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Band gaps and vibration of strongly heterogeneous Reissner–Mindlin elastic plates

Bandes interdites et vibrations dans une plaque de Reissner–Mindlin fortement hétérogène

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A B S T R A C T

We consider an elastic plate governed by the Reissner–Mindlin's model, i.e., whose equilibrium equations introduce a coupling between the vertical displacement and the rotation of the normal. This structure is made of a composite with a periodic arrangement of strongly heterogeneous materials and some characteristics of the heterogeneities are comparable to the size of the microstructures. We show that, when the size of the microstructures tends to zero, the limit homogeneous structure presents, for some wavelengths, a negative "mass density" tensor. This means that there exist intervals of frequencies – the band gaps – for which wave propagation is suppressed, or restricted to certain polarizations.
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R É S U M É

On étudie l'influence de fortes hétérogénéités sur la propagation des ondes dans une plaque de Reissner–Mindlin. On montre que lorsque certaines caractéristiques élastiques sont comparables à la taille des microstructures du composite il apparaît des bandes interdites, i.e. des intervalles de fréquences – les bandes interdites – pour lesquels la propagation des ondes ne peut pas se faire ou bien est restreinte à certaines polarisations.
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Version française abrégée

On étudie l'effet, sur la propagation des ondes phononiques, de fortes hétérogénéités dans les composites qui constituent une plaque de Reissner–Mindlin. Le champ de déplacement est donné par l'expression (1), le déplacement des points de la surface moyenne de la plaque $\boldsymbol{U} = (U_1, U_2, U_3)$ et la rotation de la normale $\boldsymbol{\Theta} = (\Theta_1, \Theta_2)$ sont donnés par le système d'évolution couplé (2). En élaticité linéarisée les tenseurs Σ_{ij} et Q_i qui expriment les effets d'extension, de flexion et de cisaillement induits par les déformations dépendent linéairement des tenseurs d'élasticité $\mathbb{C}$ et de couplage $\boldsymbol{G}$ (3). On se restreint à des solutions périodiques en temps, la longueur d'onde est notée ω, les composantes bidimensionnelles (dépendant de x_1, x_2) du déplacement membranaire, $\boldsymbol{u} = (u_1, u_2)$, du déplacement transverse u_3 et de la rotation $\boldsymbol{\theta} = (\theta_1, \theta_2)$ associées à ω sont solution du problème stationnaire (4). Bien que les quations du déplacement membranaire soient

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indépendantes des quations du déplacement transverse et de la rotation on a gardé, par souci de clarté, l'ensemble du problème couplé dans la suite de l'exposé.

On considère que la plaque qui occupe le domaine Ω est constituée par un matériau composite périodique, de période $\varepsilon > 0$, il comprend une «matrice» et des inclusions selon (5). On a donc $\Omega = \Omega_m^\varepsilon \cup \Omega_c^\varepsilon$ où Ω_m^ε est le domaine occupé par la «matrice» et Ω_c^ε celui occupé par les inclusions, et Y_c représente la partie «inclusion» dans la cellule élémentaire Y de taille $\varepsilon > 0$. Les trois quantités qui caractérisent le matériau : la densité massique $\rho^\varepsilon \in L^\infty(\Omega)$ et les deux tenseurs $\boldsymbol{G}^\varepsilon = (G_{ij}^\varepsilon), G_{ij}^\varepsilon \in L^\infty(\Omega), \mathbb{C}^\varepsilon = (C_{ijkl}^\varepsilon), C_{ijkl}^\varepsilon \in L^\infty(\Omega)$ ont les propriétés classiques de positivité, symétrie et coercivité qui permettent l'existence et l'unicité de la solution $(\boldsymbol{u}^\varepsilon, u_3^\varepsilon, \boldsymbol{\theta}^\varepsilon)$ du problème (6) telles qu'énoncées dans le Théorème 1 (alternative de Fredholm). La limite de la suite $(\boldsymbol{u}^\varepsilon, u_3^\varepsilon, \boldsymbol{\theta}^\varepsilon)$ est étudiée (cf. [2]) grâce à l'opérateur d'éclatement $\mathcal{T}^\varepsilon$ (7).

