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On the modular behaviour of the infinite product $(1-x)(1-xq)(1-xq^2)(1-xq^3)\cdots$

Sur le comportement modulaire du produit infini $(1-x)(1-xq)(1-xq^2)(1-xq^3)\cdots$

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ABSTRACT

Let $q = e^{2\pi i\tau}$, $\Im\tau > 0$, $x = e^{2\pi i\xi} \in \mathbb{C}$ and $(x;q)_\infty = \prod_{n\geqslant 0}(1-xq^n)$. Let $(q,x) \mapsto (q^*, \iota_q x)$ be the classical modular substitution given by $q^* = e^{-2\pi i/\tau}$ and $\iota_q x = e^{2\pi i\xi/\tau}$. The main goal of this Note is to study the "modular behaviour" of the infinite product $(x;q)_\infty$, this means, to compare the function defined by $(x;q)_\infty$ with that given by $(\iota_q x; q^*)_\infty$. Inspired by the work [16] of Stieltjes (1886) on some semi-convergent series, we are led to a "closed" analytic formula for the ratio $(x;q)_\infty/(\iota_q x; q^*)_\infty$ by means of the dilogarithm combined with a Laplace type integral, which admits a divergent series as Taylor expansion at $\log q = 0$. Thus, we can obtain an expression linking $(x;q)_\infty$ to its modular transform $(\iota_q x; q^*)_\infty$ and which contains, in essence, the modular formulae known for Dedekind's eta function, Jacobi theta function and also for certain Lambert series. Among other applications, one can remark that our results allow one to interpret Ramanujan's formula (Berndt, 1994) [5, Entry 6, p. 265 & Entry 6′, p. 268] (see also Ramanujan, 1957 [10, pp. 365 & 284]) as being a convergent expression for the infinite product $(x;q)_\infty$.

RÉSUMÉ

Soit $q = e^{2\pi i\tau}$, $\Im\tau > 0$, $x = e^{2\pi i\xi} \in \mathbb{C}$ et $(x;q)_\infty = \prod_{n\geqslant 0}(1-xq^n)$. Soit $(q,x) \mapsto (q^*, \iota_q x)$ la substitution modulaire classique donnée par $q^* = e^{-2\pi i/\tau}$ et $\iota_q x = e^{2\pi i\xi/\tau}$. Le principal but de la présente Note est d'étudier le « comportemant modulaire » du produit infini $(x;q)_\infty$, c'est-à-dire, de comparer la fonction définie par $(x;q)_\infty$ à celle par $(\iota_q x; q^*)_\infty$. Inspiré du travail [16] de Stieltjes (1886) sur des séries semi-convergentes, nous somme parvenus à une formule analytique « explicite » pour le rapport $(x;q)_\infty/(\iota_q x; q^*)_\infty$ au moyen du dilogarithme complété par une intégrale du type Laplace, cette dernière admettant une série divergente comme développement taylorien en $\log q = 0$. Ceci nous permet d'obtenir une expression reliant $(x;q)_\infty$ à sa transformée modulaire $(\iota_q x; q^*)_\infty$ qui contient essentiellement les formules modulaires connues pour la fonction eta de Dedekind, la fonction theta de Jacobi et aussi pour certaines séries de Lambert. Parmi d'autres applications, on remarquera que nos résultats permettent d'interpréter une formule de Ramanujan (Berndt, 1994) [5, Entry 6, p. 265 & Entry 6′, p. 268] (voir aussi Ramanujan, 1957 [10, pp. 365 & 284]) comme étant une expression convergente pour le produit infini $(x;q)_\infty$.

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Soit $q = e^{2\pi i\tau}$, $\Im\tau > 0$, $x = e^{2\pi i\xi} \in \mathbb{C}$ et considérons le produit infini figurant sur le titre de la Note, noté désormais $(x;q)_\infty$. L'étude de ce dernier peut remonter au temps d'Euler [7, Chap. XVI], à qui l'on doit l'identité remarquable (1) rappelée plus loin. Celle-ci justifie que $((q-1)x;q)_\infty$ est un q-analogue de la fonction exponentielle, compte tenu du fait que $(q;q)_n/(1-q)^n \to n!$ pour $q \to 1$. Le produit infini $(x;q)_\infty$ occupe ainsi une place centrale dans le monde des « q-séries », y compris la théorie des fonctions elliptiques [17] ou celle des équations aux q-différences [19,6,12].

