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# Stokes equations and elliptic systems with nonstandard boundary conditions

## *Équations de Stokes et systèmes elliptiques avec des conditions aux limites non standard*

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ABSTRACT

In a three-dimensional bounded possibly multiply-connected domain of class $\mathcal{C}^{1,1}$, we consider the stationary Stokes equations with nonstandard boundary conditions of the form $\boldsymbol{u} \cdot \boldsymbol{n} = g$ and $\mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{h} \times \boldsymbol{n}$ or $\boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{g} \times \boldsymbol{n}$ and $\pi = \pi_0$ on the boundary $\Gamma$. We prove the existence and uniqueness of weak, strong and very weak solutions corresponding to each boundary condition in $L^p$ theory. Our proofs are based on obtaining *Inf–Sup* conditions that play a fundamental role. And finally, we give two Helmholtz decompositions that consist of two kinds of boundary conditions such as $\boldsymbol{u} \cdot \boldsymbol{n}$ and $\boldsymbol{u} \times \boldsymbol{n}$ on $\Gamma$.



RÉSUMÉ

Dans un ouvert borné tridimensionnel, éventuellement multiplement connexe de classe $\mathcal{C}^{1,1}$, nous considérons les équations stationnaires de Stokes avec des conditions aux limites de la forme $\boldsymbol{u} \cdot \boldsymbol{n} = g$ et $\mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{h} \times \boldsymbol{n}$ ou $\boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{g} \times \boldsymbol{n}$ et $\pi = \pi_0$ sur le bord $\Gamma$. Nous prouvons l'existence et l'unicité des solutions faibles, fortes et très faibles en théorie $L^p$. Nos preuves sont basées sur l'obtention de conditions *Inf–Sup* qui jouent un rôle fondamental. Finalement, on donne deux décompositions d'Helmholtz qui tiennent compte des deux types de conditions aux limites $\boldsymbol{u} \cdot \boldsymbol{n}$ et $\boldsymbol{u} \times \boldsymbol{n}$ sur $\Gamma$.



## Version française abrégée

L'objet de cette Note consiste essentiellement à étudier en théorie $L^p$, avec $1 < p < \infty$, l'existence et l'unicité de solutions faibles, fortes et très faibles pour les équations stationnaires de Stokes $(\mathcal{S}_T)$ dans le cas des conditions aux limites $\boldsymbol{u} \cdot \boldsymbol{n} = g$ et $\mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{h} \times \boldsymbol{n}$ sur $\Gamma$ et $(\mathcal{S}_N)$ dans le cas des conditions aux limites $\boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{g} \times \boldsymbol{n}$ et $\pi = \pi_0$ sur $\Gamma$. Les résultats concernant l'existence de solutions faibles et fortes pour $(\mathcal{S}_T)$ sont donnés dans le Théorème 2.1 ; et en ce qui a trait à $(\mathcal{S}_N)$, les résultats sont donnés dans le Théorème 3.2. Pour la preuve de solutions très faibles pour $(\mathcal{S}_T)$ et $(\mathcal{S}_N)$, l'une des difficultés consiste à donner un sens aux traces sur le bord. De nombreuses applications donnent souvent lieu à des problèmes où les conditions aux limites ci-dessus interviennent naturellement sur des parties du bord du domaine.

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## 1. Introduction

Let $\Omega$ be a bounded open connected set of $\mathbb{R}^3$ of class $\mathcal{C}^{1,1}$ with boundary $\Gamma$. Let $\Gamma_i$, $0 \leqslant i \leqslant I$, denote the connected components of the boundary $\Gamma$, $\Gamma_0$ being the exterior boundary of $\Omega$. We do not assume that $\Omega$ is simply-connected but we suppose that there exist $J$ connected open surfaces $\Sigma_j$, $1 \leqslant j \leqslant J$, called 'cuts,' contained in $\Omega$, such that each surface $\Sigma_j$ is an open subset of a smooth manifold, the boundary of $\Sigma_j$ is contained in $\Gamma$. The intersection $\overline{\Sigma_i} \cap \overline{\Sigma_j}$ is empty for $i \neq j$, and finally the open set $\Omega^\circ = \Omega \setminus \bigcup_{j=1}^{J} \Sigma_j$ is simply-connected and pseudo-$\mathcal{C}^{1,1}$ (see [1]). We are interested in some questions concerning the stationary Stokes equations with nonstandard boundary conditions, that generally can be written as:

$$(\mathcal{S}_T) \quad \begin{cases} -\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f} \quad \text{and} \quad \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} \cdot \boldsymbol{n} = g \quad \text{and} \quad \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{h} \times \boldsymbol{n} & \text{on } \Gamma, \\ \langle \boldsymbol{u} \cdot \boldsymbol{n}, 1 \rangle_{\Sigma_j} = 0, \quad 1 \leqslant j \leqslant J, \end{cases} \qquad (\mathcal{S}_N) \quad \begin{cases} -\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f} \quad \text{and} \quad \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{g} \times \boldsymbol{n} \quad \text{and} \quad \pi = \pi_0 & \text{on } \Gamma, \\ \langle \boldsymbol{u} \cdot \boldsymbol{n}, 1 \rangle_{\Gamma_i} = 0, \quad 1 \leqslant i \leqslant I, \end{cases}$$

where $\boldsymbol{u}$ denotes the velocity field and $\pi$ the pressure, both being unknown, and $\boldsymbol{f}$, $g$, $\boldsymbol{h}$, $\boldsymbol{g}$ and $\pi_0$ are given. Applications often give rise to problems where the previous boundary conditions occur naturally. We can find, in a Hilbertian case, a study of the Stokes problem with mixed boundary conditions of the same type [5].