L'hypothèse de forte hétérogénéité (8) est cruciale, elle permet de mettre en évidence l'existence des bandes interdites (une hypothèse du même type avait déjà était faite dans le cas de l'élasticité tridimensionnelle [1] et de la propagation électro-magnétique [3]). Elle traduit le fait que les tenseurs d'élasticité $\mathbb{C}^\varepsilon$ et de couplage $\boldsymbol{G}^\varepsilon$ sont d'ordre 1 dans la matrice et d'ordre ε^2 dans les inclusions ($\mathbb{C}^\varepsilon$ et $\boldsymbol{G}^\varepsilon$ tendent vers 0 comme ε^2 dans les inclusions, les inclusions sont donc moins raides que la matrice) ; la densité de masee ρ^ε est d'ordre 1 dans tout le matériau, les quantités $\rho_m, \rho_c, \mathbb{C}_m, \mathbb{C}_c, \boldsymbol{G}_m, \boldsymbol{G}_c$ sont indépendantes de ε.

Le Théorème 2 s'appuie sur l'hypothèse de l'existence d'un ensemble de longueurs d'onde non vide W tel que pour tout $\omega \in W$ la suite $(\boldsymbol{u}^\varepsilon, u_3, \boldsymbol{\theta}^\varepsilon)$ est uniformément bornée en ε. Cette hypothèse est valide pour certains types de matériaux, cf. [1], ce qui nous permet d'énoncer le résultat essentiel qui suit.

Le théorème de convergence 3 étudie, pour ω fixé dans W, la convergence de la suite $(\boldsymbol{u}^\varepsilon, u_3, \boldsymbol{\theta}^\varepsilon)$ quand ε tend vers 0.

(i) Les suites $\mathcal{T}^\varepsilon(\boldsymbol{u}^\varepsilon), \mathcal{T}^\varepsilon(u_3^\varepsilon)$ and $\mathcal{T}^\varepsilon(\boldsymbol{\theta}^\varepsilon)$ convergent fortement dans $\boldsymbol{L}^2(\Omega \times Y)$ respectivement vers $\boldsymbol{u} + \hat{\boldsymbol{u}}, u_3 + \hat{u}_3$ and $\boldsymbol{\theta} + \hat{\boldsymbol{\theta}}$.

(ii) Le champ limite de déplacement et de rotation $(\boldsymbol{u}, u_3, \boldsymbol{\theta}) \in (H_0^1(\Omega))^5$ est l'unique solution de la formulation variationnelle homogène (9).

(iii) Les correcteurs $(\hat{\boldsymbol{u}}, \hat{u}_3, \hat{\boldsymbol{\theta}}) \in (L^2(\Omega; H_0^1(Y_c)))^5$ sont calculés dans les bases hilbertiennes $(\boldsymbol{\varphi}^r, \lambda^r) \in (H_0^1(Y_c))^2 \times \mathbb{R}$ and $(w^r, \mu^r) \in H_0^1(Y_c) \times \mathbb{R}$, $r = 1, 2, \ldots$, qui dépendent de la géométrie de l'inclusion élémentaire Y_c et sont données par (10). On a alors $\hat{\boldsymbol{u}} = \sum_{r \geqslant 0} \alpha_r \boldsymbol{\varphi}^r$, $\hat{u}_3 = \sum_{r \geqslant 0} \beta_r w^r$, $\hat{\boldsymbol{\theta}} = \sum_{r \geqslant 0} \gamma_r \boldsymbol{\varphi}^r$. □

On notera dans (9) que la densité de masse est maintenant remplacée par un tenseur homogène $h\boldsymbol{R}^*(\omega)$ pour le mode membrannaire, par $\frac{h^3}{3}\boldsymbol{R}^*(\omega)$ pour le mode des rotations et par le scalaire $hS^*(\omega)$ pour le mode transverse, tous dépendent de la longueur d'onde ω ; les tenseurs homogènes de l'élasticité $\mathbb{C}^*$ et de couplage $\boldsymbol{G}^*$ sont indépendants de ω et sont donnés dans (11).

