C. R. Acad. Sci. Paris, Ser. I 349 (2011) 523–528

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

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Partial Differential Equations/Numerical Analysis

# Error estimates for three-dimensional Stokes problem with non-standard boundary conditions

## *Erreurs d'estimation pour le problème de Stokes en trois dimensions avec des conditions aux limites non standards*

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ARTICLE INFO

*Article history:*
Received 20 July 2010
Accepted after revision 25 March 2011
Available online 20 April 2011

Presented by Olivier Pironneau

ABSTRACT

This work is devoted to the optimal and *a posteriori* error estimates of the Stokes problem with some non-standard boundary conditions in three dimensions. The variational formulation is decoupled into a system for the velocity and a Poisson equation for the pressure. The velocity is approximated with **curl** conforming finite elements and the pressure with standard continuous elements. Next, we establish optimal *a posteriori* estimates.



RÉSUMÉ

Dans cette Note, nous établissons des estimations d'erreur *a posteriori* pour le problème de Stokes avec certaines conditions aux limites non standards en dimension trois. La formulation variationnelle est découplée en un problème pour la vitesse et une équation de Poisson pour la pression. La vitesse est approchée par les éléments finis **rot** et la pression par les éléments continus standards. Nous établirons par la suite les estimations *a posteriori* optimales.



### Version française abrégée

On consière le problème formé par l'équation (1) avec la condition aux limites (2) (ou (3)) où $\Omega$ est un domaine borné polyhédrique et convexe de $\mathbb{R}^3$ de bord $\Gamma = \partial\Omega$. On note $\mathbf{n}$ le vecteur normal sortant sur $\Gamma$, $\mathbf{u}$ la vitesse et $p$ la pression.

Habituellement, pour le problème de Stokes, nous utilisons la condition inf-sup pour établir l'existence et l'unicité de la solution théorique ; et pour la discrétisation de la pression et de la vitesse, nous utilisons des ensembles d'éléments finis qui vérifient aussi la condition inf-sup discrète.

Dans notre travail, les conditions aux limites non standard (2) et (3) nous permettent de découpler le système variationnel en une équation de Poisson pour la pression et une autre équation pour la vitesse où l'existence et l'unicité de la solution découlent directement sans la condition inf-sup. Nous utilisons la méthode des éléments finis non-conformes $H(\mathbf{rot})$ pour la vitesse et les éléments finis $P^1$ pour la pression.

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1631-073X/$ – see front matter 
doi:10.1016/j.crma.2011.03.021

Les estimations d'erreur *a priori* peuvent être classiquement établies. Concernant les etimations d'erreur *a posteriori*, nous commençons par établir celles relatives à la pression par les méthodes usuelles. Celles de la vitesse se basent sur la décomposition suivante ;

$$\mathbf{u} - \mathbf{u}_h = \nabla\lambda + \mathbf{w}$$

où $\lambda \in H_0^1(\Omega)$ et $\mathbf{w} \in V_0$ (pour la condition aux limites (2)). Ainsi, nous obtenons le système suivant vérifié uniquement par $\lambda$ :

$$\int_\Omega \nabla\lambda\nabla\mu = -\frac{1}{2}\sum_{\kappa\in\tau_h}\left(\sum_{e\in\varepsilon_\kappa}\int_e [\mathbf{u}_h.\mathbf{n}](\mu-\mu_h)\right), \quad \forall\mu\in H_0^1(\Omega),\ \forall\mu_h\in Q_{0h}$$

et le système vérifié seulement par $\mathbf{w}$

$$\nu\int_\Omega \mathbf{curl}\,\mathbf{w}\,\mathbf{curl}\,\mathbf{v} = \sum_{\kappa\in\tau_h}\left(\int_\kappa (\mathbf{f}-\mathbf{f}_h)(\mathbf{v}-\mathbf{v}_h) + \int_\kappa (\mathbf{f}_h - \nabla p_h)(\mathbf{v}-\mathbf{v}_h) - \frac{\nu}{2}\sum_{e\in\varepsilon_\kappa}\int_e \big([\mathbf{curl}\,\mathbf{u}_h\times\mathbf{n}]\big)(\mathbf{v}-\mathbf{v}_h)\right),$$
$$\forall\mathbf{v}\in V_0,\ \forall\mathbf{v}_h\in V_{0h}.$$