To prove the existence of solutions of problems $(\mathcal{S}_T)$ and $(\mathcal{S}_N)$ (see the sketch of the proofs of Theorem 2.1 for $(\mathcal{S}_T)$ and Theorem 3.2 for $(\mathcal{S}_N)$) we begin by solving pressure $\pi$ as a solution of a Neumann problem or Dirichlet problem. Then, we are reduced to solve the following elliptic problems:

$$\begin{aligned} (E_T) \quad & -\Delta \boldsymbol{\xi} = \boldsymbol{f} \quad \text{and} \quad \operatorname{div} \boldsymbol{\xi} = 0 \quad \text{in } \Omega, \qquad \boldsymbol{\xi} \cdot \boldsymbol{n} = g \quad \text{and} \quad \mathbf{curl}\, \boldsymbol{\xi} \times \boldsymbol{n} = \boldsymbol{h} \times \boldsymbol{n} \quad \text{on } \Gamma, \\ & \langle \boldsymbol{\xi} \cdot \boldsymbol{n}, 1 \rangle_{\Sigma_j} = 0, \quad 1 \leqslant j \leqslant J, \\ (E_N) \quad & -\Delta \boldsymbol{\xi} = \boldsymbol{f} \quad \text{and} \quad \operatorname{div} \boldsymbol{\xi} = 0 \quad \text{in } \Omega, \qquad \boldsymbol{\xi} \times \boldsymbol{n} = \boldsymbol{g} \times \boldsymbol{n} \quad \text{on } \Gamma, \qquad \langle \boldsymbol{\xi} \cdot \boldsymbol{n}, 1 \rangle_{\Gamma_i} = 0, \quad 1 \leqslant i \leqslant I. \end{aligned}$$

Indeed, if $(\boldsymbol{u}, \ \pi)$ solves problem $(\mathcal{S}_T)$ (it is more simply for the problem $(\mathcal{S}_N)$), then $\Delta \pi = \operatorname{div} \boldsymbol{f}$ in $\Omega$ and formally (but we can it justify) we have for any $\varphi \in W^{2,p'}(\Omega)$

$$\left\langle \frac{\partial \pi}{\partial \boldsymbol{n}}, \varphi \right\rangle_{\Gamma} = \langle \boldsymbol{f} \cdot \boldsymbol{n} - \mathbf{curl}\,\mathbf{curl}\, \boldsymbol{u} \cdot \boldsymbol{n}, \varphi \rangle_{\Gamma} = \langle \boldsymbol{f} \cdot \boldsymbol{n}, \varphi \rangle_{\Gamma} - \langle \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{n}, \nabla \varphi \rangle_{\Gamma} = \langle \boldsymbol{f} \cdot \boldsymbol{n} + \operatorname{div}_{\Gamma}(\boldsymbol{h} \times \boldsymbol{n}), \varphi \rangle_{\Gamma},$$

where $\langle \cdot, \cdot \rangle_{\Gamma}$ denotes the duality product between $\boldsymbol{W}^{-1-\frac{1}{p},p}(\Gamma)$ and $\boldsymbol{W}^{1+\frac{1}{p},p'}(\Gamma)$. That means that $\frac{\partial \pi}{\partial \boldsymbol{n}} = \boldsymbol{f} \cdot \boldsymbol{n} + \operatorname{div}_{\Gamma}(\boldsymbol{h} \times \boldsymbol{n})$ in the sense of $\boldsymbol{W}^{-1-\frac{1}{p},p}(\Gamma)$.

In the sequel, the duality product between a space $X$ and its dual $X'$ is denoted by $\langle \cdot, \cdot \rangle_{X,X'}$. For any $1 < p < \infty$, we then define the spaces:

$$\begin{aligned} & \boldsymbol{H}^p(\mathbf{curl}, \Omega) = \{ \boldsymbol{v} \in \boldsymbol{L}^p(\Omega); \mathbf{curl}\, \boldsymbol{v} \in \boldsymbol{L}^p(\Omega) \}, \qquad \boldsymbol{H}^p(\operatorname{div}, \Omega) = \{ \boldsymbol{v} \in \boldsymbol{L}^p(\Omega); \operatorname{div} \boldsymbol{v} \in L^p(\Omega) \}, \\ & \boldsymbol{X}^p(\Omega) = \boldsymbol{H}^p(\mathbf{curl}, \Omega) \cap \boldsymbol{H}^p(\operatorname{div}, \Omega), \end{aligned}$$

which are equipped with the graph norm, and their subspaces:

$$\begin{aligned} & \boldsymbol{H}^p_0(\mathbf{curl}, \Omega) = \{ \boldsymbol{v} \in \boldsymbol{H}^p(\mathbf{curl}, \Omega); \ \boldsymbol{v} \times \boldsymbol{n} = \mathbf{0} \text{ on } \Gamma \}, \qquad \boldsymbol{H}^p_0(\operatorname{div}, \Omega) = \{ \boldsymbol{v} \in \boldsymbol{H}^p(\operatorname{div}, \Omega); \ \boldsymbol{v} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma \}, \\ & \boldsymbol{X}^p_N(\Omega) = \{ \boldsymbol{v} \in \boldsymbol{X}^p(\Omega); \ \boldsymbol{v} \times \boldsymbol{n} = \mathbf{0} \text{ on } \Gamma \}, \qquad \boldsymbol{X}^p_T(\Omega) = \{ \boldsymbol{v} \in \boldsymbol{X}^p(\Omega); \ \boldsymbol{v} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma \}, \end{aligned}$$