Soit la transformation « modulaire » $\mathcal{M} : (q,x) \mapsto (q^*, \iota_q x) = (e^{-2\pi i/\tau}, e^{2\pi i\xi/\tau})$; afin de comparer la fonction $(x;q)_\infty$ à sa transformée modulaire $(\iota_q x; q^*)_\infty$, commençons par les observations suivantes :

(i) Dans le dernier paragraphe [16, pp. 252–258] de l'article de Stieltjes intitulé « Recherches sur quelques séries semi-convergentes », une intégrale « singulière à la Cauchy », *i.e.*, avec des valeurs principales, a été donnée pour représenter une série de Lambert $P(a)$ et a permis de lier $P(a)$ à $P(1/a)$.
(ii) Le produit infini $(q;q)_\infty$, lié à la fonction eta de Dedekind ou encore à la fonction τ de Ramanujan, est muni d'une formule modulaire [14, (44), p. 154] et celui-ci est la valeur de $(x;q)_\infty$ prise au point $x = q$.
(iii) La fonction theta de Jacobi vérifie une relation modulaire et elle peut s'écrire comme le produit de $(q;q)_\infty$ par le facteur $(\sqrt{q}x;q)_\infty(\sqrt{q}/x;q)_\infty$, lequel est invariant par $x \mapsto 1/x$.

Cela étant, nous sommes amenés à la question de savoir si la méthode (i) de Stieltjes est susceptible d'être étenduée à la fonction $(x;q)_\infty$ ou $\log(x;q)_\infty$ pour obtenir une relation, de la forme

$$(x;q)_\infty = K(q,x)\big(\iota_q x; q^*\big)_\infty, \tag{K}$$

vérifiant les propriétés ci-dessous :

(iv) Le facteur multiplicatif $K(x,q)$ sera donné « aussi explicite » que possible.
(v) Losque $x = q$ ou que les facteurs $(\sqrt{q}x;q)_\infty$ et $(\sqrt{q}/x;q)_\infty$ sont mis ensemble, la relation (K) voulue ci-dessus nous conduira aux formules modulaires rappelées respectivement dans (ii) et (iii) ci-dessus.

Le principal but de la Note est de montrer que la méthode de Stieltjes évoquée ci-dessus répond à notre question et que le facteur $K(q,x)$ recherché dans la relation (K) sera donné au moyen du dilogarithme avec une intégrale de Laplace, ce qui prouvera également l'aspect « semi-convergent » de $(x;q)_\infty$ pour $q \to 1$, $|q| < 1$; voir Theorem 3.2 et la relation correspondante (14) ou les formulations (15)–(17). Par ailleurs, vu les relations (18)–(19), nous en déduirons aisément la formule modulaire connue de la fonction η de Dedekind et aussi celle de la fonction θ de Jacobi. En considérant la dérivée logarithmique par rapport à la variable x ou q pour chacun des côtés de (14), nous en déduirons des formules qui relient chaque série de Lambert (généralisée) à la série obtenue après la substitution $(q,x) \mapsto \mathcal{M}(q,x)$.

Losque q tend vers la valeur unité, q^* devient exponentiellement petit ; ainsi la série $(\iota_q x; q^*)_\infty$ échappe-t-elle à toute représentation en termes de série entière, convergente ou semi-convergente, de la variable $\log q$. Dans le *Notebooks* de Ramanujan, on trouve au moins deux formules consacrées au développement du produit $(x;q)_\infty$ au moyen du dilogarithme et d'une série entière de $\log q$; elles donnent lieu aux Entry 6, p. 265 et Entry 6′, p. 268 de [5]. Entry 6 est interprétée dans [5] comme une formule asymptotique de $\log(x;q)_\infty$ alors que le manuscrit de Ramanujan utilise constamment le symbole « = », la plupart de ses séries entières rencontrées étant « semi-convergentes ». Nous ne savons pas si Ramanujan voulait réellement chercher une expression convergente pour son objet d'étude ; nous avons l'impression que le présent travail permet de completer une telle expression analytique, en tenant compte de la partie modulaire $(\iota_q x; q^*)_\infty$, qui est pourtant exponentiellement petite. Ceci nous fait penser aux commentaires de Selberg [13] sur la formule de $p(n)$ initiée par Hardy–Ramanujan puis achevée par Rademacher (et Selberg lui-même).