## 1. Introduction, description and analysis of the model

Let $\Omega$ be a bounded convex domain of $\mathbb{R}^3$ with a polyhedral boundary $\partial\Omega = \Gamma$. Consider the following Stokes equations for an incompressible fluid, with the velocity $\mathbf{u}$ and the pressure $p$:

$$-\nu\Delta\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega, \qquad \operatorname{div}\mathbf{u} = 0 \quad \text{in } \Omega, \tag{1}$$

and the boundary conditions:

$$\mathbf{u}\times\mathbf{n} = \mathbf{0}, \qquad p = 0 \quad \text{on } \partial\Omega, \tag{2}$$

or

$$\mathbf{u}.\mathbf{n} = 0, \qquad \mathbf{curl}\,\mathbf{u}\times\mathbf{n} = \mathbf{0} \quad \text{on } \partial\Omega. \tag{3}$$

We denote by (*Prob*1) the system of Eqs. (1) and (2), and by (*Prob*2) the system of Eqs. (1) and (3). We suppose that $\mathbf{f}\in L^2(\Omega)^3$ and we denote by $C$ a generic positive constant. We introduce the spaces:

$$H(\operatorname{div},\Omega) = \left\{\mathbf{v}\in L^2(\Omega)^3,\ \operatorname{div}\mathbf{v}\in L^2(\Omega)\right\}; \qquad H_0(\operatorname{div},\Omega) = \left\{\mathbf{v}\in H(\operatorname{div},\Omega), \mathbf{v}\cdot\mathbf{n} = 0 \text{ on } \Gamma\right\};$$
$$H(\mathbf{curl},\Omega) = \left\{\mathbf{v}\in L^2(\Omega)^3,\ \mathbf{curl}\,\mathbf{v}\in L^2(\Omega)^3\right\}; \qquad H_0(\mathbf{curl},\Omega) = \left\{\mathbf{v}\in H(\mathbf{curl},\Omega),\ \mathbf{v}\times\mathbf{n} = \mathbf{0} \text{ on } \Gamma\right\};$$

normed respectively by

$$\|\mathbf{v}\|_{H(\operatorname{div},\Omega)} = \left\{\|\mathbf{v}\|_{0,\Omega}^2 + \|\operatorname{div}\mathbf{v}\|_{0,\Omega}^2\right\}^{1/2} \quad \text{and} \quad \|\mathbf{v}\|_{H(\mathbf{curl},\Omega)} = \left\{\|\mathbf{v}\|_{0,\Omega}^2 + \|\mathbf{curl}\,\mathbf{v}\|_{0,\Omega}^2\right\}^{1/2}.$$

**Theorem 1.** *(Prob1) and (Prob2) have respectively the following weak variational formulations*:

*Find* $\mathbf{u}\in H_0(\mathbf{curl},\Omega)$ *and* $p\in H_0^1(\Omega)$ *such that*

$$\nu(\mathbf{curl}\,\mathbf{u},\mathbf{curl}\,\mathbf{v}) + (\nabla p,\mathbf{v}) = (\mathbf{f},\mathbf{v}), \quad \forall\mathbf{v}\in H_0(\mathbf{curl},\Omega), \tag{4}$$
$$(\nabla q,\mathbf{u}) = 0, \quad \forall q\in H_0^1(\Omega), \tag{5}$$

*and*

*Find* $\mathbf{u}\in H(\mathbf{curl},\Omega)$ *and* $p\in H^1(\Omega)/\mathbb{R}$ *such that*

$$\nu(\mathbf{curl}\,\mathbf{u},\mathbf{curl}\,\mathbf{v}) + (\nabla p,\mathbf{v}) = (\mathbf{f},\mathbf{v}), \quad \forall\mathbf{v}\in H(\mathbf{curl},\Omega), \tag{6}$$
$$(\nabla q,\mathbf{u}) = 0, \quad \forall q\in H^1(\Omega). \tag{7}$$

Let us introduce the spaces:

$$V_0 = \left\{\mathbf{v}\in H_0(\mathbf{curl},\Omega);\ (\nabla q,\mathbf{v}) = 0,\ \forall q\in H_0^1(\Omega)\right\}$$

and

$$U = \left\{\mathbf{v}\in H(\mathbf{curl},\Omega);\ (\nabla q,\mathbf{v}) = 0,\ \forall q\in H^1(\Omega)\right\}.$$

For all $\mathbf{v}\in V_0$ or $U$, we have $\|\mathbf{curl}\,\mathbf{v}\|_{L^2(\Omega)^3} \leqslant C\|\mathbf{v}\|_{H(\mathbf{curl},\Omega)}$.

