C. R. Acad. Sci. Paris, Ser. I 349 (2011) 1097–1100

Contents lists available at SciVerse ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

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Differential Geometry

On the bounded isometry conjecture

Sur la conjecture d'isométrie bornée

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ARTICLE INFO

Article history:
Received 15 February 2011
Accepted after revision 22 August 2011
Available online 15 September 2011

Presented by the Editorial Board

ABSTRACT

We prove the bounded isometry conjecture proposed by F. Lalonde and L. Polterovich for a special class of closed symplectic manifolds.

RÉSUMÉ

Nous prouvons la conjecture d'isométrie bornée proposée par F. Lalonde et L. Polterovich pour une classe spéciale de variétés symplectiques fermées.

1. Introduction

The group $\mathrm{Ham}(M,\omega)$ of Hamiltonian diffeomorphisms of a closed symplectic manifold (M,ω) has interesting properties. Among them it has a metric called the Hofer metric (and also called the Hofer norm) and it is a normal subgroup of the group $\mathrm{Symp}(M,\omega)$ of symplectic diffeomorphisms. It follows by the definition of the Hofer norm $\|\cdot\|$ that this subgroup is invariant under conjugation by symplectic diffeomorphisms: $\|\psi\circ h\circ\psi^{-1}\| = \|h\|$ for $h\in\mathrm{Ham}(M,\omega)$ and $\psi\in\mathrm{Symp}(M,\omega)$. Then for a fixed symplectic diffeomorphism ψ, the map

$$\mathcal{C}_\psi : \mathrm{Ham}(M,\omega)\to\mathrm{Ham}(M,\omega)$$

defined by $\mathcal{C}_\psi(h) = \psi\circ h\circ\psi^{-1}$ is an isometry with respect to the Hofer metric. In [5], F. Lalonde and L. Polterovich study the isometries of the group of Hamiltonian diffeomorphisms with respect to the Hofer metric. Based on this they call a symplectic diffeomorphism $\psi\in\mathrm{Symp}_0(M,\omega)$ *bounded*, if the Hofer norm of $[\psi,h]$ remains bounded as h varies in $\mathrm{Ham}(M,\omega)$. Otherwise it is called *unbounded*.

The set $\mathrm{BI}_0(M,\omega)$ of bounded symplectic diffeomorphisms of (M,ω) is a group that contains all Hamiltonian diffeomorphisms. In fact in [5], F. Lalonde and L. Polterovich proved that these two groups are equal $\mathrm{BI}_0(M,\omega) = \mathrm{Ham}(M,\omega)$ when M is a surface Σ_g of positive genus or is a product of these surfaces. They made the conjecture that the group of bounded symplectic diffeomorphisms is equal to the group of Hamiltonian diffeomorphisms for every closed symplectic manifold. This conjecture is known as the bounded isometry conjecture. In [4], F. Lalonde and C. Pestieau proved the conjecture for symplectic manifolds of the form $\Sigma_g\times N$, where N is a closed symplectic manifold with trivial $H^1(N)$ and Σ_g a closed surface of positive genus. Recently Z. Han [3] proved the conjecture for the Kodaira–Thurston manifold.

In [1], C. Campos and A. Pedroza proved the bounded isometry conjecture for a special class of closed symplectic manifolds. Here we relax the hypothesis of [1] and we obtain a wider class of closed symplectic manifolds for which the

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[1] The author was supported by CONACYT grant No. 50662.

1631-073X/$ – see front matter
doi:10.1016/j.crma.2011.08.016

conjecture holds, in particular it includes the Kodaira–Thurston manifold. Moreover the proof that appears in [1] also applies in this case with minor changes.

Let (M,ω) be a closed symplectic manifold of dimension $2n$ so that:

(a) There are open sets $U_1,\dots,U_l \subset M$ such that each U_k is symplectomorphic to $\mathbb{T}^{2n}\setminus\mathbb{T}^j = \mathbb{T}^{2n-j}_* \times \mathbb{T}^j$ with the standard symplectic form and $j\in\{0,1,2,\dots,2n-1,2n\}$.