Notons enfin que les résultats énoncés dans la Note sont extraits d'un article mis en ligne qui est référencé en arXiv:0905.1343v1 [math.NT] ; voir [20].

1. Introduction

There are more and more studies on q-series and related topics, not only in traditional themes, but also in more recent branches, such as quantum physics, random matrices. A first non-trivial example of q-series may be the infinite product $(q;q)_\infty := \prod_{n\geqslant 1}(1-q^n)$, that is considered in Euler [7, Chap. XVI] and then is revisited by many of his successors, particularly intensively by Hardy and Ramanujan [8, pp. 238–241; pp. 276–309; pp. 310–321]. Beautiful formulae are numerous and motivations are often various: elliptic and modular functions theory, number and partition theory, orthogonal polynomials theory, etc. Concerning this wonderful history, one may think of Euler's pentagonal number theorem [2, p. 30], Jacobi's triple product identity [3, §10.4, pp. 496–501], Dedekind modular eta function [14, (44), p. 154], [1, Chapter 3], to quote only some examples of important masterpieces.

However, the infinite product $(x;q)_\infty := \prod_{n\geqslant 0}(1-xq^n)$, also already appearing as initial model in the same work [7, Chap. XVI] of Euler, receives less attention although it always plays a remarkable role in all above-mentioned subjects: thanks to Euler, one knows the following remarkable formula:

$$(x;q)_\infty = \sum_{n\geqslant 0} \frac{q^{n(n-1)/2}}{(q;q)_n}(-x)^n \quad \bigl((q;q)_n = (q;q)_\infty/\bigl(q^{n+1};q\bigr)_\infty\bigr), \tag{1}$$

which says in what manner $((q-1)x;q)_\infty$ may be viewed as a q-analog of the usual exponential function; see [19,6,12] for a point of view from the analytic theory of q-difference equations. Moreover, $(x;q)_\infty$ is immediately linked with several constants in the elliptic integral theory, Gauss' binomial formula [3, §10.2, pp. 487–491], generalized Lambert series, etc. Indeed, rapidly the situation may become more complicated and, what is really important, the modular relation does not occur for generic values of x.

In this Note, we shall point out how, up to an explicit part, the function defined by the product $(x;q)_\infty := (1-x)(1-xq)(1-xq^2)(1-xq^3)\cdots$ can be seen somewhat modular. This *non-modular part* can be considered as being represented by a divergent but Borel-summable or semi-convergent power series on variable $\log q$ near zero, that is, when q tends toward the unit value. These results, subjects of Theorems 2.1 and 3.2, give rise to new and unified approaches to treat Jacobi theta function, Lambert series, ...

As $q \to 1-$, $q^* = e^{4\pi^2/\log q}$ becomes exponentially small and the modular part cannot be represented by any semi-convergent power series of $\log q$. Remember the power series encoding the *non-modular part* appeared already in [5, Entry 6, p. 265] and [10, p. 365]. If we compare [5, Entry 6, p. 265] with [5, Entry 6′, p. 268], we would like to believe Ramanujan really wanted to give a convergent expression to $\log(x;q)_\infty$. This is very like the history of the Hardy–Ramanujan's formula on $p(n)$ which finally becomes completed by Rademacher (and Selberg); see the beautiful paper [13].

The note is organized as follows. In Section 2, we will give some integral representations about the real-valued function $\log(x;q)_\infty$ for $q \in (0,1)$ and $x \in (0,1)$. In Section 3, we will extend the above results to the complex case and therefore find the modular type formula for $(x;q)_\infty$.