and $\boldsymbol{X}^p_0(\Omega) = \boldsymbol{X}^p_N(\Omega) \cap \boldsymbol{X}^p_T(\Omega)$. We also define the space $\boldsymbol{W}^{1,p}_{\sigma}(\Omega) = \{ \boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega); \operatorname{div} \boldsymbol{v} = 0 \text{ in } \Omega \}$. For any function $q$ in $W^{1,p}(\Omega^\circ)$, $\mathbf{grad}\, q$ can be extended to $\boldsymbol{L}^p(\Omega)$. We denote this extension by $\widetilde{\mathbf{grad}}\, q$. We finally define the spaces:

$$\begin{aligned} & \boldsymbol{K}^p_T(\Omega) = \{ \boldsymbol{v} \in \boldsymbol{X}^p_T(\Omega); \ \mathbf{curl}\, \boldsymbol{v} = \mathbf{0}, \ \operatorname{div} \boldsymbol{v} = 0 \text{ in } \Omega \}, \\ & \boldsymbol{K}^p_N(\Omega) = \{ \boldsymbol{v} \in \boldsymbol{X}^p_N(\Omega); \ \mathbf{curl}\, \boldsymbol{v} = \mathbf{0}, \ \operatorname{div} \boldsymbol{v} = 0 \text{ in } \Omega \}. \end{aligned}$$

We know due to [4] (see also [1] for the case $p = 2$) that the space $\boldsymbol{K}^p_T(\Omega)$ is of dimension $J$ and that it is spanned by functions $\widetilde{\mathbf{grad}}\, q^T_j$, $1 \leqslant j \leqslant J$, where each $q^T_j \in W^{1,p}(\Omega^\circ)$. Similarly, the dimension of the space $\boldsymbol{K}^p_N(\Omega)$ is $I$ and that it is spanned by the functions $\mathbf{grad}\, q^N_i$, $1 \leqslant i \leqslant I$, where each $q^N_i \in W^{1,p}(\Omega)$. In what follows, the letter $C$ denotes a constant that does not necessarily have the same value. The detailed proofs of the results announced in this Note are given in [4].

## 2. The Stokes equations with the tangential boundary conditions

We can prove that by assuming appropriate conditions on $\boldsymbol{f}$ and $\boldsymbol{h}$, the pressure in the problem $(\mathcal{S}_T)$ may be constant, and we are reduced to solve the elliptic system $(E_T)$:

**Proposition 2.1.** *Let $\boldsymbol{f}$ belongs to $\boldsymbol{L}^p(\Omega)$ with $\operatorname{div}\boldsymbol{f}=0$ in $\Omega$, $g\in W^{1-\frac{1}{p},p}(\Gamma)$ and $\boldsymbol{h}\in\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$ verify the following compatibility conditions*:

$$\boldsymbol{f}\cdot\boldsymbol{n}+\operatorname{div}_\Gamma(\boldsymbol{h}\times\boldsymbol{n})=0\quad\text{on }\Gamma, \tag{1}$$

$$\forall\boldsymbol{v}\in\boldsymbol{K}_T^{p'}(\Omega),\quad\int_\Omega\boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x}+\langle\boldsymbol{h}\times\boldsymbol{n},\boldsymbol{v}\rangle_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)\times\boldsymbol{W}^{\frac{1}{p},p'}(\Gamma)}=0\quad\text{and}\quad\int_\Gamma g\,\mathrm{d}\boldsymbol{\sigma}=0, \tag{2}$$

*where $\operatorname{div}_\Gamma$ is the surface divergence on $\Gamma$. Then, the problem $(E_T)$ has a unique solution $\boldsymbol{u}$ in $\boldsymbol{W}^{1,p}(\Omega)$ satisfying the estimate*:

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)}\leqslant C\big(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}+\|g\|_{W^{1-1/p,p}(\Gamma)}+\|\boldsymbol{h}\times\boldsymbol{n}\|_{\boldsymbol{W}^{-1/p,p}(\Gamma)}\big).$$

*Moreover, if $g\in W^{2-1/p,p}(\Gamma)$ and $\boldsymbol{h}\in\boldsymbol{W}^{1-1/p,p}(\Gamma)$, then the solution $\boldsymbol{u}$ belongs to $\boldsymbol{W}^{2,p}(\Omega)$ and satisfies the corresponding estimate.*

**Sketch of the proof.** For the proof of weak solutions, we reduce $(E_T)$ to a problem having homogeneous normal boundary condition on $\Gamma$, where it is easy to solve it by using the *Inf–Sup* condition (see [4]):

$$\inf_{\boldsymbol{\varphi}\in\boldsymbol{V}_T^{p'}(\Omega)}\sup_{\boldsymbol{u}\in\boldsymbol{V}_T^{p}(\Omega)}\frac{\int_\Omega\operatorname{\mathbf{curl}}\boldsymbol{u}\cdot\operatorname{\mathbf{curl}}\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x}}{\|\boldsymbol{u}\|_{\boldsymbol{X}_T^p(\Omega)}\|\boldsymbol{\varphi}\|_{\boldsymbol{X}_T^{p'}(\Omega)}}>0, \tag{3}$$

with

$$\boldsymbol{V}_T^p(\Omega)=\big\{\boldsymbol{v}\in\boldsymbol{X}_T^p(\Omega);\ \operatorname{div}\boldsymbol{v}=0\text{ in }\Omega,\ \langle\boldsymbol{v}\cdot\boldsymbol{n},1\rangle_{\Sigma_j}=0,\ 1\leqslant j\leqslant J\big\}.$$

