C. R. Acad. Sci. Paris, Ser. I 349 (2011) 263–268

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Partial Differential Equations

Infinitely many solutions for a class of nonlinear eigenvalue problem in Orlicz–Sobolev spaces

Infinité de solutions pour une classe de problèmes non linéaires de valeurs propres dans les espaces d'Orlicz–Sobolev

Gabriele Bonanno [a], Giovanni Molica Bisci [b], Vicenţiu Rădulescu [c,d,1]

[a] *Department of Science for Engineering and Architecture (Mathematics Section), Engineering Faculty, University of Messina, 98166 Messina, Italy*
[b] *Department P.A.U., Architecture Faculty, University of Reggio Calabria, 89100 Reggio Calabria, Italy*
[c] *Institute of Mathematics "Simion Stoilow" of the Romanian Academy, 014700 Bucharest, Romania*
[d] *Department of Mathematics, University of Craiova, Street A.I. Cuza No. 13, 200585 Craiova, Romania*

ARTICLE INFO

Article history:
Received 16 January 2011
Accepted 5 February 2011
Available online 24 February 2011

Presented by Philippe G. Ciarlet

ABSTRACT

We study the Neumann problem $-\operatorname{div}(\alpha(|\nabla u|)\nabla u)+\alpha(|u|)u=\lambda f(x,u)$ in Ω, $\partial u/\partial\nu=0$ on $\partial\Omega$, where Ω is a smooth bounded domain in $\mathbb{R}^N$, λ is a positive parameter, f is a continuous function, and α is a real-valued mapping defined on $(0,\infty)$. The main result in this Note establishes that for all λ in a prescribed open interval, this problem has infinitely many solutions that converge to zero in the Orlicz–Sobolev space $W^1L_\Phi(\Omega)$.

RÉSUMÉ

On étudie le problème de Neumann $-\operatorname{div}(\alpha(|\nabla u|)\nabla u)+\alpha(|u|)u=\lambda f(x,u)$ dans Ω, $\partial u/\partial\nu=0$ sur $\partial\Omega$, où Ω est un domaine borné régulier de $\mathbb{R}^N$, λ est un paramètre positif, f est une fonction continue et α est une application définie sur $(0,\infty)$. Le résultat principal de cette Note montre que pour tout λ dans un certain intervalle ouvert, ce problème admet une infinité de solutions qui convergent vers zéro dans l'espace d'Orlicz–Sobolev $W^1L_\Phi(\Omega)$.

Version française abrégée

Soit Ω un ouvert borné régulier de $\mathbb{R}^N$, $N\geqslant 3$. On suppose que $f:\bar{\Omega}\times\mathbb{R}\to\mathbb{R}$ est une fonction continue, λ est un paramètre positif et $\alpha:(0,\infty)\to\mathbb{R}$ est une fonction telle que l'application $\phi:\mathbb{R}\to\mathbb{R}$ définie par $\phi(t)=\alpha(|t|)t$ si $t\neq 0$ et $\phi(0)=0$, est un homéomorphisme impaire et croissant de $\mathbb{R}$.

Le but de cette Note est d'étudier le problème de Neumann

$$\begin{cases} -\operatorname{div}\big(\alpha\big(|\nabla u|\big)\nabla u\big)+\alpha\big(|u|\big)u=\lambda f(x,u) & \text{dans } \Omega,\\ \dfrac{\partial u}{\partial \nu}=0 & \text{sur } \partial\Omega. \end{cases} \tag{1}$$

E-mail addresses: bonanno@unime.it (G. Bonanno), gmolica@unirc.it (G. Molica Bisci), vicentiu.radulescu@math.cnrs.fr (V. Rădulescu).
URL: http://www.inf.ucv.ro/~radulescu (V. Rădulescu).
[1] Address for correspondence: Department of Mathematics, University of Craiova, 200585 Craiova, Romania.

