\boldsymbol{u} = (\boldsymbol{\varphi}, \boldsymbol{\zeta}) \in \boldsymbol{W}^{1,r}(\omega) \times \boldsymbol{W}^{1,r}(\omega)$ belongs to $\boldsymbol{U}$, i.e., that

$$(\partial_1\boldsymbol{\varphi} \wedge \partial_2\boldsymbol{\varphi}) \cdot \boldsymbol{\zeta} > 0 \quad \text{and} \quad |\boldsymbol{\zeta}| = 1 \quad \text{a.e. in } \omega$$

(that the boundary conditions $\boldsymbol{\varphi} = \boldsymbol{\theta}$ and $\boldsymbol{\zeta} = \boldsymbol{a}_3$ on γ_0 are satisfied is easily seen), and that $I(\boldsymbol{u}) = \inf_{\boldsymbol{v}\in\boldsymbol{U}} I(\boldsymbol{v})$. Using Theorem 5.4 in [2] or Theorem 3.23 in [8] (either theorem may be viewed as a generalized *Fatou lemma* adapted to integrands that are not only functions of $\nabla\boldsymbol{v}$, but also of $\boldsymbol{v}$), one next shows that, thanks to the assumptions (a) and (b),

$$\begin{aligned} \inf_{\boldsymbol{v}\in\boldsymbol{U}} I(\boldsymbol{v}) &= \liminf_{\ell\to\infty}\bigg\{\int_\omega \tilde{\mathbb{W}}\big(y, \boldsymbol{v}^\ell(y), \nabla\boldsymbol{v}^\ell(y), \boldsymbol{J}\big(\nabla\boldsymbol{v}^\ell(y)\big)\big)\,\mathrm{d}y - L\big(\boldsymbol{v}^\ell\big)\bigg\} \\ &\geqslant \int_\omega \tilde{\mathbb{W}}\big(y, \boldsymbol{u}(y), \nabla\boldsymbol{u}(y), \boldsymbol{J}\big(\nabla\boldsymbol{u}(y)\big)\big)\,\mathrm{d}y - L(\boldsymbol{u}), \end{aligned}$$

where the function $\tilde{\mathbb{W}}$ is that defined in (b). Since $\inf_{\boldsymbol{v}\in\boldsymbol{U}} I(\boldsymbol{v}) < \infty$ by assumption, it follows that $\tilde{\mathbb{W}}(y, \boldsymbol{u}(y), \nabla\boldsymbol{u}(y), \boldsymbol{J}(\nabla\boldsymbol{u}(y))) < \infty$ for almost all $y \in \omega$, which eventually shows that $\boldsymbol{u} \in \boldsymbol{U}$ and that $I(\boldsymbol{u}) = \inf_{\boldsymbol{v}\in\boldsymbol{U}} I(\boldsymbol{v})$. □

Remark. The restriction that, for each nonzero $\boldsymbol{q} \in \mathbb{R}^3$, the last argument in the function $\mathbb{W}$ belong to the set $\mathbb{G}_+(\boldsymbol{q})$ allows the unknowns $\boldsymbol{\varphi}$ and $\boldsymbol{\zeta}$ to satisfy the constraint (4) (see Theorem 1), which, as already noted, is the *surface analog of the orientation-preserving condition* of Ball [1]. More specifically, it is easily seen that assumption (b) of Theorem 1 implies that, for almost all $y \in \omega$ and all $(\boldsymbol{p}, \boldsymbol{q}) \in \mathbb{R}^3 \times \mathbb{R}^3$,

$$W\big(y, (\boldsymbol{p}, \boldsymbol{q}), \boldsymbol{F}\big) = +\infty \quad \text{if and only if} \quad J_{23}(\boldsymbol{F})q_1 + J_{31}(\boldsymbol{F})q_2 + J_{12}(\boldsymbol{F})q_3 \leqslant 0,$$
$$W\big(y, (\boldsymbol{p}, \boldsymbol{q}), \boldsymbol{F}\big) \to +\infty \quad \text{if} \quad J_{23}(\boldsymbol{F})q_1 + J_{31}(\boldsymbol{F})q_2 + J_{12}(\boldsymbol{F})q_3 \to 0^+.$$

A similar existence theorem (see [7]) can also be established for the minimization problem associated with *Koiter-like shells*. In this case, the set of admissible fields takes the form

$$\boldsymbol{U} := \Big\{\boldsymbol{v} = (\boldsymbol{\psi}, \boldsymbol{\eta}) \in \boldsymbol{W}^{1,r}(\omega) \times \boldsymbol{W}^{1,r}(\omega);\ \boldsymbol{J}(\nabla\boldsymbol{v}) \in L^s(\omega; \mathbb{G}),$$
$$\partial_1\boldsymbol{\psi} \wedge \partial_2\boldsymbol{\psi} \neq \boldsymbol{0} \text{ and } \boldsymbol{\eta} = \frac{\partial_1\boldsymbol{\psi} \wedge \partial_2\boldsymbol{\psi}}{|\partial_1\boldsymbol{\psi} \wedge \partial_2\boldsymbol{\psi}|} \text{ a.e. in } \omega,\ \boldsymbol{\psi} = \boldsymbol{\theta} \text{ and } \boldsymbol{\eta} = \boldsymbol{a}_3\, \mathrm{d}\gamma\text{-a.e. on } \gamma_0\Big\}. \tag{11}$$

4. Concluding remarks

It should be clear that the existence theorems announced here do *not* apply to the original Koiter's and Naghdi's models, if only because their stored energy functions are neither polyconvex nor orientation-preserving.

Hence the next objective will consist in seeking whether their stored energy functions can be appropriately modified, so that they do not only become polyconvex and orientation-preserving and satisfy *ad hoc* growth conditions, but in addition they coincide "to within the first order" with the original stored energy functions. The derivation of such modified stored energy functions would be in the spirit of Ciarlet and Geymonat [6] (see also [3, Section 4.10]), who proposed a polyconvex, orientation-preserving, and coercive stored energy function that coincides to within the first order, i.e., "for small strains", with the stored energy function of a *Saint Venant–Kirchhoff material* (see [3, Section 3.9]). This material is often used in numerical simulations of nonlinearly elastic three-dimensional structures (just like Koiter's or Naghdi's models for the numerical simulations of nonlinearly elastic shells). Its main advantage is its simplicity, but it otherwise suffers from serious drawbacks: Just like Koiter's or Naghdi's, its stored energy function is neither polyconvex (Raoult [14]) nor orientation-preserving.

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