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Partial Differential Equations

Diffusive Back and Forth Nudging algorithm for data assimilation

Une version Diffusive du Nudging Direct et Rétrograde pour l'assimilation de données

Didier Auroux [a], Jacques Blum [a], Maëlle Nodet [b]

[a] *Laboratoire J.-A. Dieudonné, université de Nice Sophia Antipolis, parc Valrose, 06108 Nice cedex 2, France*
[b] *Université Grenoble I / INRIA, laboratoire Jean Kuntzmann, tour IRMA, 51, rue des mathématiques, domaine universitaire, BP 53X, 38041 Grenoble cedex, France*

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ABSTRACT

In this Note, we propose an improvement to the Back and Forth Nudging algorithm for handling diffusion in the context of geophysical data assimilation. We introduce the Diffusive Back and Forth Nudging, in which the sign of the diffusion term is changed in the backward integrations. We study the convergence of this algorithm, in particular for linear transport equations.

RÉSUMÉ

Dans cette Note, nous proposons une amélioration de l'algorithme du Nudging Direct et Rétrograde pour gérer la diffusion dans le contexte de l'assimilation de données en géophysique. Nous introduisons la version Diffusive du Nudging Direct et Rétrograde, dans laquelle le signe du terme de diffusion est inversé dans les équations rétrogrades. Nous étudions la convergence de cet algorithme, en particulier pour l'équation de transport linéaire.

Version française abrégée

Introduction

L'assimilation de données consiste à identifier l'état d'un système en combinant les informations contenues dans le modèle et les observations. Le nudging est l'une des premières méthodes d'assimilation de données utilisées en météorologie et océanographie opérationnelle [5,7,8]. Cette méthode consiste à rajouter dans les équations du modèle un terme de rappel Newtonien pour forcer la solution à rester proche des observations [6].

L'algorithme du nudging direct et rétrograde (ou BFN, Back and Forth Nudging) a été introduit pour permettre la prise en compte d'observations futures, en alternant des intégrations directes et rétrogrades du modèle avec un terme de nudging [1,2].

La résolution rétrograde peut s'avérer impossible en présence de diffusion [3]. En géophysique, les équations considérées sont généralement non diffusives, mais un terme de diffusion est ajouté pour stabiliser la résolution numérique et paramétriser les phénomènes sous-maille. Dans ce cadre là, nous proposons la version Diffusive du Nudging Direct et Rétrograde, dans laquelle le signe du terme de diffusion est changé dans l'intégration rétrograde, afin de conserver son rôle numérique et physique.

E-mail addresses: auroux@unice.fr (D. Auroux), jblum@unice.fr (J. Blum), maelle.nodet@inria.fr (M. Nodet).

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Une version Diffusive du Nudging Direct et Rétrograde

Nous supposons que le modèle s'écrit sous la forme de l'équation (1), et notamment que la trajectoire observée est solution de ce modèle sans diffusion. L'algorithme BFN consiste alors à itérer la résolution des équations directes et rétrogrades avec nudging (2). Si on sépare du reste du modèle le terme de diffusion (généralement faible), on obtient alors l'équation (3), où le modèle F est supposé non diffusif.

La version Diffusive du Nudging Direct et Rétrograde (algorithme D-BFN) que nous introduisons ici est alors donnée par les équations (4). Les équations directes sont les mêmes, mais dans les équations rétrogrades, le signe du terme de diffusion est inversé, comme celui du terme de nudging. Si on inverse le sens du temps, le modèle rétrograde se réécrit de façon directe (5). L'intérêt de cet algorithme est de conserver le rôle du terme de diffusion (stabilité et modélisation physique) lors de l'intégration rétrograde.