1631-073X/$ – see front matter
doi:10.1016/j.crma.2011.02.009

On définit, pour tout $t \in \mathbb{R}$, $\Phi(t) = \int_0^t \phi(s)\,\mathrm{d}s$. On suppose que les conditions suivantes soient satisfaites :

$$1 < \liminf_{t\to\infty} \frac{t\phi(t)}{\Phi(t)} \leqslant p^0 := \sup_{t>0} \frac{t\phi(t)}{\Phi(t)} < \infty; \tag{Φ_0}$$

$$N < p_0 := \inf_{t>0} \frac{t\phi(t)}{\Phi(t)} < \liminf_{t\to\infty} \frac{\log(\Phi(t))}{\log(t)}. \tag{Φ_1}$$

Soit $F(x,t) := \int_0^t f(x,s)\,\mathrm{d}s$. On définit

$$A := \liminf_{\xi\to 0^+} \frac{\int_\Omega \max_{|t|\leqslant\xi} F(x,t)\,\mathrm{d}x}{\xi^{p^0}}, \qquad B := \limsup_{\xi\to 0^+} \frac{\int_\Omega F(x,\xi)\,\mathrm{d}x}{\xi^{p_0}}.$$

Soit c la meilleure constante correspondant au prolongement compact de l'espace d'Orlicz–Sobolev $W^1L_\Phi(\Omega)$ dans $C^0(\bar{\Omega})$.

Le résultat principal de cette Note est contenu dans la propriété suivante de multiplicité :

Théorème 0.1. *Soit Φ une fonction de Young qui satisfait les hypothèses (Φ_0)–(Φ_1) et soit $\varrho > 0$ tel que*

$$\lim_{t\to 0^+} \frac{\Phi(t)}{t^{p_0}} < \varrho. \tag{Φ_ϱ}$$

De plus, on suppose que

$$\liminf_{\xi\to 0^+} \frac{\int_\Omega \max_{|t|\leqslant\xi} F(x,t)\,\mathrm{d}x}{\xi^{p^0}} < \frac{1}{(2c)^{p^0}\varrho|\Omega|} \limsup_{\xi\to 0^+} \frac{\int_\Omega F(x,\xi)\,\mathrm{d}x}{\xi^{p_0}}. \tag{h_0}$$

Alors, pour chaque λ dans l'intervalle

$$\left]\frac{\varrho|\Omega|}{B}, \frac{1}{(2c)^{p^0}A}\right[,$$

le problème (1) admet une suite de solutions qui converge vers zéro dans l'espace $W^1L_\Phi(\Omega)$.

La preuve du Théorème 0.1 repose de manière cruciale sur un résultat de Bonanno et Molica Bisci (voir [2, Theorem 2.1]), qui étend le principe variationnel de Ricceri [8].

Let Ω be a bounded domain in $\mathbb{R}^N$ $(N \geqslant 3)$ with smooth boundary and let ν denote the outer unit normal to $\partial\Omega$. Assume $f : \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a continuous function, λ is a positive parameter, and $\alpha : (0,\infty) \to \mathbb{R}$ is such that the mapping $\phi : \mathbb{R} \to \mathbb{R}$ defined by

$$\phi(t) = \begin{cases} \alpha(|t|)t, & \text{for } t \neq 0,\\ 0, & \text{for } t = 0, \end{cases}$$

is an odd, strictly increasing homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$.

In this Note we study the Neumann boundary value problem

$$\begin{cases} -\operatorname{div}\big(\alpha(|\nabla u|)\nabla u\big) + \alpha(|u|)u = \lambda f(x,u) & \text{in } \Omega,\\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{2}$$

Set $\Phi(t) = \int_0^t \phi(s)\,\mathrm{d}s$, $\Phi^\star(t) = \int_0^t \phi^{-1}(s)\,\mathrm{d}s$, for all $t \in \mathbb{R}$. We observe that Φ is a Young function, that is, $\Phi(0) = 0$, Φ is convex, and $\lim_{t\to\infty} \Phi(t) = +\infty$. We assume that Φ satisfies the following hypotheses:

$$1 < \liminf_{t\to\infty} \frac{t\phi(t)}{\Phi(t)} \leqslant p^0 := \sup_{t>0} \frac{t\phi(t)}{\Phi(t)} < \infty; \tag{Φ_0}$$

$$N < p_0 := \inf_{t>0} \frac{t\phi(t)}{\Phi(t)} < \liminf_{t\to\infty} \frac{\log(\Phi(t))}{\log(t)}. \tag{Φ_1}$$

