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Number Theory

Eisenstein cohomology and ratios of critical values of Rankin–Selberg L-functions

Cohomologie d'Eisenstein et rapports de valeurs critiques des fonctions L de Rankin–Selberg

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ABSTRACT

This is an announcement of results on rank-one Eisenstein cohomology of GL_N, with $N \geqslant 3$ an odd integer, and algebraicity theorems for ratios of successive critical values of certain Rankin–Selberg L-functions for $\mathrm{GL}_n \times \mathrm{GL}_{n'}$ when n is even and n' is odd

RÉSUMÉ

Cette Note annonce des résultats sur la cohomologie d'Eisenstein de rang 1 de GL_N, avec $N \geqslant 3$ un entier impair, et donne des théorèmes d'algébricité pour les rapports de valeurs critiques successives de certaines fonctions L de Rankin–Selberg pour $\mathrm{GL}_n \times \mathrm{GL}_{n'}$ lorsque n est pair et n' est impair.

Version française abrégée

Soit $\sigma_f \in \mathrm{Coh}(\mathrm{GL}_n, \lambda)$, ce qui signifie que σ_f est un $\mathrm{GL}_n(\mathbb{A}_f)$-facteur de la cohomologie intérieure $H_!^\bullet(S_{K_f}^{\mathrm{GL}_n}, \mathcal{E}_\lambda)$ d'un espace $S_{K_f}^{\mathrm{GL}_n} := \mathrm{GL}_n(\mathbb{Q})\backslash \mathrm{GL}_n(\mathbb{A})/K_\infty^\circ K_f$ à coefficients dans le faisceau $\mathcal{E}_\lambda$ provenant d'une représentation irréductible algébrique de plus haut poids λ, cf. Section 1. Quand n est pair et λ est régulier, un tel σ_f apparaît deux fois dans $H_!^\bullet(S_{K_f}^{\mathrm{GL}_n}, \mathcal{E}_\lambda)$ pour $\bullet = n^2/4$. En comparant ces deux copies de σ_f, on en déduit une période $\Omega^\epsilon(\sigma_f, \iota) \in \mathbb{C}^\times$, où ι est un plongement du corps de rationalité de σ_f dans la clôture algébrique de $\mathbb{Q}$ dans $\mathbb{C}$, cf. définition 2.1.

Soit maintenant $\sigma'_f \in \mathrm{Coh}(\mathrm{GL}_{n'}, \lambda')$ pour un entier impair n'. Posons $N = n + n'$. Soit $m \in \frac{1}{2} + \mathbb{Z}$ tel que m et $m+1$ soient critiques pour la fonction L de Rankin–Selberg $L(\sigma_f \times \sigma_f'^{\vee}, \iota, s)$. En supposant la validité d'un certain lemme combinatoire (voir Conjecture 5.1) notre résultat principal sur les valeurs critiques affirme que

$$\frac{1}{\Omega(\sigma_f, \iota)^{\epsilon_m \epsilon_{\sigma'}}} \frac{\Lambda(\sigma_f \times \sigma_f'^{\vee}, \iota, m)}{\Lambda(\sigma_f \times \sigma_f'^{\vee}, \iota, m+1)}$$

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est algébrique et Galois-équivariant, cf. Théorème 4.1. Ici $\Lambda(\sigma_f \times \sigma_f'^{\vee}, \iota, s)$ est la fonction L complétée.

Le Théorème 4.1 se démontre en étudiant l'image (appelée «cohomologie d'Eisenstein») de la cohomologie globale $H^\bullet(S^{\mathrm{GL}_N}, \mathcal{E}_{\tilde{\mu}})$ dans la cohomologie $H^\bullet(\partial S^{\mathrm{GL}_N}, \mathcal{E}_{\tilde{\mu}})$ de la frontière de Borel–Serre $\partial S^{\mathrm{GL}_N}$ de S^{GL_N}. Nous étudions en particulier ceci pour la cohomologie en degré $\bullet = (N^2-1)/4$ et pour un plus haut poids $\tilde{\mu}$ qui dépend des poids λ et λ' via le lemme combinatoire. Le Théorème 5.2 donne une caractérisation de cette image.