Each variational formulation is split into a system for the velocity and a Poisson equation for the pressure.

**Theorem 2.** *The problem* (4)–(5) *is equivalent to the problem*:

$$\textit{Find } \mathbf{u} \in V_0 \textit{ such that} \quad \nu(\mathbf{curl}\,\mathbf{u}, \mathbf{curl}\,\mathbf{v}) = (\mathbf{f}, \mathbf{v}), \quad \forall \mathbf{v} \in V_0. \tag{8}$$

$$\textit{Find } p \in H_0^1(\Omega) \textit{ such that} \quad (\nabla p, \nabla q) = (\mathbf{f}, \nabla q), \quad \forall q \in H_0^1(\Omega). \tag{9}$$

*The problem* (6)–(7) *is equivalent to the problem*:

$$\textit{Find } \mathbf{u} \in U \textit{ such that} \quad \nu(\mathbf{curl}\,\mathbf{u}, \mathbf{curl}\,\mathbf{v}) = (\mathbf{f}, \mathbf{v}), \quad \forall \mathbf{v} \in U. \tag{10}$$

$$\textit{Find } p \in H^1(\Omega)/\mathbb{R} \textit{ such that} \quad (\nabla p, \nabla q) = (\mathbf{f}, \nabla q), \quad \forall q \in H^1(\Omega). \tag{11}$$

*In both cases, there exists a unique solution and we have the following bounds*:

$$|p|_{1,\Omega} \leqslant \|\mathbf{f}\|_{0,\Omega}; \quad \|\mathbf{curl}\,\mathbf{u}\|_{0,\Omega} \leqslant \frac{C_1}{\nu}\|\mathbf{f}\|_{0,\Omega} \quad \textit{and} \quad \|\mathbf{u}\|_{1,\Omega} \leqslant \frac{C_2}{\nu}\|\mathbf{f}\|_{0,\Omega}.$$

For the details of the previous two theorems, we can refer to V. Girault [4], p. 206.

## 2. Finite element discretization

Let $h > 0$ be a discretization parameter and for each $h$, let $\tau_h$ be a corresponding regular (or non-degenerate) family of triangulations of $\overline{\Omega}$, consisting of tetrahedra such that any two tetrahedra are either disjoint or share a vertex or an entire edge or face. For an arbitrary $\kappa \in \tau_h$, we denote by $\eta_\kappa$ the diameter of $\kappa$ and by $\rho_\kappa$ the diameter of the sphere inscribed in $\kappa$. Then $\eta$ denotes the maximum of $\eta_\kappa$ and we assume that $\tau_\eta$ is regular in the sense of Ciarlet [2]: there exists a constant $\sigma$ independent of $\eta$ such that

$$\sup_{\kappa \in \tau_\eta} \frac{\eta_\kappa}{\rho_\kappa} = \sigma_\kappa \leqslant \sigma. \tag{12}$$

For each $\kappa$ in $\tau_h$, we introduce the spaces $\mathbb{P}_0(\kappa)$ of the restrictions to $\kappa$ of constant functions on $\mathbb{R}^3$, $\mathbb{P}_1(\kappa)$ of the restrictions to $\kappa$ of affine function on $\mathbb{R}$ and the space $\mathbb{P}_K(\kappa)$ of the restrictions to $\kappa$ of polynomials $\mathbf{v}$ of the form:

$$\mathbf{v}(x) = \mathbf{a} + \mathbf{b} \times \mathbf{x}, \quad \mathbf{a} \in \mathbb{R}^3,\ \mathbf{b} \in \mathbb{R}^3.$$

The space $\mathbb{P}_K(\kappa)$ and the corresponding finite elements are studied in [5]. Their degrees of freedom are the average flux along the edges $\int_l (\mathbf{v}.\mathbf{t})\,\mathrm{d}l$, for the six edges $l$ of $\kappa$, $t$ is the direction vector of $l$.

Hence, we associate the operator $r_\kappa$ where $r_\kappa(\mathbf{u})$ is the unique polynomial of $\mathbb{P}_K$ that has the same flux along the edges as $\mathbf{u}$. We define also the operator $I_\kappa$ where $I_\kappa(q)$ is the unique polynomial of $\mathbb{P}_1(\kappa)$ that has the same values on the vertex of $\kappa$ as $q$. Next, let us introduce the discrete spaces:

$$M_h = \{\mathbf{u}_h \in H(\mathbf{curl}, \Omega);\ \mathbf{u}_h|_\kappa \in \mathbb{P}_K(\kappa),\ \forall \kappa \in \tau_h\}, \qquad M_{0h} = M_h \cap H_0(\mathbf{curl}, \Omega), \tag{13}$$

$$Q_h = \{q_h \in C^0(\overline{\Omega});\ q_h|_\kappa \in \mathbb{P}_1(\kappa),\ \forall \kappa \in \tau_h\}, \qquad Q_{0h} = Q_h \cap H_0^1(\Omega). \tag{14}$$

With these spaces, the finite-dimensional analogues of $V_0$ and $U$ are

$$V_{0h} = \{\mathbf{v}_h \in M_{0h};\ (\nabla q_h, \mathbf{v}_h) = 0,\ \forall q_h \in Q_{0h}\}, \qquad U_h = \{\mathbf{v}_h \in M_h;\ (\nabla q_h, \mathbf{v}_h) = 0,\ \forall q_h \in Q_h\}.$$

We define the interpolation operators $r_h$ from $H^1(\Omega)^3$ onto $M_h$, $I_h$ from $H^2(\Omega)$ onto $Q_h$ by

$$r_h u = r_\kappa(u) \quad \text{on } \kappa,\ \forall \kappa \in \tau_h \text{ (similarly for } I_h).$$

We discretize (*Prob*1) by: Find $\mathbf{u}_h \in V_{0h}$ and $p_h \in Q_{0h}$ such that

$$\nu(\mathbf{curl}\,\mathbf{u}_h, \mathbf{curl}\,\mathbf{v}_h) + (\nabla p_h, \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h), \quad \forall \mathbf{v}_h \in M_{0h}. \tag{15}$$

Similarly, we discretize (*Prob*2) by: Find $\mathbf{u}_h \in U_h$ and $p_h \in Q_h/\mathbb{R}$ such that

$$\nu(\mathbf{cur}\,\mathbf{u}_h, \mathbf{curl}\,\mathbf{v}_h) + (\nabla p_h, \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h), \quad \forall \mathbf{v}_h \in M_h. \tag{16}$$

As in the continuous way, the discrete form of (*Prob*1) and (*Prob*2) can be respectively split into

$$\text{Find } \mathbf{u}_h \in V_{0h} \text{ such that} \quad \nu(\mathbf{curl}\,\mathbf{u}_h, \mathbf{curl}\,\mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h), \quad \forall \mathbf{v}_h \in V_{0h}, \tag{17}$$

$$\text{Find } p_h \in Q_{0h} \text{ such that} \quad (\nabla p_h, \nabla q_h) = (\mathbf{f}, \nabla q_h), \quad \forall q_h \in Q_{0h}. \tag{18}$$

And

$$\text{Find } \mathbf{u}_h \in U_h \text{ such that } \quad \nu(\mathbf{curl}\,\mathbf{u}_h, \mathbf{curl}\,\mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h), \quad \forall \mathbf{v}_h \in U_h, \tag{19}$$

$$\text{Find } p_h \in Q_h/\mathbb{R} \text{ such that } \quad (\nabla p_h, \nabla q_h) = (\mathbf{f}, \nabla q_h), \quad \forall q_h \in Q_h. \tag{20}$$

These two last discrete problems have unique solutions (see Girault [4], p. 216). The pressure is entirely dissociated from the velocity, i.e. can be computed without knowing the velocity. We have also for both discrete problems the following bounds:

$$\|\mathbf{curl}\,\mathbf{u}_h\|_{0,\Omega} \leqslant \frac{C}{\nu}\|\mathbf{f}\|_{0,\Omega} \quad \text{and} \quad |p_h|_{1,\Omega} \leqslant \|\mathbf{f}\|_{0,\Omega}.$$

