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Gradient vector fields with values into S^1

Champs de gradient à valeurs dans S^1

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ABSTRACT

We state the following regularity result: if a two-dimensional gradient vector field $v=\nabla\psi$ with values into the unit circle S^1 belongs to $H^{1/2}$ (or $W^{1,1}$) then v is locally Lipschitz except at a locally finite number of vortices. We also state approximation results for such vector fields.

RÉSUMÉ

Le résultat de régularité suivant a lieu : Si un champ de gradient $v=\nabla\psi$ est à valeurs dans le cercle unité S^1 et appartient à $H^{1/2}$ (ou $W^{1,1}$) alors v est localement Lipschitz en dehors d'un nombre localement fini de points singuliers. Ensuite, des résultats de densité sont énoncés pour cette classe de champs de gradient.

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Soit $\Omega\subset\mathbf{R}^2$ un domaine borné. On s'intéresse à la structure des champs de gradient mesurables $v=\nabla\psi:\Omega\to\mathbf{R}^2$ qui satisfont l'équation eikonale $|\nabla\psi|=1$ p.p. dans Ω. Le premier but est d'établir le résultat de régularité suivant : Si un tel champ de gradient v appartient à $H^{1/2}(\Omega,S^1)$ (ou $W^{1,1}(\Omega,S^1)$) alors v est localement Lipschitz sauf en un nombre localement fini de points singuliers. De plus, chaque point singulier P correspond à un vortex de degré 1 de v, i.e., il existe $\alpha\in\{-1,1\}$ tel que

$$v(x)=\alpha\frac{x-P}{|x-P|}\quad\text{pour tout } x\neq P \text{ dans chaque voisinage convexe de } P \text{ dans } \Omega.$$

En particulier, si le champ de gradient v appartient à $H^1(\Omega,S^1)$ alors v est localement Lipschitz. L'idée repose sur la formulation cinétique suivante : pour toute caractéristique $\chi(\cdot,\xi)$ associée à v (au sens faible) dans la direction $\xi\in S^1$, on a $\xi\cdot\nabla\chi(\cdot,\xi)=0$ dans $\mathcal{D}'(\Omega)$. La régularité Lipschitz est optimale dans le sens où il existe un champ de gradient $v\in W^{1,\infty}(\Omega,S^1)$ qui n'est pas C^1. La géométrie du domaine Ω influence le nombre de vortex d'un champ de gradient $H^{1/2}$ (où $W^{1,1}$). Par exemple, si Ω est un domaine convexe, alors un tel champ de gradient est soit un champ « vortex » (i.e., $v(x)=x/|x|$ à une translation et signe ±1 près) soit localement Lipschitz (donc, sans singularité vortex) ; autrement dit, les domaines convexes ne permettent la formation de plus d'un vortex. Néanmoins, il existe des domaines non convexes Ω où les champs de gradient peuvent avoir une infinité de singularités vortex.

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On s'intéresse aussi à des résultats de densité dans le cas des domaines Lipschitz Ω. Premièrement, tout champ de gradient $v \in H^{1/2}(\Omega, S^1)$ (ou $v \in W^{1,1}(\Omega, S^1)$) peut être approché dans la topologie $W^{1,p}_{loc}$ (pour $p \in [1,2)$) par des champs de gradient $|v_n| = 1$ réguliers sauf en un nombre fini de points. En particulier, pour un champ de gradient $v \in H^1(\Omega, S^1)$, la suite $\{v_n\}$ pourrait être choisie partout régulière dans Ω. Deuxièmement, on cherche à approcher des champs de gradient $v \in H^{1/2}(\Omega, S^1)$ (ou $v \in W^{1,1}(\Omega, S^1)$) par des champs de vecteurs partout réguliers $v_n \in C^\infty(\Omega, S^1)$ (pas nécessairement de rotationnel nul) dans une topologie plus faible. En effet, un champ « vortex » $v(x) = x/|x|$ ne peut pas être approché dans la topologie L^1 (forte) par des champs de gradient $v_n \in C^\infty(\Omega, S^1)$. Par contre, le résultat d'approximation dans L^1 est vrai si on relaxe la condition de rotationnel nul pour la suite régularisante : pour tout champ de gradient $v \in H^{1/2}(\Omega, S^1)$ (ou $v \in W^{1,1}(\Omega, S^1)$) il existe une suite $\{v_n\}_n \in C^\infty(\Omega, S^1)$ telle que $v_n \to v$ dans L^1 et $(\nabla \times v_n)1_\Omega \to 0$ dans $\dot{H}^{-1/2}(\mathbf{R}^2)$. (Ici, v_n ne sont pas de champs de gradient, mais cette contrainte est satisfaite à la limite dans la topologie $\dot{H}^{-1/2}$; cette topologie est optimale puisque le résultat de densité est faux en général si la contrainte de rotationnel nul est imposée à la limite dans la topologie faible $H^s(\Omega)$ pour $s > -1/2$.)