1. The general situation

Let $G/\mathbb{Q}$ be a connected split reductive algebraic group over $\mathbb{Q}$ whose derived group $G^{(1)}/\mathbb{Q}$ is simply connected. Let $Z/\mathbb{Q}$ be the center of G and let S be the maximal $\mathbb{Q}$-split torus in Z. Let C_∞ be a maximal compact subgroup of $G(\mathbb{R})$ and let $K_\infty = C_\infty S(\mathbb{R})^\circ$. The connected component of the identity of K_∞ is denoted K_∞° and $K_\infty/K_\infty^\circ = \pi_0(K_\infty) \xrightarrow{\sim} \pi_0(G(\mathbb{R}))$. Let $K_f = \prod_p K_p \subset G(\mathbb{A}_f)$ be an open compact subgroup; here $\mathbb{A}$ is the adèle ring of $\mathbb{Q}$ and $\mathbb{A}_f$ is the ring of finite adèles. The locally symmetric space of G with level structure K_f is defined as

$$S^G_{K_f} := G(\mathbb{Q})\backslash G(\mathbb{A})/K_\infty^\circ K_f.$$

(For the following see Harder [6, Chapter 3, Sections 2, 2.1, 2.2] for details.) For a dominant integral weight λ, let E_λ be an absolutely irreducible finite-dimensional representation of $G/\mathbb{Q}$ with highest weight λ, and let $\mathcal{E}_\lambda$ denote the associated sheaf on $S^G_{K_f}$. We have an action of the Hecke-algebra $\mathcal{H} = \mathcal{H}^G_{K_f} = \bigotimes_p' \mathcal{H}_p$ on the cohomology groups $H^\bullet(S^G_{K_f}, \mathcal{E}_\lambda)$.

We always fix a level, but sometimes drop it in the notation. For any finite extension $F/\mathbb{Q}$, let $E_{\lambda,F} = E_\lambda \otimes_{\mathbb{Q}} F$, then $\mathcal{E}_{\lambda,F}$ is the corresponding sheaf on $S^G_{K_f}$.

Let $\bar{S}^G_{K_f}$ be the Borel–Serre compactification of $S^G_{K_f}$, i.e., $\bar{S}^G_{K_f} = S^G_{K_f} \cup \partial\bar{S}^G_{K_f}$, where the boundary is stratified as $\partial\bar{S}^G_{K_f} = \bigcup_P \partial_P S^G_{K_f}$ with P running through the conjugacy classes of proper parabolic subgroups defined over $\mathbb{Q}$. The sheaf $\mathcal{E}_{\lambda,F}$ on $S^G_{K_f}$ naturally extends, using the definition of the Borel–Serre compactification, to a sheaf on $\bar{S}^G_{K_f}$ which we also denote by $\mathcal{E}_{\lambda,F}$. Restriction from $\bar{S}^G_{K_f}$ to $S^G_{K_f}$ in cohomology induces an isomorphism $H^i(\bar{S}^G, \mathcal{E}_\lambda) \xrightarrow{\sim} H^i(S^G, \mathcal{E}_\lambda)$.

Our basic object of interest is the following long exact sequence of $\pi_0(K_\infty) \times \mathcal{H}$-modules

$$\cdots \longrightarrow H^i_c(S^G, \mathcal{E}_\lambda) \xrightarrow{\iota^*} H^i(\bar{S}^G, \mathcal{E}_\lambda) \xrightarrow{r^*} H^i(\partial\bar{S}^G, \mathcal{E}_\lambda) \longrightarrow H^{i+1}_c(S^G, \mathcal{E}_\lambda) \longrightarrow \cdots.$$

The image of cohomology with compact supports inside the full cohomology is called *inner* or *interior* cohomology and is denoted $H^\bullet_! := \mathrm{Image}(\iota^*) = \mathrm{Im}(H^\bullet_c \to H^\bullet)$. The theory of Eisenstein cohomology is designed to describe the image of the restriction map r^*. Our goal is to study the arithmetic information contained in the above exact sequence.

The inner cohomology is a semi-simple module for the Hecke-algebra. (See Harder [6, Chap. 3, 3.3.5].) After a suitable finite extension $F/\mathbb{Q}$, where $\mathbb{Q} \subset F \subset \bar{\mathbb{Q}} \subset \mathbb{C}$, we have an isotypical decomposition

$$H^i_!(S^G_{K_f}, \mathcal{E}_{\lambda,F}) = \bigoplus_{\pi_f \in \mathrm{Coh}(G,K_f,\lambda)} H^i_!(S^G_{K_f}, \mathcal{E}_{\lambda,F})(\pi_f)$$