For the *a priori* error estimates, we can refer to [4]. If the solution of the problem (8)–(9) (resp. (10)–(11)) is sufficiently smooth we have

$$|p - p_h|_{1,\Omega} \leqslant Ch|p|_{2,\Omega} \quad \text{and} \quad \|\mathbf{curl}(\mathbf{u} - \mathbf{u}_h)\|_{0,\Omega} \leqslant Ch\big(|p|_{2,\Omega} + |\mathbf{u}|_{2,\Omega}\big). \tag{21}$$

## 3. *A posteriori* error analysis

We now intend to prove *a posteriori* error estimates between the exact solution $(\mathbf{u}, p)$ of the problem (8)–(9) and the numerical solution $(\mathbf{u}_h, p_h)$ of the problem (17)–(18). By the same way, we can prove *a posteriori* error estimates between the solution $(\mathbf{u}, p)$ of the exact problem (10)–(11) and $(\mathbf{u}_h, p_h)$ of the numerical problem (19)–(20). In all the rest of the paper, we suppose that $\mathbf{f} \in H(\operatorname{div}, \Omega)$.

We introduce for an element $\kappa$ of $\tau_h$, the bubble function $\psi_\kappa$ (resp. $\psi_e$ of the face $e$) which is equal to the product of the $d+1$ barycentric coordinates associated with the vertices of $\kappa$ (resp. of $e$) and $\mathcal{L}_e$ the lifting operator from polynomials defined on $e$ into polynomials defined on the elements $\kappa$ and $\kappa'$ containing $e$, which is constructed by affine transformations from a fixed operator on the reference element.

We first introduce the space $Z_h = \{\mathbf{g}_h \in L^2(\Omega)^3;\ \forall \kappa \in \tau_h,\ \mathbf{g}_h|_\kappa \in \mathbb{P}_0(\kappa)\}$, and we fix an approximation $\mathbf{f}_h$ of the data $\mathbf{f}$ in $Z_h$. Let us begin with *a posteriori* error for the pressure. We define the error indicator by $\eta_\kappa = \sum_{e\in\varepsilon_\kappa} h_e^{1/2}\|[(\mathbf{f}_h - \nabla p_h).\mathbf{n}]\|_{L^2(e)}$.

**Proposition 3.** *The following a posteriori estimate holds between the solution p of* (9) *and the solution $p_h$ of* (18):

$$|p - p_h|_{1,\Omega} \leqslant C\left\{\sum_{\kappa\in\tau_h}\left(\eta_\kappa^2 + h_\kappa^2\|\operatorname{div}\mathbf{f}\|_{L^2(\kappa)}^2 + \left(\sum_{e\in\varepsilon_\kappa} h_e^{1/2}\big\|\big[(\mathbf{f} - \mathbf{f}_h).\mathbf{n}\big]\big\|_{L^2(e)}\right)^2\right)\right\}^{1/2}.$$

*The error indicators verify the following optimality conditions*:

$$\eta_\kappa \leqslant C\left(|p - p_h|_{H^1(\Delta_\kappa)} + h_e\|\operatorname{div}\mathbf{f}\|_{L^2(\Delta_\kappa)} + \sum_{e\in\varepsilon_\kappa} h_e^{1/2}\big\|\big[(\mathbf{f} - \mathbf{f}_h).\mathbf{n}\big]\big\|_{L^2(e)}\right). \tag{22}$$

**Proof.** Using Eqs. (9) and (18), the error function $p - p_h$ belongs to $H_0^1(\Omega)$ and satisfies:

$$\big(\nabla(p - p_h), \nabla q\big) = \sum_{\kappa\in\tau_h}\left(\int_\kappa (\mathbf{f} - \mathbf{f}_h)\nabla(q - q_h) + \int_\kappa (\mathbf{f}_h - \nabla p_h)\nabla(q - q_h)\right). \tag{23}$$

By integrating by part and taking $q_h = R_h q$ (the image of $q$ by the Clément type regularization operator [3]), we obtain the upper bound. For the lower bounds, we integrate by part Eq. (23), and take $q_h = 0$ and $q = \mathcal{L}_e([(\mathbf{f}_h - \nabla p_h).\mathbf{n}]\psi_e)$. □