(b) Let $j_k : U_k \to M$ be the inclusion map and $j_{k,*} : H^1_c(U_k) \to H^1(M)$ the induced map in cohomology. Then

$$H^1(M) = \sum_{k=1}^{l} j_{k,*}\big(H^1_c(U_k)\big).$$

A symplectic manifold satisfying the conditions above is said to satisfy (H). Here $H^*_c(\cdot)$ stands for de Rham cohomology with compact support, by $\mathbb{T}^{2n}\setminus\mathbb{T}^0$ we mean the punctured torus $\mathbb{T}^{2n}_*$ and by $\mathbb{T}^{2n}\setminus\mathbb{T}^{2n}$ we mean a point. If (M,ω) is such that $H^1(M)$ is trivial, then it satisfies (H) by taking U to be a point.

Theorem 1.1. *Let (M,ω) be a closed symplectic manifold that satisfies (H). Then*

$$\mathrm{BI}_0(M,\omega) = \mathrm{Ham}(M,\omega).$$

Note that in [1] a similar class of closed symplectic manifolds was defined, namely the set U_k assumed to be symplectomorphic to the punctured torus $\mathbb{T}^{2n}_*$. The reason why degree-one cohomology enters the criteria in condition (H) is because the bounded isometry conjecture is equivalent to the following statement:

For every nonzero $v \in H^1(M)/\Gamma_M$ there exists $\psi \in \mathrm{Symp}_0(M,\omega)$ unbounded so that $\mathrm{Flux}(\psi) = v$.

Here Γ_M stands for the flux group of (M,ω). Recall that Hamiltonian diffeomorphisms are characterized by having trivial flux.

Along the lines of the proof of Theorem 1.1 we prove two important results, though by Proposition 3.4 they are equivalent. The first enables us to write the flux group of a product of symplectic manifolds as the direct sum of the flux groups of each symplectic manifold.

Theorem 1.2. *Let (M,ω) and (N,η) be closed symplectic manifolds that satisfy (H), then the flux group of $(M\times N,\omega\oplus\eta)$ splits: $\Gamma_{M\times N} \simeq \Gamma_M \oplus \Gamma_N$.*

This is something that one might suspect, because in the typical example of the torus $(\mathbb{T}^{2n},\omega_0)$ it is well known that the flux group is a direct sum of the flux group of 2-dimensional torus, $\Gamma_{\mathbb{T}^{2n}} = \Gamma_{\mathbb{T}^2}\oplus\cdots\oplus\Gamma_{\mathbb{T}^2}$. However, at least to the author's knowledge, there appears to be no similar result in literature regarding the splitting of the flux group for a general product of symplectic manifolds.

The final result deals with the product of symplectic diffeomorphisms. It would be interesting to find consequences of this result in the context of dynamical systems.

Theorem 1.3. *Let (M,ω) and (N,η) be closed symplectic manifolds that satisfy (H). If $\psi\in\mathrm{Symp}_0(M,\omega)$ and $\phi\in\mathrm{Symp}_0(N,\eta)$ are such that $\psi\times\phi$ is a Hamiltonian diffeomorphism, then ψ and ϕ are also Hamiltonian diffeomorphisms.*

2. Example: Kodaira–Thurston manifold

The Kodaira–Thurston manifold (M,ω) can be defined in several ways, but for our purpose it is more convenient to consider it as the quotient of $[0,1]\times S^1\times\mathbb{T}^2$ by the relation $(0,x_2,x_3,x_4)\sim(1,x_2,x_3+x_4,x_4)$. The symplectic form on (M,ω) comes from the standard symplectic form of $[0,1]\times S^1\times\mathbb{T}^2$, namely $\mathrm{d}x_1\wedge\mathrm{d}x_2+\mathrm{d}x_3\wedge\mathrm{d}x_4$. Moreover $H^1(M)$ is generated by $\mathrm{d}x_1$, $\mathrm{d}x_2$ and $\mathrm{d}x_4$. For example see [3] and [6].