2. Integral representations of $\log(x;q)_\infty$

In this section, we will suppose that $q \in (0,1)$ and $x \in (0,1)$, so that the infinite product $(x;q)_\infty$ takes values in $(0,1)$. By applying an idea of Stieltjes [16, pp. 252–258] to the real-valued function $\log(x;q)_\infty$, we firstly find the following statement:

Theorem 2.1. *Let $q = e^{-2\pi\alpha}$, $x = e^{-2\pi(1+\nu)\alpha}$ and suppose $\alpha > 0$ and $\nu > -1$. The following relation holds*:

$$\log(x;q)_\infty = -\frac{\pi}{12\alpha} + \log\frac{\sqrt{2\pi}}{\Gamma(\nu+1)} + \frac{\pi}{12}\alpha - \left(\nu+\frac{1}{2}\right)\log\frac{1-e^{-2\pi\nu\alpha}}{\nu} + \int_0^\nu \left(\frac{2\pi\alpha t}{e^{2\pi\alpha t}-1} - 1\right)dt + M(\alpha,\nu), \tag{2}$$

where

$$M(\alpha,\nu) = -\sum_{n=1}^{\infty}\frac{\cos 2n\pi\nu}{n(e^{2n\pi/\alpha}-1)} - \frac{2}{\pi}\mathcal{PV}\int_0^\infty \sum_{n=1}^{\infty}\frac{\sin 2n\nu\pi t}{n(e^{2n\pi t/\alpha}-1)}\frac{dt}{1-t^2}. \tag{3}$$

In the above, $\mathcal{PV}\int$ stands for the principal value of a singular integral in the Cauchy's sense; see [17, §6.23, p. 117]. If we let

$$B(t) = \frac{1}{e^{2\pi t}-1} - \frac{1}{2\pi t} + \frac{1}{2}, \tag{4}$$

it is easy to see that Theorem 2.1 can be stated as follows:

Theorem 2.2. *Let q, x, α, ν and $M(\alpha,\nu)$ be as given in Theorem 2.1. Then the following relation holds*:

$$\log(x;q)_\infty = -\frac{\pi}{12\alpha} - \left(\nu+\frac{1}{2}\right)\log 2\pi\alpha + \log\frac{\sqrt{2\pi}}{\Gamma(\nu+1)} + \frac{\pi}{2}(\nu+1)\nu\alpha + \frac{\pi}{12}\alpha + 2\pi\alpha\int_0^\nu\left(t-\nu-\frac{1}{2}\right)B(\alpha t)\,dt + M(\alpha,\nu).$$

As usual, let Li_2 denote the dilogarithm function; recall Li_2 can be defined as follows [3, (2.6.1-2), p. 102] or [18]:

$$\mathrm{Li}_2(x) = -\int_0^x \log(1-t)\frac{dt}{t} = \sum_{n=0}^{\infty}\frac{x^{n+1}}{(n+1)^2}. \tag{5}$$

Theorem 2.3. *The following relation holds for any $q \in (0,1)$ and $x \in (0,1)$:*

$$\log(x;q)_\infty = \frac{1}{\log q}\mathrm{Li}_2(x) + \log\sqrt{1-x} - \frac{\log q}{24} - \int_0^\infty B\left(-\frac{\log q}{2\pi}t\right)x^t\frac{\mathrm{d}t}{t} + M\left(-\frac{\log q}{2\pi}, \log_q x\right). \tag{6}$$

We shall write the singular integral part in (3) by means of contour integration in the complex plane. Fix a real $r \in (0,1)$ and, for any positive integer n, let $C_{n,r}^-$ (resp., $C_{n,r}^+$) be the half-circle passing from $n-r$ to $n+r$ by the right (resp., left) hand side. Let $\ell_r^\mp = (0, 1-r) \cup (\bigcup_{n\geqslant 1}(C_{n,r}^\mp \cup (n+r, n+1-r)))$ and define $P^\mp(\alpha,\nu)$ as follows:

$$P^\mp(\alpha,\nu) = \int_{\ell_r^\mp} \frac{\sin \nu t}{e^{t/\alpha}-1}\left(\cot\frac{t}{2} - \frac{2}{t}\right)\frac{\mathrm{d}t}{t}, \tag{7}$$

where $\alpha > 0$ and where ν may be an arbitrary real number.