Étude de convergence de l'algorithme

On suppose ici que le modèle F est linéaire. En s'intéressant à une itération de l'algorithme D-BFN, on peut définir un opérateur ψ donné par (6) qui associe la solution finale de l'itération (une intégration directe puis une rétrograde) aux conditions initiales du système et de la trajectoire observée. On voit que cet opérateur dépend linéairement des conditions initiales, et donc après k itérations de D-BFN, la condition initiale obtenue s'écrit sous la forme (8). Si l'opérateur linéaire est contractant, alors $X_k(0)$ converge vers X_∞ solution de (9).

Nous considérons maintenant une équation de transport linéaire (10) sur un domaine périodique. Pour des raisons de stabilité numérique, il peut s'avérer intéressant de rajouter un terme de diffusion avec un petit cœfficient comme dans (11). Mais on suppose que les observations u_{obs} du système sont solutions de l'équation sans diffusion. Alors l'algorithme D-BFN appliqué à ce système prend la forme du système (13), dans lequel on a déplié le temps, en mettant bout à bout les différentes itérations.

Il est alors possible de calculer la variation de l'énergie du système au cours des itérations, et celle-ci peut être majorée par un terme négatif, comme indiqué dans l'équation (14). Cela prouve que l'opérateur C associé à ce modèle est contractant, et la convergence de l'algorithme D-BFN est alors assurée.

Dans un cas simplifié où le terme de transport est constant ($a(x) = a$), on peut calculer explicitement la limite de la trajectoire construite par l'algorithme D-BFN, et en suivant les caractéristiques de l'équation de transport, la solution converge vers la solution v_∞ de l'équation (15) : la trajectoire limite est le résultat d'un lissage opéré par le Laplacien sur les observations.

Il est à noter également que des tests préliminaires avec un modèle réaliste d'océan non linéaire donnent des résultats prometteurs.

1. Introduction

Data assimilation consists in estimating the state of a system by combining via numerical methods two different sources of information: models and observations. Data assimilation makes it possible to answer a wide range of questions such as: optimal identification of the initial state of a system, perform reliable numerical forecasts, identify or extrapolate non-observed variables by using a numerical model, etc. [4].

Nudging can be seen as a degenerate Kalman filter. Also known as the Luenberger or asymptotic observer [6], it consists in applying a Newtonian recall of the state value toward its direct observation. The standard nudging algorithm, which initially appeared in meteorology [5], is the first data assimilation method used in operational oceanography [7,8]. A main disadvantage of such sequential data assimilation methods is that it only takes into account past observations at a given time, and not future ones.

Auroux and Blum proposed in [1] an original approach of backward and forward nudging (or Back and Forth Nudging, BFN), which consists in initially solving the forward equations with a nudging term, and then, using the final state as an initial condition, in solving the same equations in a backward direction with a feedback term (with the opposite sign compared to the feedback term of forward nudging). This process is then repeated iteratively until convergence. The implementation of the BFN algorithm has been shown to be very easy, compared to other data assimilation methods [2].

However, several theoretical and numerical studies showed that it was difficult to deal with diffusion processes during backward integrations, leading to instabilities or explosion of the numerical solutions [3]. We propose here an improved Back and Forth Nudging algorithm for diffusive equations in the context of meteorology and oceanography. In these applications, the theoretical equations are usually diffusive free (e.g. Euler's equation for meteorological processes). But then, in a numerical framework, a diffusive term is often added to the equations (or a diffusive scheme is used), in order to both stabilize the numerical integration of the equations, and take into consideration some subscale phenomena. In such situations, it is physically coherent to change the sign of the diffusion term in the backward integrations, in order to keep unchanged both roles of the diffusion term.

After briefly recalling the standard BFN algorithm, we introduce in Section 2 the diffusive BFN algorithm (D-BFN) for such situations. Then we present in Sections 3 and 4 some theoretical and numerical results of this algorithm in idealized situations.