Now, let us establish *a posteriori* error for the velocity. The error function $\mathbf{u} - \mathbf{u}_h$ belongs to $H_0(\mathbf{curl}, \Omega)$, there exists a function $\lambda \in H_0^1(\Omega)$ solution of the problem:

$$\forall \mu \in H_0^1(\Omega), \quad \int_\Omega \nabla\lambda\nabla\mu = \int_\Omega (\mathbf{u} - \mathbf{u}_h)\nabla\mu = -\int_\Omega \mathbf{u}_h\nabla\mu. \tag{24}$$

Then the function $\mathbf{w} = (\mathbf{u} - \mathbf{u}_h) - \nabla\lambda$ belongs to $V_0$ and we have $\mathbf{curl}\,\mathbf{w} = \mathbf{curl}(\mathbf{u} - \mathbf{u}_h)$. We obtain

$$\|\mathbf{u} - \mathbf{u}_h\|_{H(\mathbf{curl},\Omega)}^2 = \|\nabla\lambda\|_{0,\Omega}^2 + \|\mathbf{w}\|_{H(\mathbf{curl},\Omega)}^2. \tag{25}$$

In order to find the upper and lower bounds of $\|\mathbf{u} - \mathbf{u}_h\|_{H(\mathbf{curl},\Omega)}^2$, we start by finding those of the terms in the left-hand side of (25).

We introduce the indicator $\xi_\kappa = \sum_{e\in\varepsilon_\kappa} h_e^{1/2}\|[\mathbf{u}_h.\mathbf{n}]\|_{0,e}$.

**Theorem 4.** *The following bounds hold*:

$$|\lambda|_{1,\Omega} \leqslant C\left(\sum_{\kappa\in\tau_h} \xi_\kappa^2\right)^{1/2} \quad \textit{and} \quad \xi_\kappa \leqslant C|\lambda|_{1,\Delta_\kappa}. \tag{26}$$

**Proof.** Using Eq. (24), we have for any $\mu_h \in Q_{0h}$,

$$\int_\Omega \nabla\lambda\nabla\mu = -\int_\Omega \mathbf{u}_h \nabla(\mu-\mu_h) = -\frac{1}{2}\sum_{\kappa\in\tau_h}\left(\sum_{e\in\varepsilon_\kappa}\int_e [\mathbf{u}_h.\mathbf{n}](\mu-\mu_h)\right).$$

First we take $\mu=\lambda$ and $\mu_h = R_h\mu$ to obtain the upper bound. Second, we take $\mu = \mathcal{L}_e([\mathbf{u}_h.\mathbf{n}]\psi_e)$ to obtain the lower bounds. □

We introduce the indicator $\gamma_\kappa = h_\kappa\|\mathbf{f}_h - \nabla p_h\|_{0,\kappa} + \frac{\nu}{2}\sum_{e\in\varepsilon_\kappa} h_e^{1/2}\|[\mathbf{curl}\,\mathbf{u}_h \times \mathbf{n}]\|_{0,e}$.

**Theorem 5.** *The following bounds hold*:

$$\|\mathbf{w}\|_{H(\mathbf{curl},\Omega)} \leqslant C\left(\sum_{\kappa\in\tau_h}\left(h_\kappa^2\|\mathbf{f}-\mathbf{f}_h\|_{0,T}^2 + \gamma_\kappa^2\right)\right)^{1/2}, \tag{27}$$

$$\gamma_\kappa \leqslant G\left(\|\mathbf{curl}\,\mathbf{w}\|_{0,\Delta_\kappa} + (h_\kappa + h_e)\left(\|\mathbf{f}-\mathbf{f}_h\|_{0,\Delta_\kappa} + |p-p_h|_{0,\Delta_\kappa}\right)\right). \tag{28}$$

**Proof.** Using Eqs. (8) and (17), we have for all $\mathbf{v}\in V_0$,

$$\nu\int_\Omega \mathbf{curl}(\mathbf{u}-\mathbf{u}_h)\,\mathbf{curl}\,\mathbf{v} = \int_\Omega \mathbf{f}\mathbf{v} - \nu\int_\Omega \mathbf{curl}\,\mathbf{u}_h\,\mathbf{curl}\,\mathbf{v}.$$