Si, pour ω fixé, $\boldsymbol{R}^*(\omega)$ et $S^*(\omega)$ sont défini-positifs alors les ondes se propagent librement dans Ω à la fréquence associée. Dans le cas contraire on montre qu'il existe des bandes interdites : dans chaque intervalle $(\lambda^r, \lambda^{r+1})$ (respectivement (μ^r, μ^{r+1})) donnés par (10) il existe un sous-intervalle de fréquence dans lequel $\boldsymbol{R}^*(\omega)$ (respectivement $S^*(\omega)$) n'est pas positif, il n'y a pas de propagation d'ondes pour les polarisations associées [4]. Des applications extrêmement intéressantes aux cristaux phononiques sont données dans [5].

1. The Reissner–Mindlin plate model. Propagation of elastic waves

We consider a plate[1] whose reference configuration $\overline{\Omega}$ is stress-free, the domain (open, bounded, connected) $\Omega \subset \mathbb{R}^2$ is spanned by coordinates $x = (x_1, x_2)$ and has a Lipschitz-continuous boundary denoted by $\partial\Omega$. Let $T > 0$ be a positive time. Under the action of initial conditions and external loading by applied forces $\boldsymbol{F} = (F_1, F_2, F_3) : \overline{\Omega} \times (0, T) \to \mathbb{R}^3$ and moments $\boldsymbol{M} = (M_1, M_2) : \overline{\Omega} \times (0, T) \to \mathbb{R}^3$, the plate with positive mass density ρ and thickness $2h$ deforms according to the displacement field of the form:

$$U_1(x,t) + hx_3\Theta_1(x,t), \qquad U_2(x,t) + hx_3\Theta_2(x,t), \qquad U_3(x,t), \quad \text{for all } x \in \overline{\Omega},\ t \in (0,T). \tag{1}$$

By virtue of this model, material straight fibers orthogonal to the mean surface can rotate by (Θ_1, Θ_2) *independently* of this normal, however, they stay straight and inextended. Thus, only deformation of the mean surface of the plate, i.e. for $x_3 = 0$, are to be determined: the displacement field $\boldsymbol{U} : \Omega \times (0, T) \to (U_1, U_2, U_3)$ and the rotation field $\boldsymbol{\Theta} : \Omega \times (0, T) \to (\Theta_1, \Theta_2)$ are solution to the evolution problem:

[1] Latin exponents and indices take their values in the set $\{1, 2\}$, Einstein convention for repeated exponents and indices is used and bold face letters represent vectors or vector spaces. We denote the partial derivative with respect to time by superscript $'$ and the gradient with respect to space variables by ∇, for example $U'' = \frac{\partial^2 U}{\partial t^2}$, $\nabla U = (\partial_1 U, \partial_2 U)$. For brevity, we skip the dependence on t and x whenever it is not necessary.

$$\begin{cases} \rho U_i'' - \partial_j \boldsymbol{\Sigma}_{ij}(\boldsymbol{U}) = F_i & \text{in } \Omega \times (0,T),\ i=1,2,\\ \rho U_3'' - \partial_i Q_i(U_3,\boldsymbol{\Theta}) = F_3 & \text{in } \Omega \times (0,T),\\ \dfrac{h^3}{3}\rho \Theta_i'' - \dfrac{h^3}{3}\partial_j \Sigma_{ij}(\boldsymbol{\Theta}) + hQ_i(U_3,\boldsymbol{\Theta}) = M_i & \text{in } \Omega \times (0,T),\ i=1,2,\\ \text{(Dirichlet boundary conditions)}\quad \boldsymbol{U}=\boldsymbol{0},\quad \boldsymbol{\Theta}=\boldsymbol{0} & \text{on } \partial\Omega \times (0,T),\\ \text{(Cauchy initial conditions)}\quad \boldsymbol{U}(0)=\boldsymbol{U}^0,\quad \boldsymbol{U}'(0)=\boldsymbol{U}^1,\quad \boldsymbol{\Theta}(0)=\boldsymbol{\Theta}^0,\quad \boldsymbol{\Theta}'(0)=\boldsymbol{\Theta}^1 & \text{in } \Omega \end{cases} \tag{2}$$