where π_f is an isomorphism type of an absolutely irreducible $\mathcal{H}$-module, i.e., an F-vector space H_{π_f} with an absolutely irreducible action of $\mathcal{H}$. The local factors $\mathcal{H}_p$ are commutative outside a finite set $V = V_{K_f}$ of primes and the factors $\mathcal{H}_p$ and $\mathcal{H}_q$, for $p \neq q$, commute with each other. Hence for $p \notin V$ the commutative algebra $\mathcal{H}_p$ acts on H_{π_f} by a homomorphism $\pi_p : \mathcal{H}_p \to F$. Let H_{π_p} be the one dimensional vector space F with basis $1 \in F$ with the action π_p on it. Then $H_{\pi_f} = \bigotimes_{p\in V} H_{\pi_p} \bigotimes'_{p\notin V} H_{\pi_p} = \bigotimes'_p H_{\pi_p}$. The set of isomorphism classes which occur in the above decomposition is called the 'spectrum' $\mathrm{Coh}(G, K_f, \lambda)$. If we restrict the elements of the Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ to F we get the conjugate embeddings of F into $\bar{\mathbb{Q}}$; we introduce $\mathcal{I}(F) = \{\iota : F \to \mathbb{C}\} = \{\iota : F \to \bar{\mathbb{Q}}\}$. For $\iota \in \mathcal{I}(F)$ define $\iota \circ \pi_f$ as $H_{\pi_f} \otimes_{F,\iota} \mathbb{C}$. We define the rationality field of π_f as $\mathbb{Q}(\pi_f) = \{x \in F \mid \iota(x) = \iota'(x) \text{ if } \iota \circ \pi_f = \iota' \circ \pi_f\}$.

2. The case of GL_n and the definition of relative periods when n is even

Let $T/\mathbb{Q}$ be a maximal $\mathbb{Q}$-split torus in G, let $T^{(1)} = T \cap G^{(1)}$. Let $X^*(T)$ be its group of characters then restriction of characters gives an inclusion $X^*(T) \subset X^*(T^{(1)}) \oplus X^*(Z)$ and after tensoring by $\mathbb{Q}$ this becomes an isomorphism. Any $\lambda \in X^*(T)$ can be written as $\lambda^{(1)} + \delta$, $\lambda^{(1)} \in X^*(T^{(1)}) \otimes \mathbb{Q} =: X^*_{\mathbb{Q}}(T^{(1)})$, $\delta \in X^*_{\mathbb{Q}}(Z)$.

Consider the case $G = \mathrm{GL}_n/\mathbb{Q}$. Take a regular essentially self-dual dominant integral highest weight λ. Let $\rho \in X^*_{\mathbb{Q}}(T^{(1)})$ be half the sum of positive roots, and write $\lambda + \rho = a_1\gamma_1 + \cdots + a_{n-1}\gamma_{n-1} + d \cdot \det$, which is an equation in $X^*_{\mathbb{Q}}(T)$; the $\gamma_i \in X^*_{\mathbb{Q}}(T)$ restrict to the fundamental weights in $X^*(T^{(1)})$ and are trivial on the center Z. Regular, dominant and integral mean that $a_i \geqslant 2$ are integers, and essentially self-dual means $a_i = a_{n-i}$. Further, for such a weight λ we have $2d \in \mathbb{Z}$ and it satisfies the parity condition:

$$2d \equiv \mathbf{w} + n - 1 \pmod 2 \qquad (1)$$

where $\mathbf{w} = \mathbf{w}(\lambda) := \sum_i a_i$ is the 'motivic weight'; see below.

Given such a λ, there is a unique essentially unitary Harish-Chandra module $H_{\pi_\infty^\lambda}$ such that the relative Lie algebra cohomology group $H^\bullet(\mathfrak{g}, K_\infty^\circ, H_{\pi_\infty^\lambda} \otimes E_\lambda) \neq 0$. Let $L^2_d(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f, \omega_{E_\lambda}^{-1})$ denote the discrete spectrum for $G(\mathbb{A})$ in the space of L^2-automorphic forms with level structure K_f on which $Z(\mathbb{R})^\circ$ acts via the inverse of the central character of E_λ. For $\pi_f \in \mathrm{Coh}(G, K_f, \lambda)$ and $\iota \in \mathcal{I}(F)$ we consider

$$W\big(\pi_\infty^\lambda \otimes \iota \circ \pi_f\big) = \mathrm{Hom}_{(\mathfrak{g}, K_\infty^\circ)\times \mathcal{H}^G_{K_f}}\big(H_{\pi_\infty^\lambda} \otimes (H_{\pi_f} \otimes_{F,\iota} \mathbb{C}),\ L^2_d\big(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f, \omega_{E_\lambda}^{-1}\big)\big)$$

which is one-dimensional due to multiplicity-one for the discrete spectrum of GL_n; the image is in fact in the cuspidal spectrum by regularity of λ. (See, for example, Schwermer [11, Corollary 2.3].) We choose a generator Φ for $W(\pi_\infty^\lambda \times \iota \circ \pi_f)$.