The Orlicz space $L_\Phi(\Omega)$ defined by Φ (see [1]) is the space of measurable functions $u : \Omega \to \mathbb{R}$ such that

$$\|u\|_{L_\Phi} := \sup\left\{\int_\Omega u(x)v(x)\,\mathrm{d}x;\ \int_\Omega \Phi^\star\big(|v(x)|\big)\,\mathrm{d}x \leqslant 1\right\} < \infty.$$

Then $(L_\Phi(\Omega), \|\cdot\|_{L_\Phi})$ is a Banach space whose norm is equivalent with the Luxemburg norm

$$\|u\|_\Phi := \inf\left\{k > 0;\ \int_\Omega \Phi\left(\frac{u(x)}{k}\right)\mathrm{d}x \leqslant 1\right\}.$$

We denote by $W^1L_\Phi(\Omega)$ the corresponding Orlicz–Sobolev space, defined by

$$W^1L_\Phi(\Omega) = \left\{u \in L_\Phi(\Omega);\ \frac{\partial u}{\partial x_i} \in L_\Phi(\Omega),\ i = 1, \ldots, N\right\}.$$

Hypothesis (Φ_0) is equivalent with the fact that Φ and $\Phi^\star$ both satisfy the Δ_2-condition (at infinity), see [1, p. 232]. In particular, both (Φ, Ω) and $(\Phi^\star, \Omega)$ are Δ-regular, see [1, p. 232]. Consequently, the spaces $L_\Phi(\Omega)$ and $W^1L_\Phi(\Omega)$ are separable, reflexive Banach spaces, see Adams [1, p. 241 and p. 247]. Let $c > 0$ denote the best constant corresponding to the compact embedding of $W^1L_\Phi(\Omega)$ into $C^0(\bar{\Omega})$.

Set $F(x,t) := \int_0^t f(x,s)\,\mathrm{d}s$. We define

$$A := \liminf_{\xi\to 0^+} \frac{\int_\Omega \max_{|t|\leqslant\xi} F(x,t)\,\mathrm{d}x}{\xi^{p^0}}, \qquad B := \limsup_{\xi\to 0^+} \frac{\int_\Omega F(x,\xi)\,\mathrm{d}x}{\xi^{p_0}}.$$

The main result in this Note is the following multiplicity property:

Theorem 0.1. *Assume Φ is a Young function satisfying the conditions (Φ_0)–(Φ_1) and let ϱ be a positive constant such that*

$$\lim_{t\to 0^+} \frac{\Phi(t)}{t^{p_0}} < \varrho. \tag{Φ_ϱ}$$

Further, assume that

$$\liminf_{\xi\to 0^+} \frac{\int_\Omega \max_{|t|\leqslant\xi} F(x,t)\,\mathrm{d}x}{\xi^{p^0}} < \frac{1}{(2c)^{p^0}\varrho\,|\Omega|} \limsup_{\xi\to 0^+} \frac{\int_\Omega F(x,\xi)\,\mathrm{d}x}{\xi^{p_0}}. \tag{h$_0$}$$

Then, for every λ belonging to

$$\left]\frac{\varrho|\Omega|}{B}, \frac{1}{(2c)^{p^0}A}\right[,$$

problem (2) admits a sequence of pairwise distinct weak solutions which strongly converges to zero in $W^1L_\Phi(\Omega)$.

1. Proof of Theorem 0.1

Set $X := W^1L_\Phi(\Omega)$. We use in the proof the following auxiliary results (see [3,7]):