1. Introduction

Let $\Omega \subset \mathbf{R}^2$ be an open bounded set. We will focus on measurable vector fields $v : \Omega \to \mathbf{R}^2$ such that

$$|v| = 1 \quad \text{a.e. in } \Omega \quad \text{and} \quad \nabla \times v = 0 \quad \text{in } \mathcal{D}'(\Omega). \tag{1}$$

One can equivalently consider measurable vector fields $m : \Omega \to \mathbf{R}^2$ that satisfy

$$|m| = 1 \quad \text{a.e. in } \Omega \quad \text{and} \quad \nabla \cdot m = 0 \quad \text{in } \mathcal{D}'(\Omega). \tag{2}$$

(The passage from (2) to (1) is done via $m = v^\perp = (-v_2, v_1)$.) Locally, v (resp. m) can be written in terms of a stream function ψ, i.e., $v = \nabla\psi$ (resp. $m = \nabla^\perp\psi$) so that we get to the eikonal equation through ψ:

$$|\nabla\psi| = 1. \tag{3}$$

Typically, one can construct such vector fields by considering stream functions of the form $\psi = \operatorname{dist}(\cdot, K)$ for some closed set $K \subset \mathbf{R}^2$. However, not every stream function can be written as a distance function (up to a sign ± 1 and an additive constant); for example, if $\psi(x) = \max\{|x - P_1|, |x - P_2|\}$ for two different points $P_1, P_2 \in \mathbf{R}^2$, then (3) still holds.

2. Regularity

For $p \geqslant 1$ and $s > 0$, we denote by

$$W^{s,p}_{curl}(\Omega, S^1) = \{v \in W^{s,p}(\Omega, \mathbf{R}^2): \ v \text{ satisfies (1)}\}.$$

We first state the following regularity result (see [10]):

Theorem 1. *If $v \in W^{1,1}_{curl}(\Omega, S^1)$ (or $v \in H^{1/2}_{curl}(\Omega, S^1)$) then v is locally Lipschitz continuous inside Ω except at a locally finite number of singular points. Moreover, every singular point P of v corresponds to a vortex singularity of degree 1 of v, i.e., there exists a sign $\alpha = \pm 1$ such that*

$$v(x) = \alpha \frac{x - P}{|x - P|} \quad \text{for every } x \neq P \text{ in any convex neighborhood of } P \text{ in } \Omega.$$

In particular, if $v \in H^1_{curl}(\Omega, S^1)$ then v is locally Lipschitz in Ω.

Remark 1. Theorem 1 was proved by Jabin–Otto–Perthame [13] in the particular case of zero-energy states of a line-energy Ginzburg–Landau model: for $\varepsilon > 0$, one defines the functional $E_\varepsilon : H^1(\Omega, \mathbf{R}^2) \to \mathbf{R}_+$ by

$$E_\varepsilon(m_\varepsilon) = \varepsilon \int_\Omega |\nabla m_\varepsilon|^2 \, dx + \frac{1}{\varepsilon} \int_\Omega \left(1 - |m|^2\right)^2 dx + \frac{1}{\varepsilon} \|\nabla \cdot m_\varepsilon\|^2_{H^{-1}(\Omega)}, \quad m_\varepsilon \in H^1(\Omega, \mathbf{R}^2).$$

A vector field $m : \Omega \to \mathbf{R}^2$ is called zero-energy state if there exists a family $\{m_\varepsilon \in H^1(\Omega, \mathbf{R}^2)\}_{\varepsilon \to 0}$ satisfying

$$m_\varepsilon \to m \quad \text{in } L^1(\Omega) \quad \text{and} \quad E_\varepsilon(m_\varepsilon) \to 0 \quad \text{as } \varepsilon \to 0.$$

Then m satisfies (2) and $m^\perp$ shares the structure stated in Theorem 1.

