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Partial Differential Equations

Non-convex self-dual Lagrangians and variational principles for certain PDE's

Lagrangiens autoconjugués non-convexes et principes variationnels associés à certaines EDP

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A R T I C L E I N F O

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A B S T R A C T

We study the concept and the calculus of *non-convex self-dual* (*Nc-SD*) Lagrangians and their derived vector fields which are associated to many partial differential equations and evolution systems. They yield new variational resolutions for large class of partial differential equations with variety of linear and non-linear boundary conditions including many of the standard ones. This approach seems to offer several useful advantages: It associates to a boundary value problem several potential functions which can often be used with relative ease compared to other methods such as the use of Euler–Lagrange functions. These potential functions are quite flexible, and can be adapted to easily deal with both non-linear and homogeneous boundary value problems.

R É S U M É

Nous étudions le concept des Lagrangiens *autoconjugués non-convexes* ainsi que le calcul qui leur est associé, de même que leurs champs de vecteurs derivés, reliés à nombre d'équations aux dérivées partielles et de systèmes d'évolution. Nous obtenons de nouvelles formulations variationnelles pour une grande classe d'équations aux dérivées partielles et comprenant une large variété de conditions aux limites linéaires et non-linéaires qui englobe la plupart des conditions aux limites usuelles. Cette approche semble offrir certains avantages : Elle associe au problème aux limites donné un certain nombre de fonctions potentiel qui peuvent être maniées avec plus de facilité par rapport à d'autres méthodes telles celles faisant usage des fonctions d'Euler–Lagrange. Ces fonctions potentiel peuvent être aisément adaptées afin de traiter des problèmes aux limites non-linéaires et homogènes.

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1. Non-convex self-dual Lagrangians

Let V be a Banach space that is in separating duality with the Banach space V^*. Functions $\Phi : V \times V^* \to \mathbb{R} \cup \{\infty\}$ on the phase space $V \times V^*$ will be called Lagrangians. We shall consider the class of Lagrangians that are convex and lower semicontinuous on the second variable. The Fenchel–Legendre dual of Φ with respect to the second variable will be denoted by $LF_2(\Phi)$ and is a function on $V \times V$ given by

$$LF_2(\Phi)(u, v) = \sup_{p \in V^*} \big\{ \langle p, v\rangle - \Phi(u, p) \big\}.$$

We define the non-convex dual, $\Phi^\#$, of Φ by computing the Fenchel–Legendre dual of $LF_2(\Phi)(., v)$ with respect to the first variable. Therefore $\Phi^\#$ is a Lagrangian on the phase space $V \times V^*$ given by

$$\Phi^\#(v, q) = \sup_{u \in V} \big\{ \langle q, u\rangle - LF_2(\Phi)(u, v) \big\}.$$

Definition 1.1. Suppose Φ is a Lagrangian on phase space $V \times V^*$. Say that the Lagrangian Φ on $V \times V^*$ is non-convex self-dual if the following property hold:

$$\Phi^\#(u, p) = \Phi(u, p) \quad \text{for all } (u, p) \in V \times V^*.$$

Note that, $\Phi^\#$, the non-convex dual of the functional Φ is indeed inspired by the duality theory for the difference of two convex functions introduced by J.F. Toland [8] and I. Singer [7].

We now list some *Nc-SD* Lagrangians on the phase space $V \times V^*$.

Proposition 1.1. *The following statements hold*:

(1) *If $\varphi : V \to \mathbb{R}$ is convex and lower semicontinuous and φ^* its Fenchel–Legendre dual defined on V^*, then the Lagrangians $\Phi_1(u, p) := \varphi^*(p) - \varphi(u)$, $\Phi_2(u, p) := \varphi^*(p) - \langle p, u\rangle$ and $\Phi_3(u, p) := \langle p, u\rangle - \varphi(u)$ defined on $V \times V^*$ are non-convex self-dual.*

(2) *If $\Lambda : V \to V^*$ is symmetric and Φ is any Nc-SD Lagrangian then the Lagrangian $\Psi(u, p) := \Phi(u, \Lambda u + p)$ is also non-convex self-dual.*

To connect this notion to the PDE's, let us consider an inclusion of the form

$$\Lambda u \in \partial\varphi(u), \tag{1}$$

where $\Lambda : Dom(\Lambda) \subset V \to V^*$ is a linear self-adjoint operator and φ is a closed proper convex function. Note that u is a solution of inclusion (1) if and only if the pair $(\Lambda u, u)$ is a solution of one the following inclusions on the phase space $V \times V^*$:

(1) $(-\Lambda u, u) \in \partial\Phi_1(u, \Lambda u)$,
(2) $(-\Lambda u, u) \in \partial\Phi_2(u, \Lambda u)$,
(3) $(-\Lambda u, u) \in \partial\Phi_3(u, \Lambda u)$,

where Φ_1, Φ_2 and Φ_3 are the *non-convex self-dual* