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Differential Geometry

Foliated vector bundles and Riemannian foliations

Fibrés vectoriels feuilletés et feuilletages riemanniens

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ABSTRACT

In this Note we prove the equivalence between the Riemannian foliation and each of the following conditions: 1) the lifted foliation $\mathcal{F}^r$ on the bundle of r-transverse jets is Riemannian for $r \geqslant 1$; 2) the foliation $\mathcal{F}_0^r$ on the slashed $\mathcal{J}_0^r$ is Riemannian and vertically exact for $r \geqslant 1$; 3) there exists a positively admissible transverse Lagrangian on $\mathcal{J}_0^r E$, the r-transverse slashed jet bundle of a foliated bundle $E \to M$, for $r \geqslant 1$.

RÉSUMÉ

Dans cette Note on établie l'équivanence entre la propriété pour un feuilletage d'être riemannien et chacune des conditions suivantes : 1) le feuilletage rélevé $\mathcal{F}^r$ sur l'espace des jets r-transverses est riemannien pour une certaine valeur de $r \geqslant 1$; 2) le feuilletage rélevé $\mathcal{F}_0^r$ sur l'espace réduit des jets r-transverses est riemannien et verticalement exact pour une certaine valeur de $r \geqslant 1$; 3) il existe un lagrangien positif, admissible et transvers sur $\mathcal{J}_0^r E$, le fibré réduit des jets r-transverses d'un fibré vectoriel $E \to M$, pour une certaine valeur $r \geqslant 1$.

Version française abrégée

Soit $\mathcal{F}$ un feuilletage de dimension k sur une variété M. Un fibré $p : E \to M$ est *feuilleté* s'il existe un atlas fibré tel que les fonctions structurales soient basiques. Il existe aussi un feuilletage $\mathcal{F}_E$ sur E qui a la même dimension k, tel que la restriction de p à chaque feuille F_E de $\mathcal{F}_E$ soit un difféomorphisme local sur une feuille F de $\mathcal{F}$. Dans la Note on utilise surtout des fibrés feuilletés affines ou vectoriels.

Dans [8, Définition 1.1] on écrit qu'un feuilletage $\mathcal{F}$ est de *type fini* s'il existe $r \geqslant 1$ tel que $\mathcal{F}^r$ est transversalement parallélisable. De plus, si toutes les feuilles de $\mathcal{F}^r$ sont relativement compactes alors on dit que $\mathcal{F}$ est dit de *type fini compact*. Aussi dans [8, Théorème 1.2] on démontre qu'*un feuilletage de type fini compact est riemannien*. Comme un feuilletage transversalement parallélisable est riemannien, le résultat de Tarquini est amélioré par le résultat suivant :

Théorème 0.1. *Un feuilletage $\mathcal{F}^r$ est riemannien pour un certain $r \geqslant 1$, si et seulement si $\mathcal{F}$ est un feuilletage riemannien.*

Pour le feuilletage induit $\mathcal{F}_0^r$ sur le fibré vectoriel réduit $\mathcal{J}_*^r = \mathcal{J}^r \backslash \{\bar{0}\}$, ce théorème ne peut donner aucune réponse à la question suivante : *Si $\mathcal{F}_0^r$ est riemannien pour une certaine valeur $r \geqslant 1$, le feuilletage F est-il riemannien ?*

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Soit $p: E \to M$ un fibré vectoriel feuilleté. Un *lagrangien positif, admissible* sur E est une application continue $L: E \to \mathbb{R}$ dont la réstriction au fibré réduit $E_* = E\backslash\{\bar{0}\} \to M$ (où $\{\bar{0}\}$ est l'image de la section nulle) est différentiable et vérifie les conditions suivantes : 1) L est défini positif (c'est-à-dire que la forme hessienne verticale est définie positive), $L(x, y) \geqslant 0 = L(x, 0)$, $(\forall)x \in M$, $y \in E_x = p^{-1}(x)$; 2) L est localement projetable sur un lagrangien transverse $\bar{L}$; 3) il existe une fonction basique $\varphi: M \to (0, \infty)$, telle que $(\forall)$ $x \in M$, il existe au moins un $y \in E_x$ tel que $L(x, y) = \varphi(x)$. Un *finslérien* est un lagrangien homogène de degré 2 ; s'il est toujours positif, alors il est admissible. Le fibré vertical $VTE = \ker p_* \to E$ peut être considéré comme un sous-fibré vectoriel $\nu F_E \to E$ par la projection canonique $TE \to \nu F_E$, car VTE est transverse à τF_E. On dit qu'une métrique riemannienne invariante G' sur νF_E est *verticalement exacte* si sa restriction G aux sections verticales transverses est la forme hessienne verticale d'un lagrangien positif et admissible $L: E \to \mathbb{R}$; dans ces conditions, on dit aussi que le feuilletage $\mathcal{F}_E$ est *verticalement exact*. A noter que pour un fibré affine $p: E \to M$, la hessienne verticale d'un lagrangien $L: E \to \mathbb{R}$ est une forme bilinéaire sur les fibres du fibré vertical $VTE \to E$, définie par les dérivées partielles de second ordre de L, en utilisant des coordonnées sur fibres (voir [6], pour les détails).

Théorème 0.2. *Soit $\mathcal{F}$ un feuilletage sur la variété M et soit $\mathcal{F}_0^r$ le feuilletage relevé sur le fibré réduit $\mathcal{J}_0^r$ des jets d'ordre r-transverses du fibré normal $\nu\mathcal{F}$. Alors $\mathcal{F}_0^r$ est riemannien et verticalement exact pour une certaine valeur de $r \geqslant 1$, si et seulement si, $\mathcal{F}$ est riemannien.*

Pour établier la condition suffisante, on utilise le résultat suivant :

Proposition 0.1. *Une métrique invariante g sur νF donne canoniquement une métrique invariante sur νF^r verticalement exacte, pour une certaine valeur de $r \geqslant 1$.*

En particulier, une métrique invariante g sur νF engendre un lagrangien canonique sur $\mathcal{J}^r$, provenant de la partie vérticale de la métrique verticalement exacte sur νF^r. On peut se demander si la réciproque est aussi vraie : *l'existence d'un lagrangien sur J^r assure-t-elle que F est riemannen ?*

Théorème 0.3. *Soit $p: E \to M$ un fibré vectoriel feuilleté sur la variété feuilleté $(M, \mathcal{F})$. Alors il existe un lagrangien transverse, positif admissible sur $\mathcal{J}^r E$, pour une certaine valeur de $r \geqslant 1$, si et seulement si le feuilletage $\mathcal{F}$ est riemannien.*

L'outil de base dans la démonstration de la nécessité des Théorèmes 0.2 et 0.3 est intéressant en lui-même. On a la :

Proposition 0.2. *Soient $p_1: E_1 \to M$ et $p_2: E_2 \to M$ deux fibrés vectoriels feuilletés sur la variétée feuilétée $(M, \mathcal{F})$ et soit $q_2: E_{2*} \to M$ le fibré réduit. S'il existe un lagrangien transverse positif admissible $L: E_2 \to \mathbb{R}$ et une métrique b sur le fibré induit $q_2^* E_1 \to E_{2*}$, qui est transverse relativement à $\mathcal{F}_{E_{2*}}$, alors il existe une métrique sur E_1 feuilletée relativement à $\mathcal{F}$.*

Comme corollaire on peut traiter le cas particulier $E_1 = E_2 = E$, b hessienne d'un lagrangien transverse positif admissible $L: E \to \mathbb{R}$, considéré comme une métrique sur $p^* E_* \to E$, où $p: E \to M$ est un fibré vectoriel feuilleté.

Corollaire 0.1. *Soit $p: E \to M$ un fibré vectoriel feuilleté sur la variétée feuilétée $(M, \mathcal{F})$. S'il existe un lagrangien transverse positif admissible $L: E \to \mathbb{R}$, alors il existe une métrique feuilletée sur E.*

Dans le cas particulier $E = \nu F$ et L, forme quadratique d'une métrique de finslerienne feuilletée, on peut faire en sorte qu'un feuilletage muni d'une métrique finslerienne transverse soit un feuilletage riemannien (le problème est proposé dans [4] comme un cas particulier d'un problème proposé par E. Ghys dans l'Annexe E du livre [5] ; voir [4,3,6]). Un autre cas intéressant est $E = \nu^* F$, spécialement en ce qui concerne la dualité lagrangien–hamiltonien.

Enfin, il est assez naturel de se poser la question : *Dans l'hypothèse du Théorème* 0.2 *peut-on éliminer la condition sur le feuilletage $\mathcal{F}_0^r$ d'être verticalement exact ?*

1. Introduction

Let M be an n-dimensional manifold and $\mathcal{F}$ be a k-dimensional foliation on M. We denote the tangent plane field by τF and the normal bundle $\tau M/\tau F$ by νF. A bundle is called *foliated* if there is an atlas of local trivializations on E such that all the components of the structural functions are basic ones. In this case a canonical foliation $\mathcal{F}_E$ on E is induced, having the same dimension k, such that p restricted to leaves is a local diffeomorphism. In particular, we consider affine and vector bundles that are foliated. Given a foliated vector bundle, its tensor bundles are foliated vector bundles. For example, we can consider the transverse vector bundle of bilinear forms on the fibers of E. If $p: E \to M$ is a foliated bundle, then $\mathcal{J}^1 E \to M$ is a foliated bundle of 1-jets of foliated sections of E; a canonical foliation $\mathcal{F}_E^1$ on $\mathcal{J}^1 E$ can be considered. The elements of $\mathcal{J}^1 E$ are equivalence classes $[s]$ of *foliated* local sections s of E, where the equivalence relation is coincidence up to order one. The natural projection $\pi_0^1: \mathcal{J}^1 E \to E$ is that of an affine bundle over E with vector space $\mathrm{Hom}(\nu F, E)$. Indeed, if (m, e)

is an element of E, the fiber $(\pi_0^1)^{-1}(m,e)$ can be seen as the affine space of (k-dimensional) subspaces H of $T_{(m,e)}E$ such that $H \cap \ker p_* = \{0\}$ and $p_* H \cap \tau F = \{0\}$. So, there is a free transitive action of $\mathrm{Hom}(\nu_m F, E_m)$ on the fiber $(\pi_0^1)^{-1}(m,e)$. In particular, the tangent space to such a fiber is canonically isomorphic to $\mathrm{Hom}(\nu_m F, E_m)$. Analogously one can consider equivalence classes $\mathcal{J}^r E$ of *foliated* sections of E, where the equivalence relation is coincidence up to an order $r \geqslant 1$; it carries a foliation $\mathcal{F}_E^r$. For $r \geqslant 1$, the canonical projection $\pi_{r-1}^r : \mathcal{J}^r E \to \mathcal{J}^{r-1} E$ is also an affine bundle, with the director vector bundle $\mathrm{Hom}((\nu F)^r, E))$. For $r = 0$ one obtain a bundle $\pi_{-1}^r : \mathcal{J}^r E \to M$. If $p : E \to M$ is a foliated *vector* bundle, then $\pi_{-1}^r : \mathcal{J}^r E \to M$ is also a foliated vector bundle and a natural vector subbundle of $\mathcal{J}^1 \mathcal{J}^{r-1} E \to M$, the first jet bundle of $\pi_{-1}^{r-1} : \mathcal{J}^{r-1} E \to M$. Details can be found, for example, in [2]. The foliated translation is similarly to the setting used in [8], where the foliated vector bundle $\pi : \nu F \to M$ is considered. In this case, for sake of simplicity, we denote below $\mathcal{J}^r \nu \mathcal{F}$ by $\mathcal{J}^r$ and the lifted foliation on $\mathcal{J}^r$ by $\mathcal{F}^r$. According to [8, Definition 1.1], a foliation $\mathcal{F}$ is called of *finite type* if there exists $r \geqslant 1$ such that $\mathcal{F}^r$ is transversely parallelizable. If moreover all the leaves of $\mathcal{F}^r$ are relatively compact, then $\mathcal{F}$ is called a *compact finite type foliation*. In [8, Theorem 1.2.] it is proved that *any compact finite type foliation is Riemannian*. Since a transversely parallelizable foliation is a Riemannian one, the following result improves the result of Tarquini:

Theorem 1.1. *The lifted foliation $\mathcal{F}^r$ is Riemannian for some $r \geqslant 1$ iff $\mathcal{F}$ is Riemannian.*

Considering the induced foliation $\mathcal{F}_0^r$ on the slashed vector bundle $\mathcal{J}_*^r = \mathcal{J}^r \backslash \{\bar{0}\}$, then Theorem 1.1 can not give any answer to the following question: *when is $\mathcal{F}$ Riemannian if $\mathcal{F}_0^r$ is Riemannian for some $r \geqslant 1$*?