The open sets $U_1=(0,1)\times S^1\times\mathbb{T}^2$, $U_2=[0,1]\times(0,1)\times\mathbb{T}^2$ and $U_4=[0,1]\times S^1\times S^1\times(0,1)$ of $[0,1]\times S^1\times\mathbb{T}^2$ are such that $H^1_c(U_j)$ is generated by $g(x_j)\,\mathrm{d}x_j$ where $g:\mathbb{R}\to\mathbb{R}$ is a smooth function with compact support and total integral equal to 1. It follows that the Kodaira–Thurston manifold (M,ω) satisfies condition (H) and so by Theorem 1.1 it satisfies the bounded isometry conjecture. Therefore our class of closed symplectic manifolds for which the bounded isometry holds includes all the previous known examples.

3. Outline of the proof

As mentioned above, Theorems 1.2 and 1.3 are equivalent by properties of the flux morphism. Let (M,ω) be a closed symplectic manifold and $\psi=\{\psi_t\}_{0\leqslant t\leqslant 1}$, a loop that represents an element of $\pi_1(\mathrm{Symp}_0(M,\omega))$, then the flux morphism $\mathrm{Flux}_M : \pi_1(\mathrm{Symp}_0(M,\omega)) \to H^1(M)$ is defined by

$$\mathrm{Flux}_M(\psi) = \int_0^1 \big[\iota(X_t)\omega\big]\,\mathrm{d}t,$$

where the time-dependent vector field X_t is induced by the isotopy $\{\psi_t\}$ via the equation $\frac{\mathrm{d}}{\mathrm{d}t}\psi_t = X_t \circ \psi_t$. The image of Flux_M is denoted by Γ_M and is called the *flux group* of (M, ω). The flux morphism can also be defined on $\mathrm{Symp}_0(M, \omega)$, rather than on its fundamental group. In this case for a given symplectic diffeomorphism ψ, one considers a symplectic isotopy $\{\psi_t\}$ that joins $\psi_0 = 1_M$ with $\psi_1 = \psi$, with time-dependent vector field X_t as before. Then the map $\mathrm{Flux}_M : \mathrm{Symp}_0(M, \omega) \to H^1(M)/\Gamma_M$ is defined exactly as above. In this scenario there is an exact sequence of groups,

$$0 \to \mathrm{Ham}(M, \omega) \to \mathrm{Symp}_0(M, \omega) \to H^1(M)/\Gamma_M \to 0, \tag{1}$$

where the first map is inclusion and the last one is the flux morphism just defined. For more details of the flux morphism see the books of D. McDuff and D. Salamon [6] and of L. Polterovich [7].

Proposition 3.4. *Theorems* 1.3 *and* 1.2 *are equivalent.*

Proof. This result follows by analyzing the exact sequence (1) of the flux morphism. We use the exact sequence (1) for the manifolds (M, ω), (N, η) and $(M \times N, \omega \oplus \eta)$ as in the next diagram where the rows are exact.

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \mathrm{Symp}_0(M, \omega) \oplus \mathrm{Symp}_0(N, \eta) & \longrightarrow & H^1(M \times N)/\Gamma_M \oplus \Gamma_N & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & \mathrm{Symp}_0(M \times N, \omega \oplus \eta) & \longrightarrow & H^1(M \times N)/\Gamma_{M\times N} & \longrightarrow & 0 \end{array}$$

Here the horizontal maps are $\mathrm{Flux}_M \oplus \mathrm{Flux}_N$ and $\mathrm{Flux}_{M\times N}$. And the vertical maps are $(\psi, \phi) \mapsto \psi \times \phi$ and $x + \Gamma_M \oplus \Gamma_N \mapsto x + \Gamma_{M\times N}$. So defined, the diagram commutes and the result follows. □

As for the proof of Theorem 1.1, we first comment on the Hofer norm on the group $\mathrm{Ham}(M, \omega)$. In order to find lower bounds for the Hofer norm of a Hamiltonian diffeomorphism, one uses the energy-capacity inequality. Let A be an open subset of (M, ω) where M has dimension $2n$, then there are two numbers associated with A: Gromov's width $c_G(A)$ and the displacement energy $e(A)$;

$$c_G(A) = \sup\big\{\pi r^2 : B^{2n}(r) \to A \text{ is a symplectic embedding, } B^{2n}(r) \subset \mathbb{R}^{2n}\big\},$$
$$e(A) = \inf\big\{\|\psi\|:\ \psi \in \mathrm{Ham}(M, \omega) \text{ and } \psi(A) \cap A = \emptyset\big\}.$$