Theorem 2.4. *Let M be as in Theorem* 2.1 *and let P^- be as in* (7)*. For any $\nu \in \mathbb{R}$ and $\alpha > 0$, the following relation holds*:

$$M(\alpha,\nu) = \log\left(e^{2\pi\nu i - 2\pi/\alpha}; e^{-2\pi/\alpha}\right)_\infty + P^-(\alpha,\nu).$$

Consequently, the term M appearing in Theorem 2.1 can be considered as being an *almost modular term* of $\log(x;q)_\infty$; the correction term P^- given by (7) will be called *disruptive factor or perturbation term*.

3. Modular type expansion of $(x;q)_\infty$

From now on, we will work with complex variables q and x, with $q = e^{2\pi i\tau}$, $\tau \in \mathcal{H}$ and $x = e^{2\pi i\xi} \in \mathbb{C}^*$. As usual, log will stand for the principal branch of the logarithmic function over its Riemann surface $\tilde{\mathbb{C}}^*$. For any pair of real numbers $a < b$, we will define

$$S(a,b) := \left\{z \in \tilde{\mathbb{C}}^*\colon \arg z \in (a,b)\right\}; \tag{8}$$

therefore, the Poincaré's half-plane $\mathcal{H}$ will be identified to $S(0,\pi)$ while the broken plane $\mathbb{C} \setminus (-\infty, 0]$ will be seen as the subset $S(-\pi,\pi) \subset \tilde{\mathbb{C}}^*$.

Firstly, let us introduce the following modified complex version of P^- given by (7): for any $d \in (-\pi, 0)$, let

$$P^d(\tau,\xi) = \int_0^{\infty e^{id}} \frac{\sin\frac{\xi}{\tau}t}{e^{it/\tau}-1}\left(\cot\frac{t}{2} - \frac{2}{t}\right)\frac{\mathrm{d}t}{t}, \tag{9}$$

the path of integration being the half-line starting from origin to infinity with argument d. Note that P^d is analytic over the domain Ω^d if we set

$$\Omega^d = \bigcup_{\sigma\in(0,\pi)} \left(0, \infty e^{i(d+\sigma)}\right) \times \left\{\xi \in \mathbb{C}\colon \left|\Im\left(\xi e^{-i\sigma}\right)\right| < \sin\sigma\right\}.$$

The family of functions $\{P^d\}_{d\in(-\pi,0)}$ given by (9) gives rise to an analytical function over the domain

$$\Omega_- := S(-\pi,\pi) \times \left(\mathbb{C} \setminus \left((-\infty,-1] \cup [1,\infty)\right)\right) \subset \mathbb{C}^2. \tag{10}$$

Moreover, if we denote this function by $P_-(\tau,\xi)$, then the following relation holds for all $\alpha > 0$ and $\xi \in \mathbb{R}$:

$$P_-(\alpha i, \xi\alpha i) = P^-(\alpha,\xi). \tag{11}$$

On the other hand, if we take the arguments $d \in (0,\pi)$ instead of $d \in (-\pi, 0)$ in (9), we can get an analytical function, say P_+, defined over

$$\Omega_+ := S(0, 2\pi) \times \left(\mathbb{C} \setminus \left((-\infty,-1] \cup [1,\infty)\right)\right).$$

Therefore, one gets the following Stokes relation related to the above Laplace type integral (9); see [11]:

Theorem 3.1. *For any $\tau \in \mathcal{H}$, the relation*

$$P_-(\tau,\xi) - P_+(\tau,\xi) = 2i\sum_{n=1}^\infty \frac{\sin\frac{2n\xi\pi}{\tau}}{n(e^{2n\pi i/\tau}-1)}$$

holds provided that $|\Im(\xi/\tau)| < -\Im(1/\tau)$.

Secondly, the integral term involving the function B in formula (6) of Theorem 2.3 is related to the remainder term of the Stirling's formula on Γ-function. Indeed, an elementary analysis leads to introducing the following function:

$$G(\tau,\xi)=-\log\Gamma\left(\frac{\xi}{\tau}+1\right)+\left(\frac{\xi}{\tau}+\frac{1}{2}\right)\log\frac{\xi}{\tau}-\frac{\xi}{\tau}+\log\sqrt{2\pi}, \tag{12}$$

for all pair $(\tau,\xi)\in U^+:=\{(\tau,\xi)\in\mathbb{C}^*\times\mathbb{C}^*\colon\ \xi/\tau\notin(-\infty,0]\}$. One can check the following relation:

$$G(\tau,\xi)+G(\tau,-\xi)=\log\bigl(1-e^{\mp 2\pi i\xi/\tau}\bigr) \tag{13}$$

according to $\frac{\xi}{\tau}\in S(-\pi,0)$ or $S(0,\pi)$, respectively.