with two linear constitutive laws which depend upon the second order coupling tensor $\boldsymbol{G}=(G_{ij})$ and the fourth order elasticity tensor $\mathbb{C}=(C_{ijkl})$ and involve the linearized deformation tensor $\epsilon_{ij}(\boldsymbol{V}) = \frac{1}{2}(\partial_i V_j + \partial_j V_i)$,

$$\Sigma_{ij}(\boldsymbol{V}) := C_{ijkl}\epsilon_{kl}(\boldsymbol{V}) = \big[\mathbb{C}\epsilon(\boldsymbol{V})\big]_{ij}, \qquad Q_i(U_3,\boldsymbol{\Theta}) := G_{ij}(\partial_j U_3 - \Theta_j) = \big[\boldsymbol{G}(\nabla U_3 - \boldsymbol{\Theta})\big]_i. \tag{3}$$

We recall that, for positive mass density ρ, for coercive coupling tensor $\boldsymbol{G}$ and elasticity tensor $\mathbb{C}$, for regular forces $\boldsymbol{F}$, moments $\boldsymbol{M}$ and for initial conditions in appropriate functional spaces, namely $(\boldsymbol{U}^0,\boldsymbol{U}^1) \in (H_0^1(\Omega)\times L^2(\Omega))^3$ and $(\boldsymbol{\Theta}^0,\boldsymbol{\Theta}^1) \in (H_0^1(\Omega)\times L^2(\Omega))^2$ there exists a unique weak solution $(\boldsymbol{U},\boldsymbol{\Theta})$ to the evolution problem (2) such that: $(\boldsymbol{U},\boldsymbol{U}') \in (C^0(0,T;H_0^1(\Omega)\times L^2(\Omega)))^3$, $(\boldsymbol{\Theta},\boldsymbol{\Theta}') \in (C^0(0,T;H_0^1(\Omega)\times L^2(\Omega)))^2$.

Let us consider a *fixed* incident wave frequency ω which corresponds to given loads $F_j(x,t) = f_j(x)e^{\mathrm{i}\omega t}$, $j=1,2,3$, and $M_k(x,t) = m_k(x)e^{\mathrm{i}\omega t}$, $k=1,2$. We look for periodic solutions $(\boldsymbol{U}(\omega,x,t),\boldsymbol{\Theta}(\omega,x,t))$ with $U_j(\omega,x,t) = u_j(\omega,x)e^{\mathrm{i}\omega t}$, $j=1,2,3$, and $\Theta_k(\omega,x,t) = \theta_k(\omega,x)e^{\mathrm{i}\omega t}$, $k=1,2$, associated to compatible initial conditions (from now on, we omit the dependence in ω). Hence, the amplitude of the membrane elastic wave $\boldsymbol{u}=(u_1,u_2)$, of the transverse wave u_3 and of the rotation wave $\boldsymbol{\theta}=(\theta_1,\theta_2)$ are given by the stationary problem:

$$\begin{cases} \omega^2 h\rho u_i + h\partial_j \Sigma_{ij}(\boldsymbol{u}) = f_i & \text{in } \Omega,\\ \omega^2 h\rho u_3 + h\partial_i Q_i(u_3,\boldsymbol{\theta}) = f_3 & \text{in } \Omega,\\ \omega^2 \dfrac{h^3}{3}\rho\theta_i + \dfrac{h^3}{3}\partial_j \Sigma_{ij}(\boldsymbol{\theta}) + hQ_i(u_3,\boldsymbol{\theta}) = m_i & \text{in } \Omega,\\ \boldsymbol{u}=\boldsymbol{0},\quad u_3=0,\quad \boldsymbol{\theta}=\boldsymbol{0} & \text{on } \partial\Omega. \end{cases} \tag{4}$$