2. Improved Back and Forth Nudging algorithm for quasi-inviscid models

2.1. Standard Back and Forth Nudging algorithm

We first briefly recall the Back and Forth Nudging (BFN) algorithm, introduced in [1]. We assume that the time continuous model satisfies dynamical equations of the form:

$$\partial_t X = F(X), \quad 0 < t < T, \tag{1}$$

with an initial condition $X(0) = x_0$, and where F is the model operator (including spatial derivative operators). We will denote by H the observation operator, allowing one to compare the observations $X_{obs}(t)$ with the corresponding $H(X(t))$, deduced from the state vector $X(t)$.

The Back and Forth Nudging algorithm consists in first solving the forward nudging equation and then the backward nudging equation. The *initial* condition of the backward integration is the final state obtained after integration of the forward nudging equation. At the end of this process, one obtains an estimate of the initial state of the system. This process is iterated until convergence: for $k \geqslant 1$,

$$\begin{cases} \partial_t X_k = F(X_k) + K\big(X_{obs} - H(X_k)\big), \\ X_k(0) = \tilde{X}_{k-1}(0), \quad 0 < t < T, \end{cases} \qquad \begin{cases} \partial_t \tilde{X}_k = F(\tilde{X}_k) - K'\big(X_{obs} - H(\tilde{X}_k)\big), \\ \tilde{X}_k(T) = X_k(T), \quad T > t > 0, \end{cases} \tag{2}$$

with the notation $\tilde{X}_0(0) = x_0$ and where K and K' are gain matrices. We refer to [1–3] for theoretical and numerical results about this algorithm. As explained in the introduction, when the model F contains some diffusion processes, the backward integration may be ill posed, and the numerical experiments may require quite large nudging coefficients and enough observations in order to stabilize the backward integrations.

2.2. Diffusive Back and Forth Nudging (D-BFN) algorithm

In the framework of oceanographic and meteorologic problems, there is usually no diffusion in the model equations. However, the numerical equations that are solved contain some diffusion terms in order to both stabilize the numerical integration (or the numerical scheme is set to be slightly diffusive) and model some subscale processes. We can then separate the diffusion term from the rest of the model terms, and assume that the model equations read:

$$\partial_t X = F(X) + \nu \Delta X, \quad 0 < t < T, \tag{3}$$

where F has no diffusive terms, ν is the diffusion coefficient, and we assume that the diffusion is a standard second-order Laplacian (note that it could be a fourth- or sixth-order derivative in some oceanographic models, but for clarity, we assume here that it is a Laplacian operator).

We introduce the D-BFN algorithm in this framework, for $k \geqslant 1$:

$$\begin{cases} \partial_t X_k = F(X_k) + \nu \Delta X_k + K\big(X_{obs} - H(X_k)\big), \\ X_k(0) = \tilde{X}_{k-1}(0), \quad 0 < t < T, \end{cases} \qquad \begin{cases} \partial_t \tilde{X}_k = F(\tilde{X}_k) - \nu \Delta \tilde{X}_k - K'\big(X_{obs} - H(\tilde{X}_k)\big), \\ \tilde{X}_k(T) = X_k(T), \quad T > t > 0. \end{cases} \tag{4}$$

It is straightforward to see that the backward equation can be rewritten, using $t' = T - t$:

$$\partial_{t'} \tilde{X}_k = -F(\tilde{X}_k) + \nu \Delta \tilde{X}_k + K'\big(X_{obs} - H(\tilde{X}_k)\big), \qquad \tilde{X}_k\big(t' = 0\big) = X_k(T), \tag{5}$$

where $\tilde{X}$ is evaluated at time t'. Then the backward equation can easily be solved, with an initial condition, and the same diffusion operator as in the forward equation. The diffusion term both takes into account the subscale processes and stabilizes the numerical backward integrations, and the feedback term still controls the trajectory with the observations.

