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Number Theory

# On Zaremba's conjecture

## *Sur une conjecture de Zaremba*

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ABSTRACT

It is shown that there is a constant $A$ and a density one subset $S$ of the positive integers such that, for each $q \in S$, there is some $1 \leqslant p < q$, $(p, q) = 1$, so that $\frac{p}{q}$ has all its partial quotients bounded by $A$.



RÉSUMÉ

On montre qu'il existe une constante $A$ et un sous-ensemble $S$ des entiers positifs de densité un, tel que pour tout $q \in S$ il y a un entier $1 \leqslant p < q$, $(p, q) = 1$ pour lequel les quotients partiels de $\frac{p}{q}$ sont bornés par $A$.



### Version française abrégée

Une conjecture due à Zaremba [9] affirme qu'il existe pour tout entier $q \in \mathbb{Z}_+$ un entier $1 \leqslant p < q$, $(p, q) = 1$, tel que le développement $\frac{p}{q} = [a_1, \dots, a_k]$ en fraction continue a tous ses quotients partiels $a_j$ bornés par une constante absolue $A$. En fait, il est conjecturé que $A = 5$ suffit. Notre résultat principal est le suivant :

**Théorème.** *Il existe un sous-ensemble $S$ de $\mathbb{Z}_+$, de pleine densité, vérifiant la conjecture de Zaremba pour une constante absolue $A$.*

Notre approche au problème est une adaptation de la méthode introduite dans [2] pour étudier les ensembles d'entiers engendrés par les orbites d'un sous-groupe de $SL_2(\mathbb{Z})$. La différence principale est que, dans la situation présente, il s'agit d'un semi-groupe. En particulier, nous ne faisons pas usage des résultats de [3], mais plutôt de ceux de [1] basés sur l'approche symbolique.

**1.** For given $A > 0$, let $\mathcal{C}_A \subset [0, 1]$ be the Cantor-like set of real numbers $x$ in the unit interval, whose partial quotients are bounded by $A$. Thus, writing $x$ in its continued fraction expansion

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$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots \cfrac{1}{a_k + \ddots}}} = [a_1, a_2, \dots, a_k, \dots],$$

we have that all partial quotients $a_k$ are bounded by $A$. The Hausdorff dimension $\delta_A$ of $\mathcal{C}_A$ is asymptotically

$$\delta_A = 1 - \frac{6}{\pi^2 A} - \frac{72 \log A}{\pi^4 A^2} + O\left(\frac{1}{A^2}\right)$$

as $A \to \infty$ [5].

Let further $\mathcal{R}_A$ denote the set of all partial convergents $\frac{p}{q}, (p,q) = 1$ of numbers in $\mathcal{C}_A$ and let $\mathcal{Q}_A$ be the set of all continuants $q$.

Zaremba's conjecture [9] states that

$$\mathcal{Q}_A = \mathbb{Z}_+$$

for sufficiently large $A$ (possibly $A = 5$). See also [6].

It should be noted that the original motivation for this problem has to do with the theory of "good" lattice points and low-discrepancy sequences in numerical multi-dimensional integration and in pseudo-randomness, see [7].

It was shown by Niederreiter [8] that Zaremba's conjecture holds for small powers, in fact

$$\{2^j\}, \{3^k\} \subset \mathcal{Q}_3.$$

On the other hand, a result due to Hensley [4] states that there are constants $0 < c < C < \infty$ so that for $N$ sufficiently large,

$$c \cdot N^{2\delta_A} < \#\left\{\frac{p}{q} \in \mathcal{R}_A; (p,q) = 1 \text{ and } 1 \leqslant p < q \leqslant N\right\} < C \cdot N^{2\delta_A}.$$

Note also that $p + q \in \mathcal{Q}_A$ whenever $\frac{p}{q} \in \mathcal{R}_A$. An easy consequence the previous two facts is that

$$\#\mathcal{Q}_A \cap [1, N] \gg N^{\delta_A}.$$

Our main result is the following:

**Theorem.** *For $A$ sufficiently large, almost every integer satisfies Zaremba's conjecture. That is,*

$$\#\mathcal{Q}_A \cap [1, N] = N\big(1 + o(1)\big),$$

*as $N \to \infty$. ($A = 2189$ satisfies the claim.)*

## 2. A few comments about the method

Our approach is an adaptation of the technique introduced in [2] in the study of sequences of integers produced by orbits of subgroups $\Gamma$ of $\mathrm{SL}_2(\mathbb{Z})$, assuming the dimension $0 < \delta < 1$ of the limit set of $\Gamma$ close enough to 1. We proceed by the Hardy–Littlewood circle method, analyzing certain relevant exponential sums on 'minor' and 'major' arcs. While this approach is quite standard in number theoretical problems (for instance in the Goldbach problem), the ingredients involved in our situation are special.

In [2], the analysis on the minor arcs is achieved using Vinogradov-type multi-linear estimates, depending essentially on the group structure. Then a precise evaluation of the exponential sum on the major arcs is obtained by relying on the spectral and representation theory of $\Gamma\backslash\mathrm{SL}_2$, as developed in [3]. The outcome is the usual local-to-global representation formula, with a small exceptional set.

It turns out that Zaremba's conjecture admits a formulation of similar flavor. Let $\mathcal{G}_A$ be the semi-group generated by the matrices

$$\begin{pmatrix} 0 & 1 \\ 1 & a \end{pmatrix} \quad \text{with } 1 \leqslant a \leqslant A$$

and observe that

$$\begin{pmatrix} 0 & 1 \\ 1 & a_1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & a_2 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & a_k \end{pmatrix} = \begin{pmatrix} * & p \\ * & q \end{pmatrix}$$

is equivalent to $\frac{p}{q} = [a_1, \dots, a_k]$.

Hence, the orbit $\mathcal{G}_A e_2$, with $e_2 = (0, 1)$, consists precisely of the set of coprime pairs $(p, q)$ with $\frac{p}{q} \in \mathcal{R}_A$.

The main difference with [2] is that instead of the group $\Gamma$, the semi-group $\mathcal{G}_A$ is involved. It turns out however that this distinction has essentially no effect on the minor arcs analysis. On the other hand to proceed with the description of the exponential sum on the major arcs, the automorphic approach from [3] is no longer applicable. Instead we rely on the thermodynamical formalism based on the Ruelle transfer operator (which actually is already exploited in [4]). Here our aim is to establish certain equidistributional properties from a joint Archimedean/modular perspective, since this allows us to analyze the exponential sum on a major arc $\theta = \frac{r}{s} + \beta$, $(r, s) = 1$ with $s < N^\epsilon$ and $|\beta| < 1/N^{1-\epsilon}$. This type of (quantitative) results is provided by Bourgain et al. [1] in a form applicable to our problem.

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