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Partial Differential Equations

Mathematical and numerical modeling of wave propagation in fractal trees

Modélisation mathématique et numérique de la propagation d'ondes dans des arbres fractals

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ABSTRACT

We propose and analyze a mathematical model for wave propagation in infinite trees with self-similar structure at infinity. The emphasis is put on the construction and approximation of transparent boundary conditions.

RÉSUMÉ

Nous proposons et analysons un modèle mathématique pour la propagation d'ondes dans des arbres infinis qui sont auto-similaires à l'infini. L'accent est mis sur la construction et l'approximation de conditions aux limites transparentes.

Version française abrégée

Nous nous intéressons à la propagation d'ondes dans un arbre infini (cf. Fig. 1). Le modéle mathématique consiste à résoudre des équations d'ondes 1D sur chaque branche de l'arbre (1), couplées par des conditions de transmission de type Kirchhoff en chaque noeud (2). Il s'interprète comme une équation des ondes « à poids » μ le long de l'arbre (3). Ce problème, qui peut paraitre très académique, a été motivé par l'étude de la propagation acoustique dans un poumon humain qui peut être vu comme un arbre dyadique ([4,5], Fig. 2). La difficulté essentielle est le caractére « infini » de l'arbre (même si, comme nous le supposerons, il est géométriquement fini) et en particulier la définition de la condition aux limites « à l'infini » dans l'arbre. Numériquement, se pose la question de tronquer le domaine de calcul à l'aide d'une condition artificielle à distance finie. Dans cette note, nous traitons ces questions dans le cas où l'arbre est supposé p-adique (p entier supérieur à 2) et autosimilaire au delà d'une certaine génération, la fonction μ possédant également des propriétés d'auto-similarité analogues.

Les caractéristiques auto-similaires de la géométrie et du poids μ se décrivent à l'aide de deux suites de nombres positifs α (pour la géométrie) et ν (pour la fonction μ), comme décrit au début de la section 3 :

$$\alpha = (\alpha_1, \dots, \alpha_p), \qquad \nu = (\nu_1, \dots, \nu_p), \quad 0 < \alpha_j < 1,\ 0 < \mu_j$$

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l'inégalité $\alpha_j < 1$ assurant que l'arbre est géométriquement fini. Le choix de la condition aux limites «à l'infini» (Dirichlet ou Neumann) passe par la formulation faible du problème (voir (9)) et par conséquent par un cadre fonctionnel adapté s'appuyant sur la construction d'espaces de Sobolev à poids sur l'arbre (voir section 3). Deux nombres critiques jouent un rôle fondamantal dans l'analyse du problème : $M_1(\alpha, \nu)$ et $M_2(\alpha, \nu)$ définis par (6). En particulier lorsque $M_1(\alpha, \nu) < 1 < M_2(\alpha, \nu)$, ce que nous supposons dans la suite, on est assuré que les problèmes de Neumann et Dirichlet sont distincts et que l'opérateur associé au problème d'évolution a un spectre purement discret (Théorème 3.1).

La condition transparente associée à la troncature d'un sous-arbre autosimilaire, s'exprime au travers d'un opérateur (de type DtN) de convolution en temps dont le symbole $\Lambda(\omega)$ ($\omega \in \mathbb{R}$ désigne la fréquence) est solution de l'équation fonctionnelle quadratique (14). Plus précisément, on distingue les problèmes de Dirichlet et Neumann en sélectionnant la solution de cette équation qui prend une valeur prescrite à fréquence nulle (voir Théorème 4.2 et formule (16)). La formule permet de mettre au point un algorithme de calcul de la fonction $\Lambda(\omega)$ (voir Fig. 4 pour une illustration).

Lorsqu'on coupe le domaine de calcul assez loin dans chaque sous-arbre auto-similaire, on réalise qu' il suffit de disposer d'une bonne approximation de $\Lambda(\omega)$ à basse fréquence pour construite une bonne condition aux limites. Ceci permet, à partir du développement de Taylor de $\Lambda(\omega)$ à l'origine (Lemme 5.1), de construire des conditions approchées (20) dont la stabilité est garantie, si $M_1(\alpha, \nu) < 1 < M_2(\alpha, \nu)$, par une technique d' énergie.