For the regularity, we set $\boldsymbol{z}=\operatorname{\mathbf{curl}}\boldsymbol{u}$. Since $\boldsymbol{z}\times\boldsymbol{n}\in\boldsymbol{W}^{1-1/p,p}(\Gamma)$, we deduce from [4] that $\boldsymbol{z}\in\boldsymbol{W}^{1,p}(\Omega)$. Therefore, since $\boldsymbol{u}\cdot\boldsymbol{n}\in W^{2-1/p,p}(\Gamma)$, then $\boldsymbol{u}\in\boldsymbol{W}^{2,p}(\Omega)$. □

**Theorem 2.1** *(Weak and Strong solutions for $(\mathcal{S}_T)$). Let $\boldsymbol{f}$, $g$, $\boldsymbol{h}$ with*

$$\boldsymbol{f}\in\big(\boldsymbol{H}_0^{p'}(\operatorname{div},\Omega)\big)',\qquad g\in W^{1-\frac{1}{p},p}(\Gamma),\qquad\boldsymbol{h}\in\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma), \tag{4}$$

*and verify the compatibility conditions* (2)*. Then, the Stokes problem $(\mathcal{S}_T)$ has a unique solution $(\boldsymbol{u},\pi)\in\boldsymbol{W}^{1,p}(\Omega)\times L^p(\Omega)/\mathbb{R}$ satisfying the estimate*:

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)}+\|\pi\|_{L^p(\Omega)/\mathbb{R}}\leqslant C\big(\|\boldsymbol{f}\|_{(\boldsymbol{H}_0^{p'}(\operatorname{div},\Omega))'}+\|g\|_{W^{1-\frac{1}{p},p}(\Gamma)}+\|\boldsymbol{h}\times\boldsymbol{n}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)}\big).$$

*Moreover, if $\boldsymbol{f}\in\boldsymbol{L}^p(\Omega)$, $g\in W^{2-\frac{1}{p},p}(\Gamma)$, $\boldsymbol{h}\in\boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)$, the solution $(\boldsymbol{u},\pi)$ belongs to $\boldsymbol{W}^{2,p}(\Omega)\times W^{1,p}(\Omega)$ and satisfies the corresponding estimate.*

**Sketch of the proof.** We reduce $(\mathcal{S}_T)$ to a problem with the homogeneous normal boundary condition on $\Gamma$. We use again the *Inf–Sup* condition (3) in order to prove the existence of a unique $\boldsymbol{u}\in\boldsymbol{W}^{1,p}(\Omega)$ solution of $(\mathcal{S}_T)$ and by using De Rham's Theorem, we prove the existence of a unique $\pi\in L^p(\Omega)$. For the regularity of the solution, we observe that $\pi$ satisfies: $\operatorname{div}(\nabla\pi-\boldsymbol{f})=0$ in $\Omega$ and $(\nabla\pi-\boldsymbol{f})\cdot\boldsymbol{n}=-\operatorname{div}_\Gamma(\boldsymbol{h}\times\boldsymbol{n})$ on $\Gamma$ which implies that $\pi$ belongs to $W^{1,p}(\Omega)$. We deduce the regularity of $\boldsymbol{u}$ since $\boldsymbol{u}$ is a solution of a problem $(E_T)$ with the right-hand side $\boldsymbol{F}=\boldsymbol{f}-\nabla\pi$ and by using some regularity properties concerning the tangential vector fields $\boldsymbol{v}$ in $\boldsymbol{L}^p(\Omega)$ with $\operatorname{div}\boldsymbol{v}$ in $W^{1,p}(\Omega)$ and $\operatorname{\mathbf{curl}}\boldsymbol{v}$ in $\boldsymbol{W}^{1,p}(\Omega)$. □

**Remark 2.2.** We can also treat the case when the divergence operator does not vanish. So we consider the following Stokes problem

$$\begin{cases}-\Delta\boldsymbol{u}+\nabla\pi=\boldsymbol{f}\quad\text{and}\quad\operatorname{div}\boldsymbol{u}=\chi\qquad\text{in }\Omega,\\ \boldsymbol{u}\cdot\boldsymbol{n}=g\quad\text{and}\quad\operatorname{\mathbf{curl}}\boldsymbol{u}\times\boldsymbol{n}=\boldsymbol{h}\times\boldsymbol{n}\quad\text{on }\Gamma,\qquad\langle\boldsymbol{u}\cdot\boldsymbol{n},1\rangle_{\Sigma_j}=0,\quad 1\leqslant j\leqslant J.\end{cases} \tag{5}$$

If we suppose that $\chi$ belongs to $L^p(\Omega)$, $\boldsymbol{f}$, $g$, $\boldsymbol{h}$ as in (4) satisfying the first compatibility condition in (2) and such that

$$\int_\Omega\chi\,\mathrm{d}\boldsymbol{x}=\int_\Gamma g\,\mathrm{d}\boldsymbol{\sigma}, \tag{6}$$

then, we can prove that the Stokes problem (5) has a unique solution $(\boldsymbol{u},\pi)\in \boldsymbol{W}^{1,p}(\Omega)\times L^p(\Omega)$ satisfying the estimate:

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)}+\|\pi\|_{L^p(\Omega)/\mathbb{R}}\leqslant C\big(\|\boldsymbol{f}\|_{(\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega))'}+\|\chi\|_{L^p(\Omega)}+\|g\|_{W^{1-\frac{1}{p},p}(\Gamma)}+\|\boldsymbol{h}\times\boldsymbol{n}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)}\big).$$

Moreover, if we suppose that $\chi\in W^{1,p}(\Omega)$ with $\boldsymbol{f}\in \boldsymbol{L}^p(\Omega)$, $g\in W^{2-\frac{1}{p},p}(\Gamma)$, $\boldsymbol{h}\in \boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)$, then the solution $(\boldsymbol{u},\pi)$ belongs to $\boldsymbol{W}^{2,p}(\Omega)\times W^{1,p}(\Omega)$ and satisfies the corresponding estimate.

We define now the following spaces: $\boldsymbol{T}^p(\Omega)=\{\boldsymbol{\varphi}\in \boldsymbol{H}_0^p(\mathrm{div},\Omega);\ \mathrm{div}\,\boldsymbol{\varphi}\in W_0^{1,p}(\Omega)\}$, $\boldsymbol{Y}_T^p(\Omega)=\{\boldsymbol{\varphi}\in \boldsymbol{W}^{2,p}(\Omega);\ \boldsymbol{\varphi}\cdot\boldsymbol{n}=0,\ \mathrm{div}\,\boldsymbol{\varphi}=0,\ \mathbf{curl}\,\boldsymbol{\varphi}\times\boldsymbol{n}=\boldsymbol{0}$ on $\Gamma\}$ and $\boldsymbol{H}_p(\Delta;\Omega)=\{\boldsymbol{v}\in \boldsymbol{L}^p(\Omega);\ \Delta\boldsymbol{v}\in(\boldsymbol{T}^{p'}(\Omega))'\}$, endowed with the corresponding graph norms. Note that $\boldsymbol{\mathcal{D}}(\Omega)$ is dense in $\boldsymbol{T}^p(\Omega)$ and then $[\boldsymbol{T}^p(\Omega)]'$ is a subspace of $\boldsymbol{\mathcal{D}}'(\Omega)$.

**Theorem 2.3** *(Very weak solutions for* $(\mathcal{S}_T)$*). Let* $\boldsymbol{f}$, $\chi$, $g$, *and* $\boldsymbol{h}$ *with*

$$\boldsymbol{f}\in\big(\boldsymbol{T}^{p'}(\Omega)\big)',\qquad \chi\in L^p(\Omega),\qquad g\in W^{-1/p,p}(\Gamma),\qquad \boldsymbol{h}\in \boldsymbol{W}^{-1-1/p,p}(\Gamma),$$

*and satisfying the first compatibility condition in* (2) *and* (6)*. Then, the Stokes problem* (5) *has exactly one solution* $\boldsymbol{u}\in \boldsymbol{H}_p(\Delta;\Omega)$ *and* $\pi\in W^{-1,p}(\Omega)/\mathbb{R}$ *satisfying the estimate*:

$$\|\boldsymbol{u}\|_{\boldsymbol{H}_p(\Delta;\Omega)}+\|\pi\|_{W^{-1,p}(\Omega)/\mathbb{R}}\leqslant C\big(\|\boldsymbol{f}\|_{(\boldsymbol{T}^{p'}(\Omega))'}+\|\chi\|_{L^p(\Omega)}+\|g\|_{W^{-1/p,p}(\Gamma)}+\|\boldsymbol{h}\times\boldsymbol{n}\|_{\boldsymbol{W}^{-1-1/p,p}(\Gamma)}\big).$$

**Sketch of the proof.** We use here the same ideas as in [2] and [3] to prove the existence of very weak solutions. First, we prove the density of the space $\mathcal{D}(\overline{\Omega})$ in $\boldsymbol{H}_p(\Delta;\Omega)$. Second, we prove that the mapping $\gamma:\boldsymbol{u}\mapsto \mathbf{curl}\,\boldsymbol{u}|_\Gamma\times\boldsymbol{n}$ on the space $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ can be extended by continuity to a linear and continuous mapping still denoted by $\gamma$, from $\boldsymbol{H}_p(\Delta;\Omega)$ into $\boldsymbol{W}^{-1-\frac{1}{p},p}(\Gamma)$ and we have the following Green formula: for any $\boldsymbol{u}\in \boldsymbol{H}_p(\Delta;\Omega)$ and $\boldsymbol{\varphi}\in \boldsymbol{Y}_T^{p'}(\Omega)$,

$$\langle\Delta\,\boldsymbol{u},\boldsymbol{\varphi}\rangle_{(\boldsymbol{T}^{p'}(\Omega))'\times\boldsymbol{T}^{p'}(\Omega)}=\int_\Omega \boldsymbol{u}\cdot\Delta\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x}+\langle\mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n},\boldsymbol{\varphi}\rangle_{\boldsymbol{W}^{-1-\frac{1}{p},p}(\Gamma)\times\boldsymbol{W}^{1+1/p,p'}(\Gamma)}. \tag{7}$$

Finally, using the formula (7), we can write an equivalent variational formulation of the problem (5) and we are able to conclude by using a duality argument. □

## 3. The Stokes equations with the normal boundary conditions

In this section, we focus on the study of the Stokes problem $(\mathcal{S}_N)$. Observe that the pressure $\pi$ can be obtained independently of the velocity as a solution of a Dirichlet problem. So, the velocity $\boldsymbol{u}$ is a solution of an elliptic system of type $(E_N)$.

**Proposition 3.1.** *Let* $\boldsymbol{f}\in(\boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega))'$ *with* $\mathrm{div}\,\boldsymbol{f}=0$ *in* $\Omega$ *and* $\boldsymbol{g}\in \boldsymbol{W}^{1-1/p,p}(\Gamma)$ *satisfying the compatibility condition*:

$$\forall\boldsymbol{v}\in \boldsymbol{K}_N^{p'}(\Omega),\quad \langle\boldsymbol{f},\boldsymbol{v}\rangle_{[\boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega)]'\times\boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega)}=0. \tag{8}$$

*Then, the problem* $(E_N)$ *has a unique solution* $\boldsymbol{u}$ *in* $\boldsymbol{W}^{1,p}(\Omega)$ *satisfying the estimate*:

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)}\leqslant C\big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega)]'}+\|\boldsymbol{g}\times\boldsymbol{n}\|_{\boldsymbol{W}^{1-1/p,p}(\Gamma)}\big).$$

*Moreover, if* $\boldsymbol{f}\in \boldsymbol{L}^p(\Omega)$ *and* $\boldsymbol{g}\in \boldsymbol{W}^{2-1/p,p}(\Gamma)$*, then the solution* $\boldsymbol{u}$ *is in* $\boldsymbol{W}^{2,p}(\Omega)$ *and satisfies the corresponding estimate.*

**Sketch of the proof.** First, we lift the boundary condition and we write an equivalent variational formulation for the homogeneous problem as follows: find $\boldsymbol{u}\in \boldsymbol{V}_N^p(\Omega)$ such that

$$\forall\boldsymbol{\varphi}\in \boldsymbol{V}_N^{p'}(\Omega),\quad \int_\Omega \mathbf{curl}\,\boldsymbol{u}\cdot\mathbf{curl}\,\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x}=\langle\boldsymbol{f},\boldsymbol{\varphi}\rangle_\Omega, \tag{9}$$

where $\boldsymbol{V}_N^p(\Omega)=\{\boldsymbol{w}\in \boldsymbol{X}_N^p(\Omega);\ \mathrm{div}\,\boldsymbol{w}=0$ in $\Omega$ and $\langle\boldsymbol{w}\cdot\boldsymbol{n},1\rangle_{\Gamma_i}=0,\ 1\leqslant i\leqslant I\}$. Next, using a result concerning normal vector potential [4], we establish a similar *Inf–Sup* condition to (3), where the spaces $\boldsymbol{X}_T^p(\Omega)$ and $\boldsymbol{V}_T^p(\Omega)$ are replaced by the spaces $\boldsymbol{X}_N^p(\Omega)$ and $\boldsymbol{V}_N^p(\Omega)$, respectively. This conclude the proof of weak solution. For the regularity of the velocity, we need some additional properties. We prove the following trace formula for any $\boldsymbol{v}\in \boldsymbol{W}^{1,p}(\Omega)$:

$$\mathbf{curl}\,\boldsymbol{u}\cdot\boldsymbol{n}=\left(\sum_{j=1}^{2}\frac{\partial\boldsymbol{u}}{\partial s_j}\times\boldsymbol{\tau}_j\right)\cdot\boldsymbol{n}\quad\text{on }\Gamma,\quad\text{in the sense of }\boldsymbol{W}^{-1/p,p}(\Gamma).\tag{10}$$

As a consequence, if we suppose that $\boldsymbol{u}\times\boldsymbol{n}\in\boldsymbol{W}^{2-1/p,p}(\Gamma)$, then $\mathbf{curl}\,\boldsymbol{u}\cdot\boldsymbol{n}\in W^{1-1/p,p}(\Gamma)$. This implies that $\mathbf{curl}\,\boldsymbol{u}\in\boldsymbol{W}^{1,p}(\Omega)$ and thereafter from [4], we have $\boldsymbol{u}\in\boldsymbol{W}^{2,p}(\Omega)$. □

We can also treat the case of the following elliptic system, which is similar to $(E_N)$ but where we have replaced the condition $\operatorname{div}\boldsymbol{u}=0$ in $\Omega$ by $\operatorname{div}\boldsymbol{u}=0$ on $\Gamma$:

$$\left(E_N'\right)\quad -\Delta\boldsymbol{u}=\boldsymbol{f}\quad\text{in }\Omega,\qquad \operatorname{div}\boldsymbol{u}=0\quad\text{on }\Gamma,\qquad \boldsymbol{u}\times\boldsymbol{n}=\boldsymbol{0}\quad\text{on }\Gamma,\qquad \langle\boldsymbol{u}\cdot\boldsymbol{n},1\rangle_{\Gamma_i}=0\quad\text{for any }1\leqslant i\leqslant I.$$

**Theorem 3.1.** *Let $\boldsymbol{f}\in(\boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega))'$ satisfying the compatibility condition* (8)*. Then, the problem $(E_N')$ has a unique solution $\boldsymbol{u}$ in $\boldsymbol{W}^{1,p}(\Omega)$ satisfying the estimate*:

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)}\leqslant C\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega)]'}.\tag{11}$$