2. Description of the composite material

We now consider a composite made of periodically distributed microstructures of size $\varepsilon > 0$, the small parameter ε; the aim is to study the asymptotic behaviour of the composite when $\varepsilon \to 0$. Domain Ω is split into two domains Ω_m^ε (the "matrix") and Ω_c^ε (the "inclusions") occupied by two different materials, hence $\Omega = \Omega_m^\varepsilon \cup \Omega_c^\varepsilon$ with $\Omega_m^\varepsilon \cap \Omega_c^\varepsilon = \emptyset$. This decomposition is generated as a periodic lattice by the reference cell $Y = [0,1[^2$ consisting of its elementary inclusion that occupies the open set Y_c with $Y_c \subset \operatorname{int} Y$, and of the matrix situated in domain $Y_m = Y\backslash\overline{Y_c}$; we assume that the boundary of Y_c is Lipschitz-continuous. Therefore the inclusions, which are supposed not to intersect the boundary $\partial\Omega$, occupy the domain Ω_c^ε obtained by ε-periodicity and the matrix occupies the remaining domain Ω_m^ε, hence

$$\Omega_c^\varepsilon = \bigcup_{k\in\mathbb{K}^\varepsilon} \varepsilon(Y_c+k) \quad \text{with } \mathbb{K}^\varepsilon = \big\{k\in\mathbb{Z}^2,\ \varepsilon(Y_c+k)\subset\Omega\big\}, \qquad \Omega_m^\varepsilon = \Omega\backslash\overline{\Omega_c^\varepsilon}, \quad \Omega_m^\varepsilon \text{ must be connected.} \tag{5}$$

The material is described by a set of three material data which depend upon the parameter ε: the scalar $\rho^\varepsilon \in L^\infty(\Omega)$ and the two tensors $\boldsymbol{G}^\varepsilon = (G_{ij}^\varepsilon)$, $G_{ij}^\varepsilon \in L^\infty(\Omega)$ and $\mathbb{C}^\varepsilon = (C_{ijkl}^\varepsilon)$, $C_{ijkl}^\varepsilon \in L^\infty(\Omega)$ with the same positivity and coercivity properties as in the previous section.

Let us introduce the *weak formulation* of problem (4): Find the triplet $(\boldsymbol{u}^\varepsilon, u_3^\varepsilon, \boldsymbol{\theta}^{\boldsymbol{\varepsilon}}) \in (H_0^1(\Omega))^5$ such that for all $(\boldsymbol{v}, v_3, \boldsymbol{\psi}) \in (H_0^1(\Omega))^5$:

$$\begin{aligned} &\omega^2 h\int_\Omega \rho^\varepsilon \boldsymbol{u}^\varepsilon\cdot\boldsymbol{v} - h\int_\Omega \big[\mathbb{C}^\varepsilon\epsilon\big(\boldsymbol{u}^\varepsilon\big)\big]:\epsilon(\boldsymbol{v}) + \omega^2 h\int_\Omega \rho^\varepsilon u_3^\varepsilon v_3 + \omega^2\frac{h^3}{3}\int_\Omega \rho^\varepsilon\boldsymbol{\theta}^{\boldsymbol{\varepsilon}}\cdot\boldsymbol{\psi}\\ &\quad - h\int_\Omega \big[\boldsymbol{G}^\varepsilon\big(\nabla u_3^\varepsilon - \boldsymbol{\theta}^{\boldsymbol{\varepsilon}}\big)\big]\cdot(\nabla v_3 - \boldsymbol{\psi}) - \frac{h^3}{3}\int_\Omega \big[\mathbb{C}^\varepsilon\epsilon\big(\boldsymbol{\theta}^{\boldsymbol{\varepsilon}}\big)\big]:\epsilon(\boldsymbol{\psi}) = \int_\Omega (\boldsymbol{f}\cdot\boldsymbol{v} + f_3 v_3 + \boldsymbol{m}\cdot\boldsymbol{\psi}). \end{aligned} \tag{6}$$