Therefore, we are ready to extend Theorem 2.3 into the complex plane. We recall the notations $q^*=e^{-2\pi i/\tau}$ and $\iota_q x=e^{2\pi i\xi/\tau}$; we will write x^* instead of $\iota_q x$.

Theorem 3.2. *The following relation holds for any $\tau\in\mathcal{H}$ and $\xi\in\mathbb{C}\setminus((-\infty,-1]\cup[1,\infty))$ such that $\xi/\tau\notin(-\infty,0]$:*

$$(x;q)_\infty=q^{-1/24}\sqrt{1-x}\bigl(x^*q^*;q^*\bigr)_\infty\exp\left(\frac{\mathrm{Li}_2(x)}{\log q}+G(\tau,\xi)+P(\tau,\xi)\right), \tag{14}$$

where $\sqrt{1-x}$ stands for the principal branch of $e^{\frac{1}{2}\log(1-x)}$, Li_2 denotes the dilogarithm recalled in (5)*, G is given by* (12) *and where P denotes the function P_- defined over Ω_- as in* (10)–(11)*.*

If we denote by G^* the anti-symmetrization of G given by

$$G^*(\tau,\xi)=\frac{1}{2}\bigl(G(\tau,\xi)-G(\tau,-\xi)\bigr),$$

then, according to relation (13), we may rewrite (14) as follows:

$$(x;q)_\infty=K(q,x)\bigl(x^*;q^*\bigr)_\infty, \tag{15}$$

where the factor $K(q,x)$ is given in the following manner:

$$K(q,x)=q^{-1/24}\sqrt{\frac{1-x}{1-x^*}}\exp\left(\frac{\mathrm{Li}_2(x)}{\log q}+G^*(\tau,\xi)+P(\tau,\xi)\right) \tag{16}$$

if $\xi\in\tau\mathcal{H}$, and

$$K(q,x)=q^{-1/24}\frac{\sqrt{(1-x)(1-1/x^*)}}{1-x^*}\exp\left(\frac{\mathrm{Li}_2(x)}{\log q}+G^*(\tau,\xi)+P(\tau,\xi)\right) \tag{17}$$

if $\xi\in-\tau\mathcal{H}$, that is, if $\frac{\xi}{\tau}\in S(-\pi,0)$.

In the above, G^* and P are odd functions on the variable ξ:

$$G^*(\tau,-\xi)=-G^*(\tau,\xi),\qquad P(\tau,-\xi)=-P(\tau,\xi), \tag{18}$$

Li_2 satisfies the so-called *Landen's transformation* [3, Theorem 2.6.1, p. 103]:

$$\mathrm{Li}_2(1-x)+\mathrm{Li}_2\left(1-\frac{1}{x}\right)=-\frac{1}{2}(\log x)^2. \tag{19}$$

Finally, if we write $\boldsymbol{\omega}=(\omega_1,\omega_2)=(1,\tau)$ and denote by $\Gamma_2(z,\boldsymbol{\omega})$ the Barnes' double Gamma function associated to the double period $\boldsymbol{\omega}$ [4], then Theorem 3.2 and Proposition 5 of [15] imply that

$$\begin{aligned}\frac{\Gamma_2(1+\tau-\xi,\boldsymbol{\omega})}{\Gamma_2(\xi,\boldsymbol{\omega})}&=\sqrt{i}\sqrt{1-x}\exp\left(\frac{\pi i}{12\tau}+\frac{\pi i}{2}\left(\frac{\xi^2}{\tau}-\left(1+\frac{1}{\tau}\right)\xi\right)+\frac{\mathrm{Li}_2(x)}{\log q}+G(\tau,\xi)+P(\tau,\xi)\right)\\&=\sqrt{2\sin\pi\xi}\exp\left(\frac{\pi i}{12\tau}+\frac{\xi(\xi-1)\pi i}{2\tau}+\frac{\mathrm{Li}_2(e^{2\pi i\xi})}{2\pi i\tau}+G(\tau,\xi)+P(\tau,\xi)\right).\end{aligned}$$

Remark 1. By using (1) and taking into account (14), one can get the Ramanujan's asymptotic expansion formula [5, Entry 6, p. 265] and its variant [5, Entry 6′, p. 268]. For some related subjects, see also [9].