*Moreover, if $\boldsymbol{f}\in\boldsymbol{L}^p(\Omega)$, then the solution $\boldsymbol{u}$ is in $\boldsymbol{W}^{2,p}(\Omega)$ and satisfies the corresponding estimate.*

**Theorem 3.2** *(Weak and Strong solutions for $(\mathcal{S}_N)$). Let $\boldsymbol{f}$, $\boldsymbol{g}$, $\pi_0$ such that*

$$\boldsymbol{f}\in\left(\boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega)\right)',\qquad \boldsymbol{g}\in\boldsymbol{W}^{1-1/p,p}(\Gamma),\qquad \pi_0\in W^{1-1/p,p}(\Gamma),\tag{12}$$

$$\forall\boldsymbol{v}\in\boldsymbol{K}_N^{p'}(\Omega),\quad \langle\boldsymbol{f},\boldsymbol{v}\rangle_{[\boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega)]'\times\boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega)}-\int_\Gamma\pi_0\boldsymbol{v}\cdot\boldsymbol{n}\,\mathrm{d}\boldsymbol{\sigma}=0,\tag{13}$$

*then, the Stokes problem $(\mathcal{S}_N)$ has a unique solution $(\boldsymbol{u},\pi)\in\boldsymbol{W}^{1,p}(\Omega)\times W^{1,p}(\Omega)$ satisfying the estimate*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)}+\|\pi\|_{W^{1,p}(\Omega)}\leqslant C\left(\|\boldsymbol{f}\|_{(\boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega))'}+\|\boldsymbol{g}\times\boldsymbol{n}\|_{\boldsymbol{W}^{1-1/p,p}(\Gamma)}+\|\pi_0\|_{W^{1-1/p,p}(\Gamma)}\right).\tag{14}$$

*Moreover, if $\boldsymbol{f}\in\boldsymbol{L}^p(\Omega)$, $\boldsymbol{g}\in\boldsymbol{W}^{2-1/p,p}(\Gamma)$, $\pi_0\in W^{1-1/p,p}(\Gamma)$, then the solution $(\boldsymbol{u},\pi)$ belongs to $\boldsymbol{W}^{2,p}(\Omega)\times W^{1,p}(\Omega)$ and satisfies the corresponding estimate.*

**Sketch of the proof.** We note that the pressure is a solution of the following Dirichlet problem: $-\Delta\pi=\operatorname{div}\boldsymbol{f}$ in $\Omega$ and $\pi=\pi_0$ on $\Gamma$. Since $\pi_0\in W^{1-1/p,p}(\Gamma)$, then $\pi\in W^{1,p}(\Omega)$. The velocity is a solution of the problem $(E_N)$ and it suffices to apply Proposition 3.1 to obtain weak and strong solutions. □

**Theorem 3.3** *(Very weak solutions for $(\mathcal{S}_N)$). Let $\boldsymbol{f}$, $\boldsymbol{g}$, and $\pi_0$ with*

$$\boldsymbol{f}\in\left[\boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega)\right]',\qquad \boldsymbol{g}\in\boldsymbol{W}^{-1/p,p}(\Gamma),\qquad \pi_0\in W^{-1/p,p}(\Gamma),$$

*and satisfying the compatibility conditions* (13)*. Then, the Stokes problem $(\mathcal{S}_N)$ has exactly one solution $\boldsymbol{u}\in\boldsymbol{L}^p(\Omega)$ and $\pi\in L^p(\Omega)$. Moreover, there exists a constant $C>0$ depending only on $p$ and $\Omega$ such that*:

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}+\|\pi\|_{L^p(\Omega)}\leqslant C\left(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega)]'}+\|\boldsymbol{g}\|_{\boldsymbol{W}^{-1/p,p}(\Gamma)}+\|\pi_0\|_{\boldsymbol{W}^{-1/p,p}(\Gamma)}\right).\tag{15}$$

**Sketch of the proof.** We use similar arguments presented for the case of problem $(\mathcal{S}_N)$ and the main difference between the two proofs is the fact that we prove a global Green formula. More precisely, we set the space

$$\boldsymbol{M}^p(\Omega)=\left\{(\boldsymbol{u},\pi)\in\boldsymbol{Z}^p(\Omega)\times L^p(\Omega);\ -\Delta\boldsymbol{u}+\nabla\pi\in\left[\boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega)\right]'\right\},$$

with $\boldsymbol{Z}^p(\Omega)=\{\boldsymbol{v}\in\boldsymbol{L}^p(\Omega),\ \operatorname{div}\boldsymbol{v}=0\text{ in }\Omega\text{ and }\langle\boldsymbol{v}\cdot\boldsymbol{n},1\rangle_{\Gamma_i},\ 1\leqslant i\leqslant I\}$ and by establishing the density of $\boldsymbol{\mathcal{D}}_\sigma(\overline{\Omega})\times\mathcal{D}(\overline{\Omega})$ in $\boldsymbol{M}^p(\Omega)$, we prove that the trace of any $(\boldsymbol{u},\pi)\in\boldsymbol{M}^p(\Omega)$ belongs to $\boldsymbol{W}^{-1/p,p}(\Gamma)\times W^{-1/p,p}(\Gamma)$ with the following Green formula for any $\boldsymbol{\varphi}\in\boldsymbol{Y}_N^{p'}(\Omega)$:

$$\langle-\Delta\boldsymbol{u}+\nabla\pi,\boldsymbol{\varphi}\rangle_\Omega=-\int_\Omega\boldsymbol{u}\cdot\Delta\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x}+\langle\boldsymbol{u}\times\boldsymbol{n},\mathbf{curl}\,\boldsymbol{\varphi}\rangle_\Gamma-\int_\Omega\pi\operatorname{div}\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x}+\langle\pi,\boldsymbol{\varphi}\cdot\boldsymbol{n}\rangle_\Gamma,\tag{16}$$

where $\boldsymbol{Y}_N^{p'}(\Omega)=\{\boldsymbol{\varphi}\in\boldsymbol{W}^{2,p}(\Omega);\ \operatorname{div}\boldsymbol{\varphi}=0\text{ and }\boldsymbol{\varphi}\times\boldsymbol{n}=\boldsymbol{0}\text{ on }\Gamma\}$. In the first time, we prove the existence of a unique $\pi\in W^{-1,p}(\Omega)$, next we use [3] in order to prove that $\pi\in L^p(\Omega)$. □

## 4. Helmholtz decompositions

According to the two types $\boldsymbol{u}\cdot\boldsymbol{n}$ and $\boldsymbol{u}\times\boldsymbol{n}$ of boundary conditions on $\Gamma$, we give decompositions of vector fields $\boldsymbol{u}$ in $\boldsymbol{L}^p(\Omega)$. Our results may be regarded as an extension of the well-known De Rham–Hodge–Kodaira decomposition of $\mathcal{C}^\infty$-forms on compact Riemannian manifolds into $\boldsymbol{L}^p$-vector fields on $\Omega$. We can find similar decompositions in [6], where the authors consider more regular domain with $\mathcal{C}^\infty$-boundary $\Gamma$. We can see also [7] for the case $p=2$.

**Theorem 4.1.**

(i) *Let* $\boldsymbol{u}\in\boldsymbol{L}^p(\Omega)$*. Then, there exist* $\chi\in W^{1,p}(\Omega)$, $\boldsymbol{w}\in\boldsymbol{W}^{1,p}_\sigma(\Omega)\cap\boldsymbol{X}^p_N(\Omega)$, $\boldsymbol{z}\in\boldsymbol{K}^p_T(\Omega)$ *such that*: $\boldsymbol{u}=\boldsymbol{z}+\nabla\chi+\textbf{curl}\,\boldsymbol{w}$ *satisfies the estimate*:

$$\|\boldsymbol{z}\|_{\boldsymbol{L}^p(\Omega)}+\|\chi\|_{W^{1,p}(\Omega)/\mathbb{R}}+\|\boldsymbol{w}\|_{\boldsymbol{W}^{1,p}(\Omega)/\boldsymbol{K}^p_N(\Omega)}\leqslant C\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)},$$

*where* $\boldsymbol{z}$ *is unique,* $\chi$ *is unique up to an additive constant and* $\boldsymbol{w}$ *is unique up to an additive element of* $\boldsymbol{K}^p_N(\Omega)$.

(ii) *Let* $\boldsymbol{u}\in\boldsymbol{L}^p(\Omega)$*. Then, there exist* $\chi\in W^{1,p}_0(\Omega)$, $\boldsymbol{w}\in\boldsymbol{W}^{1,p}_\sigma(\Omega)\cap\boldsymbol{X}^p_T(\Omega)$, $\boldsymbol{z}\in\boldsymbol{K}^p_N(\Omega)$ *such that*: $\boldsymbol{u}=\boldsymbol{z}+\nabla\chi+\textbf{curl}\,\boldsymbol{w}$ *satisfies the estimate*:

$$\|\boldsymbol{z}\|_{\boldsymbol{L}^p(\Omega)}+\|\chi\|_{W^{1,p}(\Omega)}+\|\boldsymbol{w}\|_{\boldsymbol{W}^{1,p}(\Omega)/\boldsymbol{K}^p_T(\Omega)}\leqslant C\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)},$$

*where* $\boldsymbol{z}$ *and* $\chi$ *are unique and* $\boldsymbol{w}$ *is unique up to an additive element of* $\boldsymbol{K}^p_T(\Omega)$.

**Sketch of the proof.** We give a short proof of the first point and the proof of the second one is similar. First, we introduce the solution $\chi$ in $W^{1,p}(\Omega)$, unique up to an additive constant, of the problem: $-\Delta\chi=\operatorname{div}\boldsymbol{u}$ in $\Omega$ and $(\textbf{grad}\,\chi-\boldsymbol{u})\cdot\boldsymbol{n}=0$ on $\Gamma$. Second, we solve the problem: $-\Delta\boldsymbol{w}=\textbf{curl}\,\boldsymbol{u}$ in $\Omega$ and $\operatorname{div}\boldsymbol{w}=0$ in $\Omega$, $\boldsymbol{w}\times\boldsymbol{n}=\boldsymbol{0}$ on $\Gamma$, which has a solution $\boldsymbol{w}\in\boldsymbol{W}^{1,p}(\Omega)$, unique up to an additive element of $\boldsymbol{K}^p_N(\Omega)$. To finish, observe that the function $\boldsymbol{z}=\boldsymbol{u}-\nabla\chi-\textbf{curl}\,\boldsymbol{w}$ belongs to $\boldsymbol{K}^p_T(\Omega)$. □

**Remark 4.2.** We can prove also similar decompositions for singular vector fields $\boldsymbol{u}\in(\boldsymbol{H}^p_0(\operatorname{div},\Omega))'$ and for $\boldsymbol{u}\in(\boldsymbol{H}^p_0(\textbf{curl},\Omega))'$.

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