Lemma 1.1. *The norms*

$$\|u\|_{1,\Phi} = \big\||\nabla u|\big\|_\Phi + \|u\|_\Phi,$$
$$\|u\|_{2,\Phi} = \max\big\{\big\||\nabla u|\big\|_\Phi, \|u\|_\Phi\big\},$$
$$\|u\| = \inf\left\{\mu > 0;\ \int_\Omega \left[\Phi\left(\frac{|u(x)|}{\mu}\right) + \Phi\left(\frac{|\nabla u(x)|}{\mu}\right)\right]\mathrm{d}x \leqslant 1\right\}$$

are equivalent on X. More precisely, for every $u \in X$,

$$\|u\| \leqslant 2\|u\|_{2,\Phi} \leqslant 2\|u\|_{1,\Phi} \leqslant 4\|u\|.$$

Lemma 1.2. *Let $u \in X$. Then*

$$\int_\Omega \big[\Phi\big(|u(x)|\big) + \Phi\big(|\nabla u(x)|\big)\big]\,\mathrm{d}x \geqslant \|u\|^{p_0}, \quad \textit{if } \|u\| > 1;$$

$$\int_\Omega \big[\Phi\big(|u(x)|\big) + \Phi\big(|\nabla u(x)|\big)\big]\,\mathrm{d}x \geqslant \|u\|^{p^0}, \quad \textit{if } \|u\| < 1.$$

Lemma 1.3. *Let $u \in X$ and assume that $\int_\Omega [\Phi(|u(x)|) + \Phi(|\nabla u(x)|)]\,dx \leqslant r$, for some $0 < r < 1$. Then $\|u\| < 1$.*

Define the functionals $J, I : X \to \mathbb{R}$ by

$$J(u) = \int_\Omega \big(\Phi\big(|\nabla u(x)|\big) + \Phi\big(|u(x)|\big)\big)\,dx \quad \text{and} \quad I(u) = \int_\Omega F\big(x, u(x)\big)\,dx,$$

where $F(x,\xi) := \int_0^\xi f(x,t)\,dt$ for every $(x,\xi) \in \bar{\Omega} \times \mathbb{R}$. Set $g_\lambda(u) := J(u) - \lambda I(u)$, for all $u \in X$. Similar arguments as those used in [6, Lemma 3.4] and [4, Lemma 2.1] imply that $J, I \in C^1(X,\mathbb{R})$ and for all $u, v \in X$,

$$\langle J'(u), v\rangle = \int_\Omega \alpha\big(|\nabla u(x)|\big)\nabla u(x)\cdot\nabla v(x)\,dx + \int_\Omega \alpha\big(|u(x)|\big)u(x)v(x)\,dx,$$
$$\langle I'(u), v\rangle = \int_\Omega f\big(x, u(x)\big)v(x)\,dx.$$

Moreover, since Φ is convex, it follows that J is a convex functional, hence J is sequentially weakly lower semi-continuous. Finally, we observe that J is coercive. Indeed, a straightforward computation shows that for any $u \in X$ with $\|u\| > 1$ we have $J(u) \geqslant \|u\|^{p_0}$. On the other hand, since X is compactly embedded into $C^0(\bar{\Omega})$, then the operator $I' : X \to X^\star$ is compact. Consequently, the functional $I : X \to \mathbb{R}$ is sequentially weakly (upper) continuous, see Zeidler [9, Corollary 41.9].

Let $\{c_n\}$ be a sequence of real numbers such that $\lim_{n\to\infty} c_n = 0$ and

$$\lim_{n\to\infty} \frac{\int_\Omega \max_{|t|\leqslant c_n} F(x,t)\,dx}{c_n^{p^0}} = A.$$

Set $r_n = (\frac{c_n}{2c})^{p^0}$ for all $n \in \mathbb{N}$. Thus, by Lemmas 1.2 and 1.3,

$$\big\{v \in X:\ J(v) < r_n\big\} \subseteq \left\{v \in X:\ \|v\| < \frac{c_n}{2c}\right\}.$$

Due to the compact embedding of X into $C(\bar{\Omega})$ combined with Lemma 1.1, we have

$$\big|v(x)\big| \leqslant \|v\|_\infty \leqslant c\|v\|_{1,\Phi} \leqslant 2c\|v\| \leqslant c_n, \quad \forall x \in \bar{\Omega}.$$

Hence

$$\left\{v \in X:\ \|v\| < \frac{c_n}{2c}\right\} \subseteq \big\{v \in X:\ |v| \leqslant c_n\big\}.$$

We also observe that for all $n \in \mathbb{N}$,

$$\varphi(r_n) = \inf_{J(u)<r_n} \frac{\sup_{J(v)<r_n}\int_\Omega F(x,v(x))\,dx - \int_\Omega F(x,u(x))\,dx}{r_n - J(u)} \leqslant \frac{\sup_{J(v)<r_n}\int_\Omega F(x,v(x))\,dx}{r_n}$$
$$\leqslant \frac{\int_\Omega \max_{|t|\leqslant c_n} F(x,t)\,dx}{r_n} = (2c)^{p^0}\frac{\int_\Omega \max_{|t|\leqslant c_n} F(x,t)\,dx}{c_n^{p^0}}.$$

Next, we observe that our assumption (h_0) implies $A < +\infty$. Therefore

$$\delta \leqslant \liminf_{n\to\infty} \varphi(r_n) \leqslant (2c)^{p^0} A < +\infty.$$

Now, take

$$\lambda \in \left]\frac{\varrho|\Omega|}{B}, \frac{1}{(2c)^{p^0}A}\right[.$$

We prove in what follows that 0, which is the unique global minimum of J, is not a local minimum of g_λ. For this purpose, let $\{\zeta_n\}$ be a sequence of positive numbers such that $\lim_{n\to\infty} \zeta_n = 0$ and

$$\lim_{n\to\infty} \frac{\int_\Omega F(x,\zeta_n)\,dx}{\zeta_n^{p_0}} = B. \tag{3}$$

Set $w_n(x) := \zeta_n$, for all $x \in \Omega$. Then $w_n \in X$, for all $n \in \mathbb{N}$. Hence

$$J(w_n)=\int_\Omega \big(\Phi\big(|\nabla w_n(x)|\big)+\Phi\big(|w_n(x)|\big)\big)\,dx=\int_\Omega \Phi(\zeta_n)\,dx=\Phi(\zeta_n)|\Omega|.$$

Moreover, by (Φ_ϱ) and taking into account that $\lim_{n\to\infty} w_n=0$, we deduce that there exist $\delta>0$ and $\nu_0\in\mathbb{N}$ such that $w_n\in\,]0,\delta[$ and $\Phi(w_n)<\varrho w_n^{p_0}$, for every $n\geqslant\nu_0$.

We first assume that $B<+\infty$. Fix $\epsilon\in\,]\frac{\varrho|\Omega|}{\lambda B},1[$. By (3), there exists ν_ϵ such that for all $n>\nu_\epsilon$, $\int_\Omega F(x,\zeta_n)\,dx>\epsilon B\zeta_n^{p_0}$. Thus, for all $n\geqslant\max\{\nu_0,\nu_\epsilon\}$,

$$g_\lambda(w_n)=J(w_n)-\lambda I(w_n)\leqslant \varrho w_n^{p_0}|\Omega|-\lambda\epsilon B w_n^{p_0}=w_n^{p_0}\big(\varrho|\Omega|-\lambda\epsilon B\big)<0.$$

Next, we assume that $B=+\infty$. Fix $M>\frac{\varrho|\Omega|}{\lambda}$. By (3), there exists ν_M such that for all $n>\nu_M$, $\int_\Omega F(x,\zeta_n)\,dx>M\zeta_n^{p_0}$. Moreover, for all $n\geqslant\max\{\nu_0,\nu_M\}$,

$$g_\lambda(w_n)=J(w_n)-\lambda I(w_n)\leqslant \varrho w_n^{p_0}|\Omega|-\lambda M w_n^{p_0}=w_n^{p_0}\big(\varrho|\Omega|-\lambda M\big)<0.$$

It follows that in both cases, $g_\lambda(w_n)<0$ for every n sufficiently large. Since $g_\lambda(0)=J(0)-\lambda I(0)=0$, then 0 is not a local minimum of g_λ. Thus, owing that J has 0 as unique global minimum, Theorem 2.1 in [2] ensures the existence of a sequence $\{v_n\}$ of pairwise distinct critical points of the functional g_λ, such that $\lim_{n\to\infty} J(v_n)=0$. By Lemma 1.2 we have $\|v_n\|^{p^0}\leqslant J(v_n)$ for every n sufficiently large. Then $\lim_{n\to\infty}\|v_n\|=0$ and this completes the proof. □