The summand $H^\bullet_!(S^G_{K_f}, \mathcal{E}_{\lambda,F})(\pi_f)$ can be decomposed for the action of $\pi_0(G(\mathbb{R})) = \mathbb{Z}/2\mathbb{Z}$ as

$$H^\bullet_!\big(S^G_{K_f}, \mathcal{E}_{\lambda,F}\big)(\pi_f) = \bigoplus_{\epsilon:\pi_0(G(\mathbb{R}))\to\mathbb{Z}/2\mathbb{Z}} H^\bullet_!\big(S^G_{K_f}, \mathcal{E}_{\lambda,F}\big)(\pi_f)(\epsilon).$$

The action of $\pi_0(G(\mathbb{R})) = \pi_0(K_\infty) = K_\infty/K_\infty^\circ$ is via its action on $H^\bullet(\mathfrak{g}, K_\infty^\circ, H_{\pi_\infty^\lambda} \otimes E_\lambda)$. (See, for example, Borel and Wallach [1, I.5].) Therefore, we get

$$\bigoplus_\epsilon W\big(\pi_\infty^\lambda \otimes \iota \circ \pi_f\big) \otimes H^\bullet\big(\mathfrak{g}, K_\infty^\circ, H_{\pi_\infty^\lambda} \otimes E_\lambda\big)(\epsilon) \otimes H_{\pi_f} \otimes_{F,\iota} \mathbb{C} \to \bigoplus_\epsilon H^\bullet_!\big(S^G_{K_f}, \mathcal{E}_{\lambda,F}\big)(\pi_f) \otimes_{F,\iota} \mathbb{C}(\epsilon).$$

Let $b_n = n^2/4$ if n is even, and $(n^2-1)/4$ if n is odd. Since π is cuspidal, it is well known (see, for example, Clozel [2]) that π_∞^λ is irreducibly induced from essentially discrete series representations and that

$$H^{b_n}\big(\mathfrak{g}, K_\infty^\circ, H_{\pi_\infty^\lambda} \otimes E_\lambda\big) = \begin{cases} H^{b_n}(\mathfrak{g}, K_\infty^\circ, H_{\pi_\infty^\lambda} \otimes E_\lambda)_+ \oplus H^{b_n}(\mathfrak{g}, K_\infty^\circ, H_{\pi_\infty^\lambda} \otimes E_\lambda)_- & \text{if } n \text{ is even;} \\ H^{b_n}(\mathfrak{g}, K_\infty^\circ, H_{\pi_\infty^\lambda} \otimes E_\lambda)_\epsilon & \text{if } n \text{ is odd,} \end{cases}$$

where each piece on the right-hand side is one-dimensional, and ϵ is a canonical sign (see [10, Section 3.3]).

Now let n be even. We will define certain periods that we call *relative periods*. We define consistent choices of generators

$$\omega_+ \in \mathrm{Hom}_{K_\infty^\circ}\big(\Lambda^{b_n}(\mathfrak{g}/\mathfrak{k}), H_{\pi_\infty^\lambda} \otimes E_\lambda\big)_+, \qquad \omega_- \in \mathrm{Hom}_{K_\infty^\circ}\big(\Lambda^{b_n}(\mathfrak{g}/\mathfrak{k}), H_{\pi_\infty^\lambda} \otimes E_\lambda\big)_-,$$

from which we get isomorphisms

$$(\Phi \otimes \omega_\pm) : \iota \circ \pi_f \to H^{b_n}_!\big(S^G_{K_f}, \mathcal{E}_{\lambda,F}\big)(\pi_f)_\pm \otimes_{F,\iota} \mathbb{C}.$$

Composing the inverse of one with the other gives a canonical transcendental isomorphism

$$T^{\mathrm{trans}}(\pi_f, \iota) = (\Phi \otimes \omega_-) \circ (\Phi \otimes \omega_+)^{-1} : H^{b_n}_!\big(S^G_{K_f}, \mathcal{E}_{\lambda,F}\big)(\pi_f)_+ \otimes_{F,\iota} \mathbb{C} \to H^{b_n}_!\big(S^G_{K_f}, \mathcal{E}_{\lambda,F}\big)(\pi_f)_- \otimes_{F,\iota} \mathbb{C}. \tag{2}$$

This isomorphism does not depend on the choice of Φ or the pair (ω_+, ω_-) because these are unique up to scalars which cancel out. On the other hand, we have an arithmetic isomorphism of $\mathcal{H}^G_{K_f}$-modules

$$T^{\mathrm{arith}}(\pi_f) : H^{b_n}_!\big(S^G_{K_f}, \mathcal{E}_{\lambda,F}\big)(\pi_f)_+ \to H^{b_n}_!\big(S^G_{K_f}, \mathcal{E}_{\lambda,F}\big)(\pi_f)_- \tag{3}$$

which is unique up to an element in $\mathbb{Q}(\pi_f)^\times$. Comparing (2) with (3) we get the following definition:

Definition 2.1. There is an array of complex numbers $\Omega(\pi_f) = (\ldots, \Omega(\pi_f, \iota), \ldots)_{\iota \in \mathcal{I}(F)}$ defined by

$$\Omega(\pi_f, \iota) T^{\mathrm{trans}}(\pi_f, \iota) = T^{\mathrm{arith}}(\pi_f) \otimes_{F,\iota} \mathbb{C}.$$

Changing $T^{\mathrm{arith}}(\pi_f)$ by an element $a \in \mathbb{Q}(\pi_f)^\times$ changes the array into $(\ldots, \Omega(\pi_f, \iota)\iota(a), \ldots)_{\iota:F\to\mathbb{C}}$.