We replace $\mathbf{u}-\mathbf{u}_h$ by $\mathbf{w}+\nabla\lambda$, integrate by part and remark that $\int_\Omega \nabla(p-p_h)\mathbf{v} = 0$ to obtain

$$\nu\int_\Omega \mathbf{curl}\,\mathbf{w}\,\mathbf{curl}\,\mathbf{v} = \sum_{\kappa\in\tau_h}\left\{\int_\kappa(\mathbf{f}-\mathbf{f}_h)(\mathbf{v}-\mathbf{v}_h) + \int_\kappa(\mathbf{f}_h-\nabla p_h)(\mathbf{v}-\mathbf{v}_h) - \frac{\nu}{2}\sum_{e\in\varepsilon_\kappa}\int_e\left([\mathbf{curl}\,\mathbf{u}_h\times\mathbf{n}]\right)(\mathbf{v}-\mathbf{v}_h)\right\}. \tag{29}$$

We take $\mathbf{v}_h = \mathcal{R}_h\mathbf{v}$ ($\mathcal{R}_h$ is the Raviart–Thomas operator [6]) and $\mathbf{v}=\mathbf{w}$ to obtain the upper bound. For the lower bound, we take $\mathbf{v}_h = \mathbf{0}$ and $\mathbf{v} = (\mathbf{f}_h - \nabla p_h)\psi_\kappa$ to obtain the first inequality

$$\|\mathbf{f}_h - \nabla p_h\|_{0,\kappa} \leqslant C\left(h_\kappa^{-1}\|\mathbf{curl}\,\mathbf{w}\|_{0,\kappa} + |p-p_h|_{1,\kappa} + \|\mathbf{f}-\mathbf{f}_h\|_{0,\kappa}\right),$$

and $\mathbf{v} = \mathcal{L}_e(([\mathbf{curl}\,\mathbf{u}_h\times\mathbf{n}])\psi_e)$ to obtain the second one

$$\left\|[\mathbf{curl}\,\mathbf{u}_h\times\mathbf{n}]\right\|_{0,e} \leqslant C\left\{h_e^{-1/2}\|\mathbf{curl}\,\mathbf{w}\|_{0,\kappa\cup\kappa'} + h_e^{1/2}\left(|p-p_h|_{1,\kappa\cup\kappa'} + \|\mathbf{f}-\mathbf{f}_h\|_{0,\kappa\cup\kappa'} + \|\mathbf{f}_h-\nabla p_h\|_{0,\kappa\cup\kappa'}\right)\right\}.$$

Using the definition of $\gamma_\kappa$ we obtain the relation (28). □

**Corollary 6.** *The optimal a posteriori estimate holds*:

$$\|\mathbf{u}-\mathbf{u}_h\|_{H_0(\mathbf{curl},\Omega)} + |p-p_h|_{1,\Omega} \leqslant \left\{\sum_{\kappa\in\tau_h}\left(\gamma_\kappa^2 + \xi_\kappa^2 + \eta_\kappa^2 + h_\kappa^2\left(\|\mathbf{f}-\mathbf{f}_h\|_{0,\kappa}^2\right) + \|\operatorname{div}\mathbf{f}\|_{L^2(\kappa)}^2\right) + \left(\sum_{e\in\varepsilon_\kappa} h_e^{1/2}\left\|\left[(\mathbf{f}-\mathbf{f}_h).\mathbf{n}\right]\right\|_{L^2(e)}\right)^2\right\}^{1/2}, \tag{30}$$

*where $\gamma_\kappa$, $\xi_\kappa$ and $\eta_\kappa$ are given by the formulas* (22), (26) *and* (28).

## 4. Conclusion

We observe that the estimate (30) is optimal: up to the terms involving the data, the full error is bounded by a constant times the sum of all indicators. Estimates (22), (26) and (28) are local, i.e., only involve the error in a neighborhood of $\kappa$ or $e$. The indicators $\eta_\kappa$, $\xi_\kappa$ and $\gamma_\kappa$ can be viewed as a measure for the error of the space discretization and can be used to adapt the mesh-size in space. For all details of the proofs, we refer to [1].

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