We illustrate this abstract existence result with the following example. Fix $p>N+1$ and consider the mapping

$$\phi(t)=\frac{|t|^{p-2}}{\log(1+|t|)}t\quad\text{for } t\neq 0,\quad\text{and}\quad \phi(0)=0.$$

By [5, p. 243] we deduce that

$$p_0=p-1<p^0=p=\liminf_{t\to\infty}\frac{\log(\Phi(t))}{\log(t)}.$$

Thus, conditions (Φ_0) and (Φ_1) are verified. Hypothesis (Φ_ϱ) also holds, since

$$\lim_{t\to 0^+}\frac{1}{t^{p-1}}\int_0^t \frac{s|s|^{p-2}}{\log(1+|s|)}\,ds=\frac{1}{p-1}.$$

Let $g:\mathbb{R}\to[0,\infty)$ be a continuous function and set $G(\xi):=\int_0^\xi g(t)\,dt$. Moreover, let $h:\bar\Omega\to\mathbb{R}$ be a continuous and positive function.

Applying Theorem 0.1 we obtain the following result:

Corollary 1.4. *Assume that*

$$\liminf_{\xi\to 0^+}\frac{G(\xi)}{\xi^p}=0\quad\textit{and}\quad\limsup_{\xi\to 0^+}\frac{G(\xi)}{\xi^{p-1}}=+\infty. \tag{h''_0}$$

Then, for all $\lambda>0$, the Neumann problem

$$\begin{cases} -\operatorname{div}\left(\dfrac{|\nabla u|^{p-2}}{\log(1+|\nabla u|)}\nabla u\right)+\dfrac{|u|^{p-2}}{\log(1+|u|)}u=\lambda h(x)g(u) & \text{in } \Omega,\\ \dfrac{\partial u}{\partial\nu}=0 & \text{on } \partial\Omega \end{cases} \tag{4}$$

admits a sequence of pairwise distinct weak solutions which strongly converges to zero in $W^1L_\Phi(\Omega)$.

We refer to Bonanno, Molica Bisci, and Rădulescu [3] for detailed proofs, examples, and related results.

Acknowledgements

V. Rădulescu acknowledges the support through Grant CNCSIS PCCE-8/2010 “Sisteme diferenţiale în analiza neliniară şi aplicaţii”.

References

[1] R.A. Adams, Sobolev Spaces, Academic Press, New York, 1975.
[2] G. Bonanno, G. Molica Bisci, Infinitely many solutions for a boundary value problem with discontinuous nonlinearities, Bound. Value Probl. 2009 (2009) 1–20.
[3] G. Bonanno, G. Molica Bisci, V. Rădulescu, Arbitrarily small weak solutions for a nonlinear eigenvalue problem in Orlicz–Sobolev spaces, Monatsh. Math., doi:10.1007/s00605-010-0280-2, in press.
[4] Ph. Clément, M. García-Huidobro, R. Manásevich, K. Schmitt, Mountain pass type solutions for quasilinear elliptic equations, Calc. Var. 11 (2000) 33–62.
[5] Ph. Clément, B. de Pagter, G. Sweers, F. de Thélin, Existence of solutions to a semilinear elliptic system through Orlicz–Sobolev spaces, Mediterr. J. Math. 1 (2004) 241–267.
[6] M. Garciá-Huidobro, V.K. Le, R. Manásevich, K. Schmitt, On principal eigenvalues for quasilinear elliptic differential operators: an Orlicz–Sobolev space setting, Nonlinear Differential Equations Appl. (NoDEA) 6 (1999) 207–225.
[7] A. Kristály, M. Mihăilescu, V. Rădulescu, Two non-trivial solutions for a non-homogeneous Neumann problem: an Orlicz–Sobolev space setting, Proc. Roy. Soc. Edinburgh Sect. A 139 (2009) 367–379.
[8] B. Ricceri, A general variational principle and some of its applications, J. Comput. Appl. Math. 113 (2000) 401–410.
[9] E. Zeidler, Nonlinear Functional Analysis and Its Applications, vol. III, Springer-Verlag, Berlin, 1985.