The main interest of this new algorithm is that for many geophysical applications, the non-diffusive part of the model is reversible, and the backward model is then stable. Moreover, the forward and backward equations are now consistent in the sense that they will be both diffusive in the same way (as if the numerical schemes were the same in forward and backward integrations), and only the non-diffusive physical model is solved backward. Note that in this case, it is reasonable to set $K' = K$.

3. Convergence of the algorithm

3.1. General result

In this section only, we suppose that the model F and the observation operator H are linear, and that the Cauchy problem for Eq. (3) is well posed with an initial data $X(0) = X_0 \in E$, where E is a suitable Hilbert space. We now consider only one iteration of the D-BFN algorithm (e.g. for $k = 1$). Let us define the following operator that corresponds to one forward and one backward integration:

$$\psi : E \times E \to E, \qquad \big(X_1(0), X_{obs}(0)\big) \mapsto \tilde{X}_1(0), \tag{6}$$

where X_k and $\tilde{X}_k$ satisfy equations (4), and X_{obs} is solution of Eq. (1). This operator is linear in the initial conditions, so that there exist C and D linear operators on E such that

$$X_2(0) = \psi\big(X_1(0), X_{obs}(0)\big) = \psi\big(X_1(0), 0\big) + \psi\big(0, X_{obs}(0)\big) = CX_1(0) + DX_{obs}(0). \tag{7}$$

So that the initial state $X_{k+1}(0)$ of the $(k+1)$th D-BFN iteration satisfies:

$$X_{k+1}(0) = C^k x_0 + \left(\sum_{m=0}^{k-1} C^m\right) DX_{obs}(0). \tag{8}$$

If the spectrum of C is included in the disk $B(0,\rho)$, with $\rho < 1$, then $C^k \to 0$ and $\sum_{m=0}^{k} C^m \to (I-C)^{-1}$ when $k \to \infty$. Therefore, in that case, $X_k(0)$ converges as k goes to infinity to X_∞ solution of

$$X_\infty = (I-C)^{-1} DX_{obs}(0). \tag{9}$$

3.2. Application to a linear transport equation

Let us consider a simple situation, in which the physical model is a linear transport equation on a periodic domain $\Omega = \mathbb{R}/\mathbb{Z}$:

$$\partial_t u + a(x)\partial_x u = 0, \quad t \in [0,T],\ x \in \Omega, \qquad u(t=0) = u_0 \in L^2(\Omega) \tag{10}$$

with periodic boundary conditions, and we assume that $a \in W^{1,\infty}(\Omega)$.

Numerically, for both stability and subscale modeling, the following equation would be solved:

$$\partial_t u + a(x)\partial_x u = \nu \partial_{xx} u, \quad t \in [0,T],\ x \in \Omega, \qquad u(t=0) = u_0 \in L^2(\Omega), \tag{11}$$

where $\nu \geqslant 0$ is assumed to be constant and rather small.

Let us assume that the observations satisfy the physical model (without diffusion):

$$\partial_t u_{obs} + a(x)\partial_x u_{obs} = 0, \quad t \in [0,T],\ x \in \Omega, \qquad u_{obs}(t=0) = u_{obs}^0 \in L^2(\Omega). \tag{12}$$

We assume in this idealized situation that the system is fully observed (and H is then the identity operator).

Then the D-BFN algorithm applied to this problem gives, for $k \geqslant 1$:

$$\begin{cases} \partial_t u_k + a(x)\partial_x u_k = \nu\partial_{xx} u_k + K(u_{obs,k} - u_k), \\ t \in \big[2(k-1)T, 2(k-1)T + T\big], \quad x \in \Omega, \\ u_k\big(2(k-1)T, x\big) = \tilde{u}_{k-1}\big(2(k-1)T, x\big), \end{cases} \qquad \begin{cases} \partial_t \tilde{u}_k - a(x)\partial_x \tilde{u}_k = \nu\partial_{xx}\tilde{u}_k + K(\tilde{u}_{obs,k} - \tilde{u}_k), \\ t \in [2kT - T, 2kT], \quad x \in \Omega, \\ \tilde{u}_k(2kT - T, x) = u_k(2kT - T, x), \end{cases} \tag{13}$$

where we changed the time variable so that the kth iteration of the BFN is solved on $[2(k-1)T, 2kT]$:

$$u_{obs,k}(t,x) = u_{obs}\big(t - 2(k-1)T, x\big), \qquad \tilde{u}_{obs,k}(t,x) = \tilde{u}_{obs}\big(t - 2(k-1)T, x\big) = u_{obs}(2kT - t, x).$$