If we pass from λ to $\lambda - l \cdot \det$ for an integer l, then we have a canonical isomorphism

$$H^\bullet_!\big(S^G_{K_f}, \mathcal{E}_{\lambda,F}\big)(\pi_f) \to H^\bullet_!\big(S^G_{K_f}, \mathcal{E}_{\lambda - l\cdot\det,F}\big)\big(\pi_f \otimes |\,|^l\big)$$

under which the $\pm$ components are switched by $(-1)^l$. We get the following period relation:

$$\Omega(\pi_f, \iota) = \Omega\big(\pi_f \otimes |\,|^l, \iota\big)^{(-1)^l}. \tag{4}$$

Remark 1. Since cuspidal automorphic representations of GL_n are globally generic we can also define periods by comparing rational structures on Whittaker models and cohomological realizations. The periods were denoted $p^{\pm}(\pi_f)$ in Raghuram and Shahidi [10] and they appear in algebraicity results for the central critical value of Rankin–Selberg L-functions for $\mathrm{GL}_n \times \mathrm{GL}_{n-1}$; see Raghuram [9, Theorem 1.1]. The periods $p^{\pm}(\pi_f)$ depend on a choice of a nontrivial character of $\mathbb{Q}\backslash\mathbb{A}$ which is implicit in any discussion concerning Whittaker models. However, one may check that if we change this character then the period changes only by an element of $\mathbb{Q}(\pi_f)^{\times}$. Further, it is an easy exercise to see that $\Omega(\pi_f) = p^{+}(\pi_f)/p^{-}(\pi_f)$ up to elements in $\mathbb{Q}(\pi_f)^{\times}$. On the other hand, the definition of the relative periods $\Omega(\pi_f)$ does not require Whittaker models suggesting that it is far more intrinsic to the representation viewed as a Hecke-summand of global cohomology.

3. The case $G = \mathrm{GL}_n \times \mathrm{GL}_{n'}$ with n even and n' odd

Let $\sigma_f \in \mathrm{Coh}(\mathrm{GL}_n, \lambda)$ and $\sigma'_f \in \mathrm{Coh}(\mathrm{GL}_{n'}, \lambda')$. The level structures will be suppressed from our notation from now on. As before, the weights are written as $\lambda + \rho = a_1\gamma_1 + \cdots + a_{n-1}\gamma_{n-1} + d \cdot \det$, and similarly $\lambda' + \rho' = a'_1\gamma'_1 + \cdots + a'_{n'-1}\gamma'_{n'-1} + d' \cdot \det'$, where $a_i = a_{n-i}$, $a'_i = a'_{n'-i}$, and again we assume regularity for both the weights. Let $G = \mathrm{GL}_n \times \mathrm{GL}_{n'}$, $\mu = \lambda + \lambda'$ and $\pi_f = \sigma_f \times \sigma'_f$. By the Künneth formula we get

$$H^{\bullet}_{!}\big(S^G, \mathcal{E}_{\mu,F}\big)(\pi_f) = H^{\bullet}_{!}\big(S^{\mathrm{GL}_n}, \mathcal{E}_{\lambda',F}\big)(\sigma_f) \otimes H^{\bullet}_{!}\big(S^{\mathrm{GL}_n}, \mathcal{E}_{\lambda',F}\big)\big(\sigma'_f\big).$$

Using Grothendieck's conjectural theory of motives, one supposes that there are motives $\mathbf{M}_{\mathrm{eff}}$ (resp., $\mathbf{M}'_{\mathrm{eff}}$) that are conjecturally attached to σ_f (resp., σ'_f). (See, for example, [7].) We call a pair of integers (p,q) a Hodge-pair for a motive $\mathbf{M}$ if the Hodge number $h^{p,q}(\mathbf{M}) \neq 0$. The Hodge-pairs of the motives $\mathbf{M}_{\mathrm{eff}}$ (resp., $\mathbf{M}'_{\mathrm{eff}}$) are expected to be $\{(\mathbf{w},0), (\mathbf{w}-a_1, a_1), \ldots, (0,\mathbf{w})\}$ (resp., $\{(\mathbf{w}',0), (\mathbf{w}'-a'_1, a'_1), \ldots, (0,\mathbf{w}')\}$) where $\mathbf{w} = \sum_{i=1}^{n-1} a_i$ (resp., $\mathbf{w}' = \sum_{i'=1}^{n'-1} a'_{i'}$) are the motivic weights. The motives $\mathbf{M}_{\mathrm{eff}}$ (resp., $\mathbf{M}'_{\mathrm{eff}}$) are suitable Tate-twists of the motives expected to be attached to σ_f (resp., σ'_f) as in Clozel [2, Conjecture 4.5]. The assertion about Hodge pairs may be verified by working with the representations at infinity and their associated local L-factors which determine the Γ-factors at infinity. The set of Hodge-pairs for $\mathbf{M}_{\mathrm{eff}} \otimes \mathbf{M}'_{\mathrm{eff}}$ are all the pairs of the form $(\mathbf{w} - a_1 \ldots - a_s + \mathbf{w}' - a'_1 \ldots - a'_{s'}, a_1 + \cdots + a_s + a'_1 + \cdots + a'_{s'})$.