Remark 2. From Theorem 3.2 and the relations (18)–(19), one can easily deduce the modular formula for Dedekind η-function and that for Jacobi θ-function [3, §10.4, pp. 496–501]. Moreover, when taking the logarithmic derivative with respective to the variable x in (14), one can get modular type relations for (generalized) Lambert series.

The results announced in this Note are included in a paper that one can find online as arXiv:0905.1343v1 [math.NT]; see [20].

References

[1] T.M. Apostol, Modular Functions and Dirichlet Series in Number Theory, second edition, Gradute Texts in Mathematics, vol. 41, Springer-Verlag, New York, 1990.
[2] G.E. Andrews, The Theory of Partitions, Cambridge University Press, Cambridge, 1976.
[3] G.E. Andrews, R. Askey, R. Roy, Special Functions, Cambridge University Press, 2000.
[4] E.W. Barnes, The theory of the double gamma function, Phil. Trans. R. Soc. Lond. A 196 (1901) 265–388.
[5] B.C. Berndt, Ramanujan's Notebooks, Part IV, Springer-Verlag, New York, 1994.
[6] L. Di Vizio, C. Zhang, On q-summation and confluence, Ann. Inst. Fourier 59 (2009) 347–392.
[7] L. Euler, Introducio in Analysin Infinitorum, 1748.
[8] G.H. Hardy, Ramanujan: Twelve Lectures on Subjects Suggested by His Life and Work, Chelsea Publishing Company, New York, 1940 (originally published by Cambridge University Press).
[9] M. Katsurada, Asymptotic expansions of certain q-series and a formula of Ramanujan for specific values of the Riemann zeta function, Acta Arith. 107 (3) (2003) 269–298.
[10] S. Ramanujan, Notebooks (2 volumes), Tata Institute of Fundamental Research, Bombay, 1957.
[11] J.-P. Ramis, Séries divergentes et théories asymptotiques, in: Panoramas et Synthèses (Suppl.), Bull. Soc. Math. France 121 (1993).
[12] J.-P. Ramis, J. Sauloy, C. Zhang, Local analytic classification of q-difference equations, arXiv:0903.0853.
[13] A. Selberg, Reflections on Ramanujan centenary, in: B.C. Berndt, R.A. Rankin (Eds.), Ramanujan: Essays and Surveys, in: History of Mathematics, vol. 22, 1989, pp. 203–214.
[14] J.-P. Serre, Cours d'arithmétique, Deuxième édition revue et corrigé, Presses Universitaires de France, Paris, 1977.
[15] T. Shintani, On a Kronecker limit formula for real quadratic fields, J. Fac. Sci. Univ. Tokyo Sect. IA Math. 24 (1) (1977) 167–199.
[16] T.J. Stieltjes, Recherches sur quelques séries semi-convergentes, Ann. Sci. École Norm. Sup. (3) 3 (1886) 201–258;
See also: T.J. Stieltjes, Colleted Papers, vol. II, Springer-Verlag, New York, 1993.
[17] E.T. Whittaker, G.N. Watson, A Course of Modern Analysis, fourth edition, Cambridge University Press, New York, 1962.
[18] D. Zagier, The dilogarithm function in geometry and number theory, in: Number Theory and Related Topics, Tata Inst. Fund. Res., Bombay, in: Tata Inst. Fund. Res. Stud. Math., vol. 12, Oxford University Press, Oxford, 1989, pp. 231–249.
[19] C. Zhang, Une sommation discrète pour des équations aux q-différences linéaires et à coefficients analytiques : théorie générale et exemples, in: B.L.J Braaksma, et al. (Eds.), Differential Equations and the Stokes Phenomenon, World Scientific, 2002, pp. 309–329.
[20] C. Zhang, A modular type formula for Euler infinite product $(1-x)(1-xq)(1-xq^2)(1-xq^3)\cdots$, arXiv:0905.1343v1.