The hypothesis $v \in W^{1,1}$ (or $v \in H^{1/2}$) in Theorem 1 is a critical regularity assumption in order to avoid line-singularities for vector fields v satisfying (1). Hence, Theorem 1 yields

$$\{v \in W^{1,1}_{loc}(\Omega, \mathbf{R}^2):\ v \text{ satisfies (1)}\} = \{v \in H^{1/2}_{loc}(\Omega, \mathbf{R}^2):\ v \text{ satisfies (1)}\}.$$

Let us now discuss the optimality of the result in Theorem 1: Firstly, Lipschitz regularity cannot be improved, i.e., there exist Lipschitz vector fields $v : \Omega \to \mathbf{R}^2$ that satisfy (1) and they are not C^1 in Ω. In general, a vector field $v \in W^{1,1}_{curl}(\Omega, S^1)$ (or $v \in H^{1/2}_{curl}(\Omega, S^1)$) (without interior vortex singularities) is only locally Lipschitz, and not necessary globally Lipschitz in Ω. This is the case of a "boundary vortex" vector field (e.g., $v(x) = \frac{x-P}{|x-P|}$ with the vortex center P lying on the boundary $\partial\Omega$). Notice that for a domain Ω with a cusp, a "boundary vortex" vector field could even belong to $H^1(\Omega, \mathbf{R}^2)$; moreover, there even exist convex domains Ω and $v \in H^1_{curl}(\Omega, S^1)$ such that v is not globally Lipschitz in Ω.

The geometry of Ω influences the number of vortices of $W^{1,1}$ (or $H^{1/2}$)-gradient vector fields with values into S^1. For example, if Ω is convex, than every vector field $v \in W^{1,1}_{curl}(\Omega, S^1)$ (or $v \in H^{1/2}_{curl}(\Omega, S^1)$) is either a "vortex" vector field (i.e., $v(x) = \pm\frac{x-P}{|x-P|}$ for every $x \in \Omega \setminus \{P\}$ where P is some point in Ω), or locally Lipschitz (i.e., without any interior vortex); therefore, convex domains do not allow for more than one interior vortex. However, there exist bounded simply-connected nonconvex domains Ω and vector fields $v \in W^{1,p}_{curl}(\Omega, S^1)$ for every $p \in [1, 2)$ that have infinitely many vortices $\{P_1, P_2, \ldots\}$. (Recall that $W^{1,p}_{loc}(\Omega, S^1) \subset H^{1/2}_{loc}(\Omega, S^1)$ for $p > 1$, and the embedding fails for $p = 1$.)

If one imposes the condition $v \cdot \nu^\perp = 0$ on the boundary of a smooth simply-connected domain Ω (here, ν is the outward normal unit vector field at $\partial\Omega$), then there are few geometries Ω so that $W^{1,1}_{curl}(\Omega, S^1) \neq \emptyset$ (resp. $H^{1/2}_{curl}(\Omega, S^1) \neq \emptyset$); more precisely, Ω is either a disk $B(P, r)$ and the only vector fields $v \in W^{1,1}_{loc}$ (or $v \in H^{1/2}_{loc}$) satisfying (1) and $v \cdot \nu^\perp = 0$ on $\partial\Omega$ is the vortex configuration $v(x) = \pm\frac{x-P}{|x-P|}$, or Ω is a strip and v is a constant vector field (perpendicular to $\partial\Omega$). If $\Omega = \mathbf{R}^2$, both alternatives hold (see [13]).

The main ingredient of Theorem 1 resides in the following kinetic formulation (see [10]):

Proposition 2 *(Kinetic formulation). Let $v \in W^{1,1}_{curl}(\Omega, S^1)$ (or $v \in H^{1/2}_{curl}(\Omega, S^1)$). For every direction $\xi \in S^1$, we define $\chi(\cdot, \xi) : \Omega \to \{0, 1\}$ (resp. $\tilde{\chi}(\cdot, \xi) : S^1 \to \{0, 1\}$) by*

$$\chi(x, \xi) = \tilde{\chi}\big(v(x), \xi\big) = \begin{cases} 1 & \text{for } v(x) \cdot \xi^\perp > 0, \\ 0 & \text{for } v(x) \cdot \xi^\perp \leqslant 0. \end{cases}$$

Then the following equation holds for every $\xi \in S^1$:

$$\xi \cdot \nabla \chi(\cdot, \xi) = 0 \quad \text{in } \mathcal{D}'(\Omega). \tag{4}$$

Here, χ corresponds to the concept of characteristics of a weak solution v satisfying (1). Indeed, if v is smooth in a neighborhood of a point $x \in \Omega$, the (classical) characteristic of v at x is given by $\dot{X}(t, x) = v(X(t, x))$ with the initial condition $X(0, x) = x$; due to (1), the orbit $\{X(t, x)\}_t$ is a straight line along which v is constant. Denoting this direction by $\xi := v(x) \in S^1$, then locally either $\nabla\chi(\cdot, \xi)$ vanishes, or is a measure concentrated on $\{X(t, x)\}_t$ and oriented by $\xi^\perp$. The knowledge of $\chi(\cdot, \xi)$ in every direction $\xi \in S^1$ determines completely the vector field v due to the straightforward formula

$$v(x) = \frac{1}{2} \int_{S^1} \xi^\perp \chi(x, \xi)\, \mathrm{d}\xi \quad \text{for a.e. } x \in \Omega. \tag{5}$$

Remark 2. Classical kinetic averaging lemma (see e.g. Golse, Lions, Perthame and Sentis [8]) shows that a measurable vector-field $v : \Omega \to S^1$ satisfying (4) belongs to $H^{1/2}_{loc}$ (due to (5)). Moreover, Jabin, Otto and Perthame (see Theorem 1.3 in [13]) proved that such a vector field has stronger regularity, i.e., it shares the structure described in Theorem 1. Therefore, the proof of Theorem 1 strongly relies on Jabin, Otto and Perthame's result [13] via Proposition 2.

Remark 3. The proof of Proposition 2 relies on the structure of lifting of vector fields $v \in W^{1,1}(\Omega, S^1)$ (resp. $v \in H^{1/2}(\Omega, S^1)$) and an appropriate chain rule. More precisely, if $v \in W^{1,1}(\Omega, S^1)$, then there exists a lifting $\varphi \in SBV(\Omega, \mathbf{R})$ such that $v = e^{i\varphi}$ a.e. in Ω (see e.g. [3,4,7,9]). While if $v \in H^{1/2}(\Omega, S^1)$, then one can find a lifting $\varphi = \varphi_1 + \varphi_2$ of v with $\varphi_1 \in H^{1/2}$, $\varphi_2 \in SBV$ and $e^{i\varphi_2} \in H^{1/2} \cap W^{1,1}$ (see [2]). Recall that $SBV(\Omega, \mathbf{R}^n)$ is the subspace of vector fields $v \in BV(\Omega, \mathbf{R}^n)$ whose differential Dv has vanishing Cantor part $D^c v$ (i.e., $D^c v \equiv 0$ as a measure in Ω).

We address the following open problem:

Open Problem 1. Is it true that every $v \in BV(\Omega, \mathbf{R}^2)$ with (1) satisfies $v \in SBV$?

3. Density results

The second goal of this Note is to state density results for vector fields $W^{1,1}_{curl}(\Omega, S^1)$ (or $H^{1/2}_{curl}(\Omega, S^1)$): these subsets are formed either by vector fields that are smooth except at a finite number of singular points and the approximation result holds in the strong $W^{1,1}$ (or $H^{1/2}$)-topology, or by everywhere smooth vector fields (not necessary curl-free) and the approximation result holds in a weaker topology. We start by extending Bethuel–Zheng's density result (see [1]) for $W^{1,1}(\Omega, S^1)$, respectively Riviere's density result (see [14]) for $H^{1/2}(\Omega, S^1)$ to the case of curl-free vector fields (see [10]):