1. Introduction

We are interested in studying the wave propagation on a tree with many generations (we consider a tree as a network with the additional notion of successive generations, as a genealogical tree—cf. Fig. 1), seen as an infinite tree, the idea being that the infinite tree should be easier than the large finite tree! The mathematical model consists in solving the 1D wave equations on each branch of the tree, these being coupled by node conditions of the Kirchhoff type and can be reinterpreted as a weighted wave equation on the tree involving a positive weight function μ (see Section 2). This may seem to be a very academic problem but this problem has been motivated by the application to the human lung (for studying fluid flow) that can be seen, modulo reasonable approximations, as such a (dyadic) tree [4,5] (see also Fig. 2). The present work is for a part an extension of [4] to wave propagation. It is also closely connected (although independent) to a series of works by Y. Achdou and his collaborators about PDE's in fractal domains (see for instance [1]). As the reader expects, the main difficulty in the modeling is the fact that the tree is infinite even though, as we shall assume, it is geometrically finite. From the mathematical point of view, the definition of a good boundary condition "at infinity" is a delicate question. From the numerical point of view, the difficulty is to restrict the calculations to a finite number of generations by replacing each eliminated sub-tree by a DtN like transparent condition. Both questions will be treated in this paper if one assumes that the tree has some additional structure. First, from a given generation n, the sub-trees are contractive self-similar p-adic trees: at any generation, each branch is divided into p branches which are homothetic to the generating branch with a ratio less than 1, independent of the generation. Second, the weight function μ has corresponding self-similarity properties.

2. The retained mathematical model

We consider here a tree $\mathcal{T}$ in $\mathbb{R}^d$ whose branches (or edges), denoted $\{\Gamma_i,\ i \in \mathcal{I}\}$, are straight segments. We denote by s_i the abscissa along Γ_i and use the notation s for the collection of the s_i's, which can be seen as a generalized coordinate along $\mathcal{T}$. If $v(s)$ denotes a function defined on $\mathcal{T}$, v_i will denote its restriction to Γ_i.

By definition, a solution of the wave equation on $\mathcal{T}$ will be a function $u(s,t): \mathcal{T} \times \mathbb{R}^+ \to \mathbb{R}$ satisfying:

$$\partial_t^2 u_i - \partial_{s_i}^2 u_i = 0, \quad \text{on } \Gamma_i \times \mathbb{R}^+,\ \forall i \in \mathcal{I} \tag{1}$$

and at each node $M_\ell = \bigcap_{i \in \mathcal{I}_\ell} \Gamma_i$ of $\mathcal{T}$

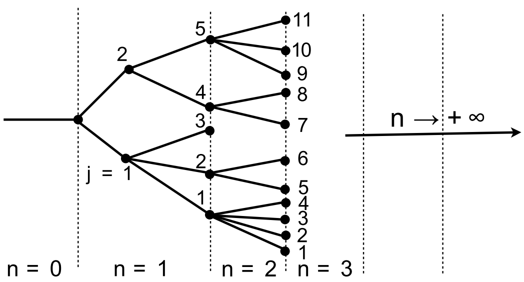

Fig. 1. An infinite tree.

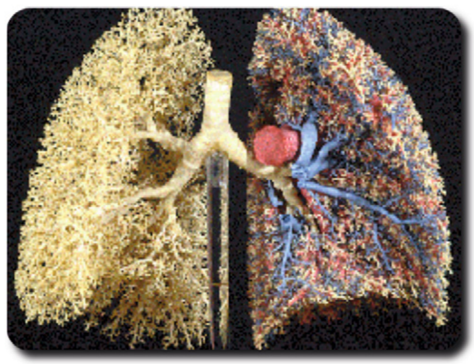

Fig. 2. Molding of a human lung (from [5]).

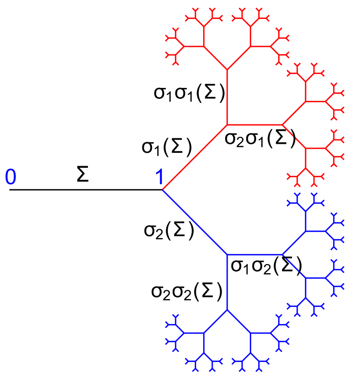

Fig. 3. A dyadic tree in $\mathbb{R}^2$ ($\alpha_1 = \alpha_2 = 0.6$).