Existence: Theorem 1 (Standard composites). For a given $\varepsilon > 0$ and for any ω different from the resonance values (which depend upon ε) the weak formulation (6) of problem (4) possesses a unique solution (Fredholm's alternative). □

We study the sequences $(\boldsymbol{u}^\varepsilon, u_3^\varepsilon, \boldsymbol{\theta}^{\boldsymbol{\varepsilon}})$ for $\varepsilon \to 0$ using (cf. [2]) the unfolding operator $\mathcal{T}^\varepsilon$ defined for all $v \in L^1(\Omega)$ (extended by 0 outside Ω) by:

$$\mathcal{T}^\varepsilon(v)(x,y) := v\left(\varepsilon\left[\frac{x}{\varepsilon}\right] + \varepsilon y\right), \quad x \in \Omega,\ y \in Y \tag{7}$$

associated to the unique decomposition $x = \varepsilon[\frac{x}{\varepsilon}] + \varepsilon\{\frac{x}{\varepsilon}\}$ into the "integer-lattice" (rough) part $[\frac{x}{\varepsilon}] \in \mathbb{Z}^2$ and the remaining (fine) part $x - \varepsilon[\frac{x}{\varepsilon}] \in Y$.

3. Limit problem and band gaps

3.1. The limit problem

For "standard" composites with material parameters being positive and bounded uniformly with respect to $\varepsilon > 0$ (as we considered in Theorem 1) it is classic to show that the sequences $(\boldsymbol{u}^\varepsilon, u_3^\varepsilon, \boldsymbol{\theta}^\varepsilon)$ converge (in a certain sense with $\varepsilon \to 0$) to the solution of a problem where homogenized coefficients satisfy the same positivity features. As a consequence, there is no wave dispersion for the limit homogenized material. The main result presented here, however, is a model of homogenized plates with "large contrasts" in $\mathbb{C}^\varepsilon$ and $\boldsymbol{G}^\varepsilon$, such that wave dispersion occurs and frequency bands of suppressed vibrations (band gaps) exist. For that purpose, we consider a two-phase composite with locally diminishing elasticity in one of the phases when $\varepsilon \to 0$.

The material is determined by the following three pairs *independent of* ε and defined in Y: the mass density (ρ_m, ρ_c), the coupling tensors $(\boldsymbol{G}_m, \boldsymbol{G}_c)$ and the elasticity tensors $(\boldsymbol{C}_m, \boldsymbol{C}_c)$. Using these pairs, the unfolded material parameters (two-scale functions) are defined by relations:

$$\begin{cases} \mathcal{T}^\varepsilon(\rho^\varepsilon)(x,y) = \rho_m(y), \quad \mathcal{T}^\varepsilon(\mathbb{C}^\varepsilon)(x,y) = \mathbb{C}_m(y), \quad \mathcal{T}^\varepsilon(\boldsymbol{G}^\varepsilon)(x,y) = \boldsymbol{G}_m(y), & x \in \Omega,\ y \in Y_m, \\ \mathcal{T}^\varepsilon(\rho^\varepsilon)(x,y) = \rho_c(y), \quad \mathcal{T}^\varepsilon(\mathbb{C}^\varepsilon)(x,y) = \varepsilon^2\mathbb{C}_c(y), \quad \mathcal{T}^\varepsilon(\boldsymbol{G}^\varepsilon)(x,y) = \varepsilon^2\boldsymbol{G}_c(y), & x \in \Omega,\ y \in Y_c. \end{cases} \tag{8}$$

It is crucial to note that the scaling ε^2 reflects the *strong heterogeneity*, i.e. large contrasts in the elasticity and the coupling tensors, when comparing the properties of the matrix and that of the inclusions, which are made "very soft". However, the densities are comparable in both subdomains, cf. [3] for analogous treatment of photonic crystals.