Theorem 3. *Let Ω be a Lipschitz bounded simply-connected domain and $v \in W^{1,1}_{curl}(\Omega, S^1)$ (or $v \in H^{1/2}_{curl}(\Omega, S^1)$). Then v has a finite number $k \geqslant 0$ of vortices $\{P_1, \dots, P_k\}$ in Ω and v can be approximated in $W^{1,p}_{loc}$ (for any $p \in [1, 2)$) by curl-free vector fields $v_n \in C^\infty(\Omega \setminus \{P_1, \dots, P_k\}, S^1)$ that are smooth except at the vortex points $\{P_1, \dots, P_k\}$ of v. In particular, if $v \in H^1_{curl}(\Omega, S^1)$, the sequence of curl-free vector fields $\{v_n\}$ can be chosen to be smooth everywhere in Ω and the approximation result holds in H^1_{loc}.*

In various applications (see e.g. Remark 4 below), we need to approximate vector fields v (with the structure given in Theorem 1) by $H^1(\Omega, S^1)$-vector fields v_n, so that v_n cannot allow for vortices. (Recall that the vortex configuration $v(x) = x/|x|$ in the unit disk B^2 satisfies $v \in W^{1,p}_{curl}(B^2, S^1)$ if and only if $p < 2$.) Therefore, an approximation result by everywhere smooth S^1-valued vector fields is needed in some weak topology. What is the optimal weak topology where such a density result holds? The following result shows that L^1-topology is too strong for having density of smooth gradient vector fields with values into S^1 (see [10]).

Proposition 4. *Let $v : B^2 \to S^1$ be the vortex configuration $v(x) = \frac{x}{|x|}$ in the unit disk B^2. Then there exists no sequence of vector fields $v_n \in C^\infty(\Omega, \mathbf{R}^2)$ with (1) such that $v_n \to v$ a.e. in B^2.*

We now generalize this property: the density result still fails if we relax the curl-free constraint on the approximated smooth vector fields, but we impose this restriction in the limit in L^1-topology (or H^{-s} weak topology for some $s \in [0, \frac{1}{2})$) (see [10]).

Proposition 5. *Let $v : B^2 \to S^1$ be the vortex configuration $v(x) = \frac{x}{|x|}$ in B^2. Then there exists no sequence $v_n \in C^\infty(\Omega, S^1)$ such that $v_n \to v$ a.e. in B^2 and one of the following two conditions holds:*

(a) $\nabla \times v_n \to 0$ *in* $L^1(B^2)$;
(b) $\nabla \times v_n \rightharpoonup 0$ *weakly in* $H^{-s}(B^2)$ *for some* $s \in [0, \frac{1}{2})$.

Finally, we state an approximation result in L^1-topology of smooth vector fields with values into S^1, that are not necessarily curl-free, but the curl-free constraint holds in the limit in the $\dot{H}^{-1/2}$-topology. Due to Proposition 5(b), this topology is optimal (see [10]):

Theorem 6. *Let Ω be a Lipschitz bounded simply-connected domain and $v \in W^{1,1}_{curl}(\Omega, S^1)$ (or $v \in H^{1/2}_{curl}(\Omega, S^1)$). Then there exists a sequence of vector fields $v_n \in C^\infty(\Omega, S^1)$ such that $v_n \to v$ a.e. in Ω and $(\nabla \times v_n)\mathbf{1}_\Omega \to 0$ in $\dot{H}^{-1/2}(\mathbf{R}^2)$.*

Remark 4. The motivation of Theorem 6 comes from thin-film micromagnetics. The following 2D energy $F_\varepsilon : H^1(\Omega, S^1) \to \mathbf{R}_+$ is considered for $\varepsilon > 0$ (see e.g., [6,5,12,11]):

$$F_\varepsilon(m_\varepsilon) = \varepsilon \int_\Omega |\nabla m_\varepsilon|^2 \, dx + \left\| (\nabla \cdot m_\varepsilon)\mathbf{1}_\Omega \right\|^2_{\dot{H}^{-1/2}(\mathbf{R}^2)}, \quad m_\varepsilon \in H^1(\Omega, S^1).$$

In particular, it is proved in [11] that a vortex configuration $m_0(x) = \frac{x^\perp}{|x|}$ in B^2 is a zero-energy state, i.e., there exists a family $\{m_\varepsilon \in H^1(B^2, S^1)\}$ such that $m_\varepsilon \to m_0$ a.e. in B^2 and $F_\varepsilon(m_\varepsilon) \to 0$ as $\varepsilon \to 0$. The role of Theorem 6 is to generalize this approximation result for every vector field $m \in W^{1,1}$ (resp. $m \in H^{1/2}$) satisfying (2).

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