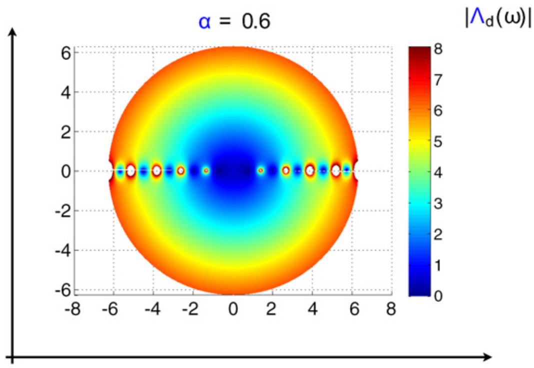

Fig. 4. The modulus of $\Lambda_\mathfrak{d}(\omega)$ in the complex plane.

$$\begin{cases} u_i(M_\ell,t) = u_{i'}(M_\ell,t), \quad \forall (i,i') \in \mathcal{I}_\ell^2 & \text{(i)}, \\ \displaystyle\sum_{i\in\mathcal{I}_\ell} \mu_i\, \partial_{s_i} u_i(M_\ell,t) = 0 & \text{(ii)}. \end{cases} \tag{2}$$

(2)(i) is nothing but the continuity of u at M_l while (2)(ii) is a generalized Kirchhoff condition. The μ_i's are supposed to be strictly positive numbers and, in (2)(ii), we have implicitly considered that s_i is (locally) oriented towards M_ℓ (see [3,2] for various justifications). Eqs. (1) and (2) can be collected in a single equation, using a (very intuitive) notion of distributional derivative along $\mathcal{T}$, as follows ($\mu : \mathcal{T} \to \mathbb{R}_*^+$ is the function that takes the value μ_i on Γ_i)

$$\mu \partial_t^2 u - \partial_s(\mu \partial_s u) = 0, \quad \text{on } \mathcal{T} \times \mathbb{R}^+. \tag{3}$$

3. Theoretical aspects of the problem

We define our reference self-similar p-adic tree $\mathcal{T}$ from a generating segment $\Sigma \equiv [\mathbf{0}, \mathbf{1}]$ of length 1, and p contractant similitudes $\{\sigma_i,\ 1 \leqslant i \leqslant p\}$ of ratio $0 < \alpha_i < 1$ such that $\sigma_i(\mathbf{0}) = (\mathbf{1})$ for any i. $\mathcal{T}$ is defined by induction via its successive generations (see Fig. 3 for an illustration when $p = 2$):

$$\mathcal{G}^0 = \Sigma, \qquad \mathcal{G}^{n+1} = \bigcup_{j=1}^{p} \sigma_j(\mathcal{T}^n), \quad \mathcal{T} = \bigcup \mathcal{G}^n. \tag{4}$$

To get a self-similar wave equation, we shall assume that the function μ also has self-similarity properties, i.e. that there exists $\{\nu_i > 0,\ 1 \leqslant i \leqslant p\}$ such that

$$\forall s \in \mathcal{T}, \qquad \mu\big(\sigma_i(s)\big) = \nu_i \mu(s), \quad \forall 1 \leqslant i \leqslant p. \tag{5}$$

Two critical numbers play a role in our analysis:

$$M_1(\alpha,\nu) := \sum_{j=1}^{p} \nu_i \alpha_i < M_2(\alpha,\nu) := \sum_{j=1}^{p} \frac{\nu_i}{\alpha_i}. \tag{6}$$

In order to construct a transparent boundary condition corresponding to the self-similar tree $\mathcal{T}$, we need to solve (3) completed by zero initial conditions, a non-homogeneous Dirichlet condition at the origin $\mathbf{0}$ of $\mathcal{T}$:

$$u(\mathbf{0}, t) = u_0(t), \tag{7}$$

and a condition "at infinity". This condition will be imposed via the weak formulation of the problem. To do so we first introduce the weighted L^2-norm on $\mathcal{T}$:

$$\|u\|_{L^2_\mu}^2 = \int_{\mathcal{T}} \mu |u|^2 := \sum_{i\in\mathcal{I}} \mu_i \int_{\Gamma_i} |u_i|^2,$$

and define the weighted Sobolev spaces on $\mathcal{T}$ ($\partial_s u$ denotes the function whose restriction to Γ_i is $\partial_{s_i} u_i$):

$$\mathrm{H}^1_\mu(\mathcal{T}) = \big\{ v \in C^0(\mathcal{T}) / \|u\|^2_{L^2_\mu} + \|\partial_s u\|^2_{L^2_\mu} < \infty \big\}$$

and the closed subspace of functions that "vanish at infinity"

$$\mathrm{H}^1_{\mu,0}(\mathcal{T}) = \overline{\mathrm{H}^1_{\mu,c}(\mathcal{T})}, \tag{8}$$

where $\mathrm{H}^1_{\mu,c}(\mathcal{T})$ is made of compactly supported functions: $\mathrm{H}^1_{\mu,c}(\mathcal{T}) = \{v \in \mathrm{H}^1_\mu(\mathcal{T})/\exists N,\ v = 0 \text{ on } \mathcal{G}_n \text{ if } n \geqslant N\}$. We define the solution $u_\mathfrak{d}$ (resp. $u_\mathfrak{n}$) of (3, 7) which satisfies the Dirichlet (resp. Neumann) condition at infinity, as the solution of the problem ($\mathfrak{a} = \mathfrak{d}$ or $\mathfrak{n}$): find $u_\mathfrak{a} : \mathbb{R}^+ \to V_\mathfrak{a}$ such that (7) is satisfied and for any $v \in V_\mathfrak{a}$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} \int_\mathcal{T} \mu u_\mathfrak{a}(t) v + \int_\mathcal{T} \mu \partial_s u_\mathfrak{a}(t) \partial_s v = 0, \tag{9}$$

with $V_\mathfrak{d} = \mathrm{H}^1_{\mu,0}(\mathcal{T})$ and $V_\mathfrak{n} = \mathrm{H}^1_\mu(\mathcal{T})$. This formulation gives a precise meaning to (3) and contains implicitly (2).

Theorem 3.1. *One has $V_\mathfrak{d} \subsetneq V_\mathfrak{n}$. and therefore $u_\mathfrak{d} \neq u_\mathfrak{n}$, if and only if $M_2(\alpha, \nu) > 1$. Moreover, the embedding $V_\mathfrak{n} \subset L^2_\mu$ is compact as soon as $M_1(\alpha, \nu) < 1$.*

We shall assume in the sequel that $M_2(\alpha, \nu) > 1$ and $M_1(\alpha, \nu) < 1$ are satisfied so that the Neumann and Dirichlet problems are different and are associated to a pure discrete spectrum.

4. Construction of transparent boundary conditions

From (9), we introduce the associated DtN operator:

$$\Lambda_\mathfrak{a} : u_0(t) \longrightarrow (\Lambda_\mathfrak{a} u_0)(t) = -\partial_s u_\mathfrak{a}(\mathbf{0}, t) \tag{10}$$

which is a convolution operator whose kernel can be characterized by its Fourier–Laplace transform $\mathcal{F}$ (namely the symbol of $\Lambda_\mathfrak{a}$). Using the notation

$$u(\cdot, t) \xrightarrow{\mathcal{F}} \hat{u}(\cdot, \omega), \quad \mathcal{I}m\,\omega > 0, \tag{11}$$

we see that the symbol $\Lambda_\mathfrak{a}(\omega)$ is defined by

$$\Lambda_\mathfrak{a}(\omega) := -\partial_s \hat{u}_\mathfrak{a}(\mathbf{0}, \omega) \tag{12}$$

where $\hat{u}_\mathfrak{a}(\cdot, \omega)$ is for each ω the solution of: find $\hat{u}_\mathfrak{a}(\omega) \in V_\mathfrak{a} / \hat{u}_\mathfrak{a}(\mathbf{0}, \omega) = 1$ and $\forall v \in V_\mathfrak{a}$

$$-\omega^2 \int_\mathcal{T} \mu \hat{u}_\mathfrak{a}(\omega) v + \int_\mathcal{T} \mu \partial_s \hat{u}_\mathfrak{a}(\omega) \partial_s v = 0. \tag{13}$$

Theorem 4.1. *Assuming $M_2(\alpha, \nu) > 1$ and $M_1(\alpha, \nu) < 1$, the functions $\Lambda_\mathfrak{a}(\omega)$, ($\mathfrak{a} = \mathfrak{d}, \mathfrak{n}$), can be extended into even meromorphic functions in the complex planes with real poles $\pm\omega_n^\mathfrak{a}$ satisfying*:

$$\omega_n^\mathfrak{a} > 0, \qquad \omega_{n+1}^\mathfrak{a} > \omega_n^\mathfrak{a}, \qquad \lim_{n \to +\infty} \omega_n^\mathfrak{a} = +\infty.$$

We now characterize $\Lambda_\mathfrak{d}(\omega)$ and $\Lambda_\mathfrak{n}(\omega)$. Let us introduce the quadratic functional equation:

$$\text{Find } \Lambda(\omega) : \mathbb{C} \longrightarrow \mathbb{C} \quad \text{such that} \quad \Lambda(\omega) \cos\omega - \omega \sin\omega = \sum_{j=0}^{p-1} \frac{\nu_j}{\alpha_j} \left(\cos\omega + \frac{\Lambda(\omega)}{\omega} \sin\omega \right) \Lambda(\alpha_i \omega). \tag{14}$$