The motivic L-function $L(\mathbf{M}_{\mathrm{eff}} \otimes \mathbf{M}'_{\mathrm{eff}}, \iota, s)$ is defined as in Deligne [3, (1.2.2)]. Intimately related to it is a 'cohomological' L-function $L^{\mathrm{coh}}(\sigma_f \times \sigma'_f, \iota, s)$ which is defined as an Euler product, where each Euler factor is expressed in terms of eigenvalues of certain normalized Hecke-operators acting on integral cohomology groups. Assume that the middle Hodge number of $\mathbf{M}_{\mathrm{eff}} \otimes \mathbf{M}'_{\mathrm{eff}}$ is zero, i.e., $h^{(\mathbf{w}+\mathbf{w}')/2,(\mathbf{w}+\mathbf{w}')/2} = 0$. Put $p(\mu) := \min\{p \mid \mathbf{w}+\mathbf{w}' \geqslant p > (\mathbf{w}+\mathbf{w}')/2,\ h^{p,\mathbf{w}+\mathbf{w}'-p} \neq 0\}$. Let $\sigma'^{\vee}$ denote the contragredient of σ'. The critical points of $L^{\mathrm{coh}}(\sigma_f \times \sigma'^{\vee}_f, \iota, s)$ are the integers

$$\big\{p(\mu), p(\mu)-1, \ldots, \mathbf{w}+\mathbf{w}'+1-p(\mu)\big\}. \tag{5}$$

Note that this decreasing list of integers is centered around $(\mathbf{w}+\mathbf{w}'+1)/2$ which is the center of symmetry of the cohomological L-function. The total number of critical integers is $2p(\mu) - (\mathbf{w}+\mathbf{w}')$. The cohomological L-function is up to a shift in the s-variable the usual automorphic Rankin–Selberg L-function $L(\sigma_f \times \sigma'^{\vee}_f, \iota, s) := L((\iota \circ \sigma_f) \times (\iota \circ \sigma'^{\vee}_f), s)$ for which the functional equation is between s and $1-s$. More precisely, we have

$$L^{\mathrm{coh}}\big(\sigma_f \times \sigma'^{\vee}_f, \iota, s\big) = L\left(\sigma_f \times \sigma'^{\vee}_f, \iota, s - \frac{(\mathbf{w}+\mathbf{w}')}{2} + a(\mu)\right) \tag{6}$$

where $a(\mu) = d - d'$. The parity condition (1) when applied to both the weights λ and λ' implies that the shift $-\frac{(\mathbf{w}+\mathbf{w}')}{2} + a(\mu)$ in the s-variable is always a half-integer. Observe that the cohomological L-function is invariant under changing σ to $\sigma \otimes |\ |^{l}$ or σ' to $\sigma' \otimes |\ |^{l'}$.

A celebrated conjecture of Deligne predicts the existence of two periods $\Omega_{\pm}(\mathbf{M}_{\mathrm{eff}} \otimes \mathbf{M}'_{\mathrm{eff}})$ obtained from the Betti and de Rham realizations of this motive that capture, up to prescribable powers of $(2\pi i)$, the possibly transcendental parts of the critical values of $L(\mathbf{M}_{\mathrm{eff}} \otimes \mathbf{M}'_{\mathrm{eff}}, \iota, s)$. See [3, Conjecture 2.7, (3.1.2) and (5.1.8)] for a precise statement. Our main result on L-values is to be viewed from this perspective.