As previously, to obtain the convergence, we need information on operator C, which is the mapping of $u_k(2(k-1)T,.)$ to $\tilde{u}_k(2kT,.)$ with $u_{obs} = 0$. We write the energy estimate for Eq. (13), with $u_{obs} = 0$ and notations $\|.\| = \|.\|_{L^2(\Omega)}$ and $\|.\|_\infty = \|.\|_{L^\infty(\Omega)}$:

$$d_t\|u\|^2 \leqslant -2\nu\|\partial_x u\|^2 - \big(2K - \|\partial_x a\|_\infty\big)\|u\|^2 \leqslant -\delta\|u\|^2, \tag{14}$$

where $\delta = 2K - \|\partial_x a\|_\infty$ is non-negative for K large enough. Therefore $\|Cu_0\|^2 \leqslant e^{-2\delta T}\|u_0\|^2$, so that $\|C\| < 1$ and convergence is assured.

In the special case where $a(x) = a \in \mathbb{R}$, we can change variables to straighten characteristics as follows. Setting $v_k(t,y) = u_k(t, y + a(t - 2(k-1)T))$ and $\tilde{v}_k(t,z) = \tilde{u}_k(t, z - a(t - 2kT))$ leads to

$$\partial_t v_k = \nu\partial_{yy} v_k + K\big(u_{obs}^0(y) - v_k\big), \qquad \partial_t \tilde{v}_k = \nu\partial_{zz}\tilde{v}_k + K\big(u_{obs}^0(z) - \tilde{v}_k\big).$$

Therefore, at the limit $k \to \infty$, v_k and $\tilde{v}_k$ tend to $v_\infty(x)$ solution of

$$\nu\partial_{xx} v_\infty + K\big(u_{obs}^0(x) - v_\infty\big) = 0, \quad x \in \mathbb{R}. \tag{15}$$

Note that Eq. (15) is well known in signal or image processing, as being the standard linear diffusion restoration equation. In some sense, v_∞ is the result of a smoothing process on the observations u_{obs}, where the degree of smoothness is given by the ratio $\frac{\nu}{K}$.