Existence: Theorem 2 (Strongly heterogeneous composites). We assume the existence of a non-empty set W such that for all $\omega \in W$ and ε small enough problem (6) has a solution $(\boldsymbol{u}^\varepsilon, u_3^\varepsilon, \boldsymbol{\theta}^\varepsilon)$ which is uniformly bounded in $(H_0^1(\Omega))^5$. □

The existence of such a set W has been proved for certain materials, cf. [1]. We are now ready to state our main result.

Convergence: Theorem 3. For all values $\omega \in W$ there exist three limit fields $(\boldsymbol{u}, u_3, \boldsymbol{\theta}) \in (H_0^1(\Omega))^5$ and three correctors fields $(\hat{\boldsymbol{u}}, \hat{u}_3, \hat{\boldsymbol{\theta}}) \in (L^2(\Omega; H_0^1(Y_c)))^5$ such that:

(i) The sequences $\mathcal{T}^\varepsilon(\boldsymbol{u}^\varepsilon)$, $\mathcal{T}^\varepsilon(u_3^\varepsilon)$ and $\mathcal{T}^\varepsilon(\boldsymbol{\theta}^\varepsilon)$ converge strongly in $\boldsymbol{L}^2(\Omega \times Y)$ to $\boldsymbol{u} + \hat{\boldsymbol{u}}$, $u_3 + \hat{u}_3$ and $\boldsymbol{\theta} + \hat{\boldsymbol{\theta}}$, respectively.

(ii) The triplet $(\boldsymbol{u}, u_3, \boldsymbol{\theta})$ is the unique solution of the variational problem:

$$\omega^2 h \int_\Omega \boldsymbol{R}^*(\omega)\boldsymbol{u}\cdot\boldsymbol{v} - h\int_\Omega \left[\mathbb{C}^*\epsilon(\boldsymbol{u})\right] : \epsilon(\boldsymbol{v}) + \omega^2\frac{h^3}{3}\int_\Omega \boldsymbol{R}^*(\omega)\boldsymbol{\theta}\cdot\boldsymbol{\psi} + \omega^2 h\int_\Omega S^*(\omega)u_3 v_3$$
$$- h\int_\Omega \left[\boldsymbol{G}^*(\nabla u_3 - \boldsymbol{\theta})\right]\cdot(\nabla v_3 - \boldsymbol{\psi}) - \frac{h^3}{3}\int_\Omega \left[\mathbb{C}^*\epsilon(\boldsymbol{\theta})\right] : \epsilon(\boldsymbol{\psi}) = L(\omega; \boldsymbol{v}, v_3, \boldsymbol{\theta}) \quad \forall(\boldsymbol{v}, v_3, \boldsymbol{\psi}) \in \left(H_0^1(\Omega)\right)^5 \tag{9}$$

where *the homogenized elasticity tensor* $\mathbb{C}^*$ and *the homogenized coupling tensor* $\boldsymbol{G}^*$ depend just on the elastic properties of the "perforated-like" matrix represented by Y_m and are independent of the frequency ω, while *the homogenized generalized masses* $\boldsymbol{R}^*(\omega)$ and $S^*(\omega)$ are determined in the whole domain Y and depend upon ω, see (11) for their expression.[2]

(iii) Let us consider the Hilbert bases $\{\boldsymbol{\varphi}^r\}_r$ and $\{w^r\}_r$ where $(\boldsymbol{\varphi}^r, \lambda^r) \in (H_0^1(Y_c))^2 \times \mathbb{R}$ and $(w^r, \mu^r) \in H_0^1(Y_c) \times \mathbb{R}$, $r = 1, 2, \ldots$, are obtained by solving the following eigenvalue problems:

$$\begin{cases} \displaystyle\int_{Y_c} \left[\mathbb{C}_c\epsilon_y(\boldsymbol{\varphi}^r)\right] : \epsilon_y(\boldsymbol{\psi}) = \lambda^r \int_{Y_c} \rho_c\boldsymbol{\varphi}^r\cdot\boldsymbol{\psi} \quad \forall\boldsymbol{\psi} \in \left(H_0^1(Y_c)\right)^2 \quad \forall\boldsymbol{\psi} \in \left(H_0^1(Y_c)\right)^2, \\ \displaystyle\int_{Y_c} \left[\boldsymbol{G}_c\nabla_y w^r\right]\cdot\nabla_y v = \mu^r \int_{Y_c} \rho_c w^r v \quad \forall v \in H_0^1(Y_c) \quad \forall v \in H_0^1(Y_c). \qquad \square \end{cases} \tag{10}$$