A *positively admissible Lagrangian* on a foliated vector bundle $p : E \to M$ is a continuous map $L : E \to \mathbb{R}$ that is asked to be differentiable at least when it is restricted to the total space of the slashed bundle $E_* = E \backslash \{\bar{0}\} \to M$, where $\{\bar{0}\}$ is the image of the null section, such that the following conditions hold: 1) L is positively defined (i.e. its vertical Hessian is positively defined) and $L(x,y) \geqslant 0 = L(x,0)$, $(\forall) x \in M$ and $y \in E_x = p^{-1}(x)$; 2) L is locally projectable on a transverse Lagrangian $\bar{L}$; 3) there is a basic function $\varphi : M \to (0,\infty)$, such that for every $x \in M$ there is $y \in E_x$ such that $L(x,y) = \varphi(x)$. If a positively transverse Lagrangian F is 2–homogeneous (i.e. $F(x,\lambda y) = \lambda^2 F(x,y)$, $(\forall)\lambda > 0$), then F is called a *Finslerian*; it is also a positively admissible Lagrangian, taking $\varphi \equiv 1$, or any positive constant. We can see the vertical bundle $VTE = \ker p_* \to E$ as a vector subbundle of $\nu F_E \to E$ by mean of the canonical projection $TE \to \nu F_E$, since VTE is transverse to τF_E. We say that an invariant Riemannian metric G' on νF_E is *vertically exact* if its restriction to the vertical foliated sections is the transverse vertical Hessian of a positively admissible Lagrangian $L : E \to \mathbb{R}$; in this case, we say that the foliation $\mathcal{F}_E$ is *vertically exact*. Notice that if $p : E \to M$ is an affine bundle, then the vertical Hessian $\mathrm{Hess}\, L$ of a Lagrangian $L : E \to \mathbb{R}$ is a symmetric bilinear form on the fibers of the vertical bundle VTE, given by the second order derivatives of L, using the fiber coordinates (see [6,7] for more details using coordinates).

Theorem 1.2. *Let $\mathcal{F}$ be a foliation on a manifold M and $\mathcal{F}_0^r$ be the lifted foliation on the slashed bundle of r-jets of sections of the normal bundle $\nu\mathcal{F}$. Then $\mathcal{F}_0^r$ is Riemannian and vertically exact for some $r \geqslant 1$ iff $\mathcal{F}$ is Riemannian.*

In particular, it follows that any invariant metric g on νF gives rise to a canonical Lagrangian on $\mathcal{J}^r$, coming from the vertical part of the vertically exact invariant Riemannian metric on νF^r. So, it is natural to ask for the converse: does the existence of a Lagrangian on $\mathcal{J}^r$ guaranties that $\mathcal{F}$ is Riemannian?

Theorem 1.3. *Let $p : E \to M$ be a foliated vector bundle over a foliated manifold $(M,\mathcal{F})$. There is a positively admissible Lagrangian on $\mathcal{J}^r E$ for some $r \geqslant 1$ iff the foliation $\mathcal{F}$ is Riemannian.*

2. Proof of the main results

Proof of Theorem 1.1. The sufficiency is given below by Proposition 2.1. We prove the necessity. By construction, the tangent plane field to $\mathcal{F}^r$ is sent to τF by $(\pi_{-1}^{r-1})_*$. So, in particular, $(\pi_{-1}^{r-1})_*$ induces a surjective map $f : \nu F^r \to \nu F$. More precisely, for each $m \in M$ and $(m,\lambda) \in \mathcal{J}^r$, f is surjective from $(\nu F^r)_{(m,\lambda)}$ to $(\nu F)_m$. We know by assumption there exists a (holonomy) invariant metric g on νF^r. Let HF^r denote the g-orthogonal of $\ker f$. Because $\nu F^r = \ker f \oplus HF^r$ and f is surjective, we have, for all (m,λ) as above, $(HF^r)_{(m,\lambda)} \simeq (\nu F)_m$. This can be reformulated as $HF^r \simeq (\pi_{-1}^{r-1})^* \nu F$. Recall that the elements of $(\mathcal{J}^r)_m$ are equivalence classes of foliated sections of $\nu\mathcal{F}$ defined near m. Therefore, for each m one can consider the equivalence class of the zero section of νF. We denote by $s_0 : M \to \mathcal{J}^r$ the corresponding section. We have $\pi_{-1}^{r-1} \circ s_0 = Id_M$ so that $\nu F = (\pi_{-1}^{r-1} \circ s_0)^* \nu F = s_0^*((\pi_{-1}^{r-1})^* \nu F) = s_0^* HF^r$. So the metric g restricted to HF^r gives a holonomy invariant metric on νF. □

Each sufficiency of Theorems 1.1, 1.2 and 1.3 is implied by the following result:

Proposition 2.1. *Any invariant metric g on νF gives a canonical vertically exact invariant Riemannian metric on νF^r, for any $r \geqslant 1$.*

Proof. We proceed by induction over $r \geqslant 1$. If ∇ is the Levi-Civita connection of the invariant metric g on νF and $\bar{g}$ is the induced metric tensor on $\mathrm{End}(\nu F) = \mathrm{Hom}(\nu F, \nu F)$, then we can consider the invariant metric $g^1([s_1],[s_2]) = g(s_1,s_2) + \bar{g}(\nabla s_1, \nabla s_2)$ and the invariant linear connection $D^1_X[s] = [\nabla_X s]$ on the foliated vector bundle $\mathcal{J}^1 \to M$. Using the decomposition $\nu F^1 = V\nu\mathcal{F}^1 \oplus H\nu\mathcal{F}^1$ given by the linear connection D^1 and the isomorphisms $V\nu F \cong p^*\nu F$, $H\nu F \cong p^*\nu F$, we consider the metric $G^1 = p^*g \oplus p^*g$ on νF^1.

Let us assume that a Riemannian metric g^r and a linear connection D^r have been constructed on the fibers of the vector bundle $\mathcal{J}^r \to M$, for $r \geqslant 1$. Let us consider the induced metric tensor $\bar{g}^r$ on $\mathrm{Hom}(\nu F, \mathcal{J}^r)$. The formulas $\tilde{g}^r([s_1],[s_2]) = g^r(s_1,s_2) + \bar{g}^r(\nabla s_1, \nabla s_2)$ and $\bar{D}^1_X[s] = [\nabla_X s]$ define an invariant metric and a linear connection respectively on the vector bundle $J^1\mathcal{J}^r \to M$. Now as in [1], on the vector subbundle $\mathcal{J}^{r+1} \subset J^1\mathcal{J}^r$, we consider the induced metric g^{r+1} and the invariant linear connection $D^r_X[s] = p'(\bar{D}^1_X[s])$, where $p' : J^1\mathcal{J}^r \to \mathcal{J}^{r+1}$ is the orthogonal projection. Using the decomposition $\nu F^{r+1} = V\nu\mathcal{F}^{r+1} \oplus H\nu\mathcal{F}^{r+1}$ given by the linear connection D^{r+1} and the isomorphisms $V\nu F^{r+1} \cong p^*\nu F^{r+1}$, $H\nu F^{r+1} \cong p^*\nu F$, we consider the invariant metric $G^{r+1} = p^*g^{r+1} \oplus p^*g$ on νF^{r+1} that is vertically exact. □

The main technical tool to prove the necessity of each Theorems 1.2 and 1.3 has independent interest, as follows:

Proposition 2.2. *Let $p_1 : E_1 \to M$ and $p_2 : E_2 \to M$ be foliated vector bundles over a foliated manifold $(M,\mathcal{F})$ and $q_2 : E_{2*} \to M$ be the slashed bundle. If there are a positively admissible Lagrangian $L : E_2 \to \mathbb{R}$ and a metric b on the pull back bundle $q_2^*E_1 \to E_{2*}$, foliated with respect to $\mathcal{F}_{E_{2*}}$, then there is a foliated metric on E_1, with respect to $\mathcal{F}$.*

Proof. For each $(m,e_2) \in E_{2*}$ we have a metric (here seen as a quadratic form) $b_{(m,e_2)} : (E_1)_{(m,e_2)} \to \mathbb{R}$. We want a metric $\bar{b}_m : (E_1)_m \to \mathbb{R}$. The idea is to integrate the dependency on e_2, using the fact that metrics form a convex set in the space of quadratic forms. We set:

$$B_m = \left\{ e_2 \in (E_2)_m;\ \frac{1}{2}\varphi(m) \leqslant L(e_2) \leqslant \varphi(m) \right\}.$$

The assumptions on L guaranty that each B_m has finite and non-zero measure with respect to any Lebesgue measure Leb on $(E_2)_m$. Indeed B_m has to be proper because it is convex and vanishes at the origin. So B_m is compact and non-empty because $\varphi(m)$ is in the image of B_m, by assumption. The interior of B_m is non-void because of conditions on L. We now set:

$$\bar{b}_m = \frac{1}{\mathrm{Leb}(B_m)} \int\limits_{B_m} b_{(m,e_2)}\, \mathrm{dLeb}(e_2).$$

Note that there is a unique Lebesgue measure on a real vector space up to multiplicative constant and this indeterminacy is absorbed when we divide by $\mathrm{Leb}(B_m)$. □

Before using this proposition to prove Theorems 1.2 and 1.3, we state as a corollary the case when $E_1 = E_2 = E$ and b is the Hessian of a positively admissible Lagrangian on E, seen as a metric on $p^*E_* \to E$ for some foliated bundle $p : E \to M$.

Corollary 2.1. *Let $p : E \to M$ be a foliated vector bundle over a foliated manifold $(M,\mathcal{F})$. If $L : E \to \mathbb{R}$ is a positively admissible Lagrangian, then there is a foliated metric on E.*

Specializing further to the case $E = \nu\mathcal{F}$ and L is a foliated Finsler metric we get back that any foliation having an invariant transverse Finsler structure is Riemannian (the problem is proposed in [4] and is a special case of a problem presented by E. Ghys in Appendix E of P. Molino's book [5]; see [4,3,6]). Another interesting special case is when $E = \nu^*F$, specially concerning the duality Lagrangian–Hamiltonian. Finally, we return to Theorems 1.2 and 1.3.

Proof of Theorems 1.2 and 1.3. The sufficiency for both theorems follows by Proposition 2.1. We prove first the necessity of Theorem 1.3. Thanks to Proposition 2.2 with $E_1 = \nu^*\mathcal{F}$ and $E_2 = \mathcal{J}^r E$, it suffices to construct a metric on $(\pi^r_{-1})^*_0(\nu^*\mathcal{F})$ (again we won't use anything near the zero section of $\mathcal{J}^r E$) which is foliated with respect to $\mathcal{F}^r$. At every $[s] \in \mathcal{J}^r E_{(0)}$ we have $\mathrm{Hess}_{[s]} L$ which is a metric on the vertical part $\ker(\pi^r_0)_*$ of the tangent bundle of $\mathcal{J}^r E$. This vertical part contains $\ker(\pi^r_{r-1})_*$ since $\pi^r_0 = \pi^{r-1}_0 \circ \pi^r_{r-1}$, where $\pi^0_0 = p : E \to M$, $\mathcal{J}^0 = E$. The vector bundle $\ker(\pi^r_{r-1})_*$ is associated with the affine bundle $\pi^r_{r-1} : \mathcal{J}^r \to \mathcal{J}^{r-1}$, thus $\ker(\pi^r_{r-1})_* \simeq (\nu^*\mathcal{F})^r \otimes E$. So it makes sense to set, for any $\lambda \in \nu^*_m\mathcal{F}$, $b_{(m,[s])}(\lambda) = (\mathrm{Hess}_{[s]} L)(\lambda^r \otimes \pi^r_0([s]))$, where b and the vertical Hessian are seen as quadratic forms and $\lambda^r = \lambda \otimes \cdots \otimes \lambda$ (r times). Thus the necessity of Theorem 1.3 follows. Finally, the necessity of Theorem 1.2 follows thanks to Theorem 1.3 using the Lagrangian on $\mathcal{J}^r$ given by the vertical part of the vertically exact invariant Riemannian metric on νF^r. □

Finally, the following question arises: *can we drop in Theorem* 1.2 *the condition that $\mathcal{F}^r_0$ be vertically exact?*

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References

[1] V. Cruceanu, Sur la théorie des sous-fibrés vectoriels, C. R. Acad. Sci. Paris, Ser. I 302 (20) (1986) 705–708.
[2] G. Giachetta, L. Mangiarotti, G. Sardanashvily, New Lagrangian and Hamiltonian. Methods in Field Theory, World Scientific, Singapore, 1997.
[3] M. Józefowicz, R. Wolak, Finsler foliations of compact manifolds are Riemannian, Differential Geometry and its Applications 26 (2) (2008) 224–226.
[4] A. Miernowski, W. Mozgawa, Lift of the Finsler foliation to its normal bundle, Differential Geometry and its Applications 24 (2006) 209–214.
[5] P. Molino, Riemannian Foliations, Progress in Mathematics, vol. 73, Birhäuser, Boston, 1988.
[6] M. Popescu, P. Popescu, Lagrangians adapted to submersions and foliations, Differential Geometry and its Applications 27 (2) (2009) 171–178.
[7] Z. Shen, Differential Geometry of Spray and Finsler Spaces, Kluwer Academic Publishers, 2001.
[8] C. Tarquini, Feuilletages de type fini compact, C. R. Acad. Sci. Paris, Ser. I 339 (2004) 209–214.