4. The main result on ratios of critical L-values

Theorem 4.1. *Let $\sigma_f \in \mathrm{Coh}(\mathrm{GL}_n, \lambda)$ and $\sigma'_f \in \mathrm{Coh}(\mathrm{GL}_{n'}, \lambda')$. Assume that n is even and n' is odd. Let $m = 1/2 + m_0 \in 1/2 + \mathbb{Z}$ be a half-integer such that both m and $m+1$ are critical for $L(\sigma_f \times \sigma'^{\vee}_f, \iota, s)$. Assuming the validity of a Combinatorial Lemma (see below) we have*

$$\frac{1}{\Omega(\sigma_f,\iota)^{\epsilon_m \epsilon_{\sigma'}}} \frac{\Lambda(\sigma_f \times \sigma'^{\vee}_f, \iota, m)}{\Lambda(\sigma_f \times \sigma'^{\vee}_f, \iota, m+1)} \in \iota(F),$$

for any $\iota \in \mathcal{I}(F)$. Moreover, for all $\tau \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$

$$\tau\left(\frac{1}{\Omega(\sigma_f,\iota)^{\epsilon_m\epsilon_{\sigma'}}}\frac{\Lambda(\sigma_f\times\sigma_f'^{\vee},\iota,m)}{\Lambda(\sigma_f\times\sigma_f'^{\vee},\iota,m+1)}\right)=\frac{1}{\Omega(\sigma_f,\tau(\iota))^{\epsilon_m\epsilon_{\sigma'}}}\frac{\Lambda(\sigma_f\times\sigma_f'^{\vee},\tau(\iota),m)}{\Lambda(\sigma_f\times\sigma_f'^{\vee},\tau(\iota),m+1)}.$$

Here $\epsilon_{\sigma'}$ is a sign determined by σ', $\epsilon_m=(-1)^{m_0}$ and $\Lambda(\sigma_f\times\sigma_f'^{\vee},\iota,s)$ is the completed Rankin–Selberg L-function.

See the main theorem of Harder [4] for the simplest nontrivial case ($n=2$ and $n'=1$) of the above theorem.

5. Eisenstein cohomology and sketch of proof of Theorem 4.1

Consider the group $\tilde{G}=\mathrm{GL}_N/\mathbb{Q}$ where $N=n+n'\geqslant 3$ is an odd integer. Let P (resp., Q) be the standard maximal parabolic subgroup of $\tilde{G}$ whose Levi quotient is $M_P=\mathrm{GL}_n\times\mathrm{GL}_{n'}$ (resp., $M_Q=\mathrm{GL}_{n'}\times\mathrm{GL}_n$). We will try to find a highest weight $\tilde{\mu}$, such that $H_!^{b_n+b_{n'}}(S^{M_P},\mathcal{E}_{\mu,F})(\sigma_f\otimes\sigma_f')\oplus H_!^{b_n+b_{n'}}(S^{M_Q},\mathcal{E}_{\mu,F})(\sigma_f'\otimes\sigma_f)$ occurs as isotypical summand in the cohomology of the boundary $H^{b_N}(\partial S_{K_f}^{\tilde{G}},\mathcal{E}_{\tilde{\mu}})$. Recall our notation that $b_N=(N^2-1)/4$, hence $b_N=b_n+b_{n'}+\dim(U_P)/2$. Therefore, we need a dominant weight $\tilde{\mu}$ and a Kostant representative $w\in W^P$ (defined as in Borel and Wallach [1, III.1.2]) of length $l(w)=\dim(U_P)/2$ such that $w\cdot\tilde{\mu}:=w(\tilde{\mu}+\tilde{\rho})-\tilde{\rho}=\mu=\lambda+\lambda'$. We believe, having checked it in infinitely many cases ($n=2$ or $n'=1$), that the following assertion is true:

Conjecture 5.1 *(Combinatorial Lemma). For a given $\mu=\lambda+\lambda'$, there exists a dominant weight $\tilde{\mu}$ and a Kostant representative $w\in W^P$ with $l(w)=\dim(U_P)/2$ and $w\cdot\tilde{\mu}=\mu$ if and only if*

$$\frac{(\mathbf{w}+\mathbf{w}')}{2}-p(\mu)+1-\frac{N}{2}\leqslant a(\mu)\leqslant-\frac{(\mathbf{w}+\mathbf{w}')}{2}+p(\mu)-1-\frac{N}{2}.$$

(*The number of possibilities for $a(\mu)$ is $2p(\mu)-(\mathbf{w}+\mathbf{w}')-1$, which is one less than the total number of critical points.*)

Assuming that μ satisfies the condition in the Combinatorial Lemma, we know that there is a $\tilde{\mu}$ such that

$$H_!^{b_n+b_{n'}}\big(S^{M_P},\mathcal{E}_{\mu,F}\big)\big(\sigma_f\otimes\sigma_f'\big)\oplus H_!^{b_n+b_{n'}}\big(S^{M_Q},\mathcal{E}_{\mu,F}\big)\big(\sigma_f'\otimes\sigma_f\big)\subset H^{b_N}\big(\partial S^{\tilde{G}},\mathcal{E}_{\tilde{\mu}}\big),$$

and it is actually an isotypical subspace. Hence, there is a Hecke-invariant projector R_{π_f} to this subspace. The theory of Eisenstein cohomology gives a description of the image of the restriction map

$$r^*:H^{b_N}\big(S^{\tilde{G}},\mathcal{E}_{\tilde{\mu}}\big)\to H^{b_N}\big(\partial S^{\tilde{G}},\mathcal{E}_{\tilde{\mu}}\big).$$