The frequency $\omega = 0$ plays obviously a particular role in this equation since we get

$$\Lambda(0)\big[1 - M_1(\alpha, \nu)\big[1 + \Lambda(0)\big]\big] = 0 \tag{15}$$

equation in $\Lambda(0)$ whose two solutions are

$$\Lambda(0) = \Lambda^0_\mathfrak{a} := \big(1 - M_1(\alpha, \nu)\big)^{-1} \quad \text{or} \quad \Lambda(0) = \Lambda^0_\mathfrak{n} := 0, \tag{16}$$

which are shown to correspond to the DtN map associated respectively to the Dirichlet and Neumann Laplace's problems on $\mathcal{T}$. Our main result is the following:

Theorem 4.2. *For $\mathfrak{a} = \mathfrak{d}, \mathfrak{n}$, the function $\Lambda_\mathfrak{a}(\omega)$ is the unique meromorphic function solution of* (14) *which satisfies $\Lambda_\mathfrak{a}(0) = \Lambda^0_\mathfrak{a}$ given by* (16)*.*

The proof relies on rewriting as a transmission problem between the generating segment Σ and the p-subtrees $\sigma_i(\mathcal{T})$. Eq. (14) is nothing but the Kirchhoff condition (2)(ii) (after a scaling argument).

We have designed an algorithm, based on Theorem 4.2, for the numerical approximation of $\Lambda_{\mathfrak{d}}(\omega)$ and $\Lambda_{\mathfrak{n}}(\omega)$. In Fig. 4, we present the result of a computation for the dyadic tree of Fig. 3 with $\nu_j = \alpha_j,\ j = 1, 2$. The modulus of $\Lambda_{\mathfrak{d}}(\omega)$ is represented. The bright spots correspond to the location of the singularities (cf. Theorem 4.1).

5. Approximate local boundary conditions

If the length of the generating segment of $\mathcal{T}$ is ℓ instead of 1, a simple scaling argument shows that, in Fourier–Laplace, the transparent boundary condition corresponding to (9) reads

$$\partial_s \hat{u}(\mathbf{0}, \omega) + \ell^{-1} \Lambda_{\mathfrak{a}}(\ell\omega) \hat{u}(\mathbf{0}, \omega) = 0. \tag{17}$$

Let us assume that ℓ is small. This will happen if one cuts the tree after many generations: the corresponding ℓ decays exponentially fast with the number n of generations that are kept in the computational domain. Eq. (18) shows that it suffices to have a good approximation of $\Lambda_{\mathfrak{a}}(\ell\omega)$ for small ω to get a good approximate boundary condition. Studying Eq. (14) near $\omega = 0$, we get the following:

Lemma 5.1.

$$\Lambda_{\mathfrak{a}}(\omega) = \Lambda_{\mathfrak{a}}^0 + \Lambda_{\mathfrak{a}}^2 \omega^2 + O\big(\omega^4\big) \quad (\omega \to 0) \tag{18}$$

where $\Lambda_{\mathfrak{a}}^0$ *is given by* (16) *and*

$$\Lambda_{\mathfrak{d}}^2 = \frac{1 + M_2(\alpha, \nu) + M_2(\alpha, \nu)^2}{3(M_2(\alpha, \nu)^2 - M_1(\alpha, \nu))} \qquad \Lambda_{\mathfrak{n}}^2 = \big(1 - M_1(\alpha, \nu)\big)^{-1}. \tag{19}$$

Keeping in (18) only the first two terms of the expansion of $\Lambda_{\mathfrak{a}}(\omega)$ and going back to the time domain, we get the following (second order) approximate boundary condition:

$$\partial_s u(\mathbf{0}, t) + \ell^{-1} \Lambda_{\mathfrak{a}}^0 u(t, \mathbf{0}) - \ell \Lambda_{\mathfrak{a}}^2 \frac{\partial^2 u}{\partial t^2}(t, \mathbf{0}) = 0. \tag{20}$$

From Eqs. (19) and (6), we deduce that

$$\Lambda_{\mathfrak{d}}^2 > 0 \quad \text{and} \quad \Lambda_{\mathfrak{n}}^2 > 0 \quad \big(\text{thanks to } M_1(\alpha, \nu) < 1\big).$$

As a consequence, one can prove by an energy argument that, coupled to the wave equation in the (truncated) tree, the boundary condition (20) leads to a well-posed initial boundary value problem.

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