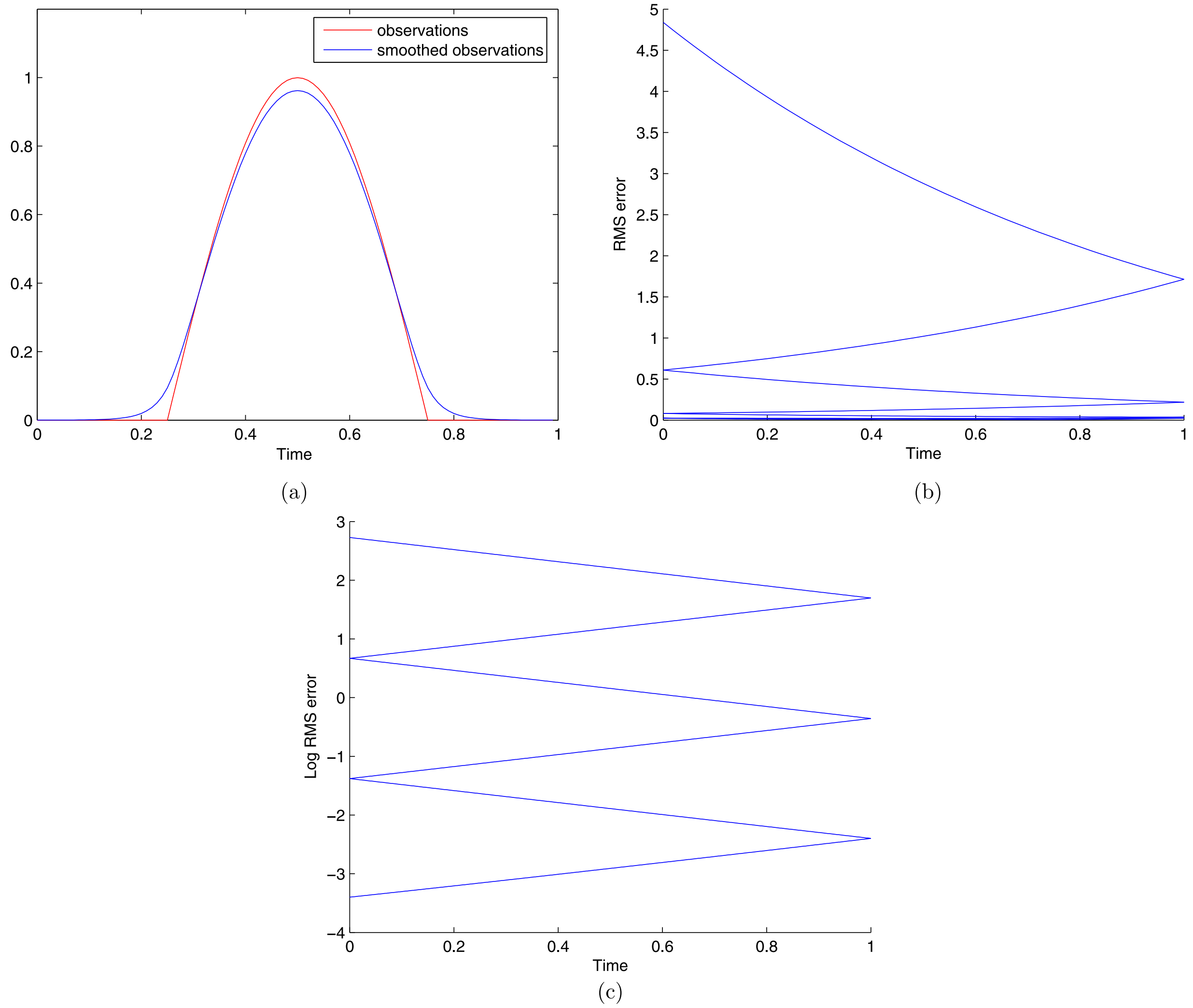

Fig. 1. (a) Initial condition of the observation and corresponding smoothed solution; (b) RMS difference between the BFN iterates and the smoothed observations; (c) same as (b) in semi-log scale.

4. Numerical results

Fig. 1 shows the results of numerical experiments on this simple case. We assume that the domain is periodic (and represented by the torus $[0, 1)$), $a = 0.5$, $T = 1$, $\nu = 0.001$, $K = 1$, and we performed 4 iterations of the D-BFN algorithm. Note that we used a stable and non-diffusive numerical scheme to simulate the observations. The left figure shows the initial condition of the observation u_{obs} (which is the positive part of a sine function), and the corresponding smoothed function v_∞. The center and right figures show the root mean square difference between the BFN iterates and the solution u_∞, in normal and semi-log scales respectively. We initialized the algorithm with a constant function (equal to zero).

As the diffusion is very small, in the first iterations, the theoretical decrease rate of the root mean square error is K, as previously shown in the theoretical computations. We can see in Fig. 1(c) that numerically the slope of the log RMS error is also K (as $K = 1$ in our experiment).

Let us note also that algorithm DBFN is currently being implemented into a non-linear realistic ocean model, showing promising results.

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