The energy-capacity inequality ensures that

$$\frac{1}{2} c_G(A) \leqslant e(A)$$

for all $A \subset M$. Hence if ψ is a Hamiltonian diffeomorphism so that $\psi(A) \cap A = \emptyset$, it follows from the energy-capacity inequality that $c_G(A) \leqslant \|\psi\|$. Thus if we want a Hamiltonian diffeomorphism ψ with a large Hofer norm we must find a *big* set A, in the sense of $c_G(\cdot)$, so that $\psi(A) \cap A = \emptyset$. But notice that if M is compact then $c_G(A)$ cannot achieve large values. Therefore in order to get a large lower bound for the Hofer norm on (M, ω), we must lift the Hamiltonian diffeomorphism to the universal cover of the symplectic manifold.

A key step in the proof of Theorem 1.1, is to check that the bounded isometry conjecture holds for $(\mathbb{T}^{2n} \setminus \mathbb{T}^j, \omega_0)$. The case of $(\mathbb{T}^2_*, \omega_0)$ was done in [5] by F. Lalonde and L. Polterovich. But their arguments work for the general case without major changes. Here is where one considers the universal cover $\mathbb{R}^{2n}$ of $\mathbb{T}^{2n}$ to get the large lower bound for the desired Hamiltonian diffeomorphism. That is, if ψ is a Hamiltonian diffeomorphism of $\mathbb{T}^{2n}$, then one considers the lift $\tilde{\psi}$ of ψ to $\mathbb{R}^{2n}$. By [2] the lift will also be a Hamiltonian diffeomorphism and will be unique. Moreover by the definition of the Hofer norm, $\|\tilde{\psi}\|_{\mathbb{R}^{2n}} \leqslant \|\psi\|_{\mathbb{T}^{2n}}$.

Sketch of proof of Theorem 1.1. The argument of the proof is similar to that of [1]; we only need to show that if $U = \mathbb{T}^{2n} \setminus \mathbb{T}^j$ then, for every $v \in H^1_c(U)/\Gamma_U$ there is a strongly unbounded symplectic diffeomorphism $\psi \in \mathrm{Symp}^c_0(U, \omega)$ such that $\mathrm{Flux}(\psi) = v$.

Note that $\mathbb{T}^{2n} \setminus \mathbb{T}^j = \mathbb{T}^{2n-j}_* \times \mathbb{T}^j$, hence if $j = 2k$ then we have the product of symplectic manifolds $\mathbb{T}^{2(n-k)}_* \times \mathbb{T}^{2k}$. By [1] the bounded isometry conjecture holds for both $\mathbb{T}^{2(n-k)}_*$ and $\mathbb{T}^{2k}$, and hence also for their product. For the case when j is odd it suffices to prove the assertion for $(S^1 \times (0, 1), \mathrm{d}x \wedge \mathrm{d}y)$. The proof of this particular case follows the same arguments of [5] where they prove the bounded isometry conjecture for the case of the punctured torus $(\mathbb{T}^2_*, \omega)$.

For simplicity we assume that (M, ω) is such that there is a symplectic embedding $j : (U, \omega_0) \to (M, \omega)$ where $U = \mathbb{T}^{2n} \setminus \mathbb{T}^j$ and $j_* : H^1_c(U) \to H^1(M)$ is surjective. Then a symplectic diffeomorphism on U with compact support can be thought of as a symplectic diffeomorphism on M. Therefore we get, $j_*(\Gamma_U) \subset \Gamma_M$, and the induced surjective map $j_* : H^1_c(U)/\Gamma_U \to H^1(M)/\Gamma_M$. Thus there is $v_0 \in H^1_c(U)/\Gamma_U$, such that $j_*(v_0) = v$.

Moreover since $\mathrm{BI}_0(U, \omega_0) = \mathrm{Ham}^c(U, \omega_0)$, there is ψ_0, an unbounded symplectic diffeomorphism on (U, ω_0) with compact support. Now consider ψ_0 as a symplectic diffeomorphism on (M, ω), then $\mathrm{Flux}_M(\psi_0) = v$ and it only remains to check that ψ_0 is unbounded as a symplectic diffeomorphism on (M, ω). For this see Proposition 4.8 of [1]. □

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