Our main result on Eisenstein cohomology is the following:

Theorem 5.2. *The image of $R_{\pi_f}\circ r^*$ is given by*

$$R_{\pi_f}\circ r^*\big(H^{b_N}\big(S^{\tilde{G}},\mathcal{E}_{\tilde{\mu}}\big)\big)\bigotimes_{F,\iota}\otimes\mathbb{C}=\left\{\psi+\frac{C(\mu)}{\Omega(\sigma_f,\iota)^{\epsilon_{\nu_0}\epsilon_{\sigma'}}}\frac{\Lambda^{\mathrm{coh}}(\sigma_f\times\sigma_f'^{\vee},\iota,\nu_0)}{\Lambda^{\mathrm{coh}}(\sigma_f\times\sigma_f'^{\vee},\iota,\nu_0+1)}T^{\mathrm{arith}}(\pi_f,\iota)(\psi)\right\},$$

where ψ is any class in $H_!^{b_n+b_{n'}}(S^{M_P},\mathcal{E}_{\mu,F})(\pi_f)$ with $\pi_f=\sigma_f\otimes\sigma_f'$; the operator $T^{\mathrm{arith}}(\pi_f,\iota)$ is defined as $T^{\mathrm{arith}}(\sigma_f,\iota)\otimes 1_{\sigma_f'}$ after using the Künneth-formula; $C(\mu)$ is a non-zero rational number; and the point of evaluation is $\nu_0=\frac{\mathbf{w}+\mathbf{w}'}{2}-a(\mu)-\frac{N}{2}$. (Note that $\Lambda^{\mathrm{coh}}(\sigma_f\times\sigma_f'^{\vee},\iota,\nu_0)=\Lambda(\sigma_f\times\sigma_f'^{\vee},\iota,-N/2)$.)

Theorem 5.2 implies the rationality result stated in Theorem 4.1 for $m=-N/2$ because the ratio of L-values together with the period is the 'slope' of a rationally defined map. For an integer l, let us change σ to $\sigma\otimes|\ |^l$, then λ changes to $\lambda-l\cdot\det$ and $a(\mu)$ changes to $a(\mu)-l$, however the possibilities for l are restricted by the inequalities in the Combinatorial Lemma since $\mathbf{w},\mathbf{w}'$ and $p(\mu)$ do not change. It may be verified using (5) that as $a(\mu)$ runs through all the possible values it can take as prescribed by the Combinatorial Lemma, the pair of numbers ν_0 and ν_0+1 run through all the successive critical arguments; Theorem 4.1 follows while using the period relations (4) for σ_f. The Combinatorial Lemma says that the theory of Eisenstein cohomology allows one to prove a rationality result for a ratio of successive L-values exactly when both the L-values are critical. (See also [5].)

The condition on μ imposed by the Combinatorial Lemma has certain strong implications on the situation that underlies Eisenstein cohomology. First, using Speh's results (see, for example, [8, Theorem 10b]) on reducibility for induced representations for $\mathrm{GL}_N(\mathbb{R})$, one sees that the representation ${}^{\mathrm{a}}\mathrm{Ind}_{P_\infty}^{\mathrm{GL}_N(\mathbb{R})}(\sigma_\infty^{\lambda}\otimes\sigma_\infty'^{\lambda'})$ of $\mathrm{GL}_N(\mathbb{R})$ obtained by un-normalized parabolic induction is irreducible. Next, using Shahidi's results [12] on local factors and the fact that ν_0 and ν_0+1 are critical, we deduce that the standard intertwining operator A_∞ from the above induced representation to the representation similarly induced from Q_∞ is both holomorphic and nonzero at $s=\nu_0$. The choice of bases $\omega_\pm$ fixes a basis for the one-dimensional space $H^{b_N}(\mathfrak{gl}_N, K_\infty^\circ, {}^{\mathrm{a}}\mathrm{Ind}_{P_\infty}^{\mathrm{GL}_N(\mathbb{R})}(\sigma_\infty^\lambda \otimes \sigma_\infty'^{\lambda'}) \otimes E_{\tilde{\mu}})$. The map induced by A_∞ at the level of $(\mathfrak{gl}_N, K_\infty^\circ)$-cohomology is then a nonzero scalar. This scalar is a power of $(2\pi i)$ times a rational number $C(\mu)$. The power of $(2\pi i)$ gives the ratio of L-factors at infinity hence giving us a statement for completed L-functions, and the quantity $C(\mu)$ is expected to be a simple number as was verified for GL_3 by Harder [4].

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