[2] At the right-hand side, $L(\omega; \boldsymbol{v}, v_3, \boldsymbol{\theta})$ is a linear functional involving homogenized forces and moments.

Then the two-scale corrector fields $(\hat{\boldsymbol{u}}, \hat{u}_3, \hat{\boldsymbol{\theta}})$ referred to in Theorem 3 can be expressed as an infinite expansion $\hat{\boldsymbol{u}} = \sum_{r \geqslant 0} \alpha_r \boldsymbol{\varphi}^r$, $\hat{u}_3 = \sum_{r \geqslant 0} \beta_r w^r$, $\hat{\boldsymbol{\theta}} = \sum_{r \geqslant 0} \gamma_r \boldsymbol{\varphi}^r$. Using the solutions of (10) we can compute the homogenized inertia for the membrane modes $h\boldsymbol{R}^*(\omega)$ and the homogenized inertia for the rotation modes $\frac{h^3}{3}\boldsymbol{R}^*(\omega)$ as well as the scalar inertia for the transversal modes $hS^*(\omega)$; the following expressions are derived:

$$\begin{cases} \boldsymbol{R}^*(\omega) = \boldsymbol{I}\left(\displaystyle\int_{Y_c} \rho_c + \int_{Y_m} \rho_m\right) - \sum_r \dfrac{\omega^2}{\omega^2 - \lambda^r} \displaystyle\int_{Y_c} \rho_c \boldsymbol{\varphi}^r \otimes \int_{Y_c} \rho_c \boldsymbol{\varphi}^r, \\ S^*(\omega) = \displaystyle\int_{Y_c} \rho_c + \int_{Y_m} \rho_m - \sum_r \dfrac{\omega^2}{\omega^2 - \mu^r} \left| \int_{Y_c} \rho_c w^r \right|^2. \end{cases} \tag{11}$$

3.2. *Wave dispersion and band-gaps*

As the result of our homogenization procedure, we obtain problem (9) where $(\boldsymbol{u}, u_3, \boldsymbol{\theta})$ are the local amplitudes of harmonic waves excited by harmonic "homogenized" loads with frequency ω. Let us note that, when for some ω the tensor $\boldsymbol{R}^*(\omega)$ is positive definite and the scalar $S^*(\omega)$ is positive, then also free structure vibrations (i.e. stationary waves in domain Ω) can be excited. However, $\boldsymbol{R}^*(\omega)$ or $S^*(\omega)$ may not by positive (definite) for some ω; for the "membrane mode", cf. [1] and [4], we proved existence of whole frequency intervals – *the band gaps* – where the positivity of $\boldsymbol{R}^*(\omega)$ fails. An analogous result can be proved for the coupled rotational and deflection modes: in each interval of frequencies $(\lambda^r, \lambda^{r+1})$ (respectively (μ^r, μ^{r+1})) given by (10) there exists a sub-interval of frequencies for which $\boldsymbol{R}^*(\omega)$ (respectively $S^*(\omega)$) is not positive. In such intervals, free vibration modes are restricted, or completely suppressed.

Thus, the band gaps for stationary waves can be predicted just upon analyzing positive definiteness of $\boldsymbol{R}^*(\omega)$ and $S^*(\omega)$. Although for the membrane mode $\boldsymbol{u}$ such band gaps prediction holds also for guided plane waves in infinite plates, see [4], for the coupled modes $\boldsymbol{q} := (\boldsymbol{\theta}, u_3)$ the dispersion analysis is more complex. Interesting applications can be found in [5].

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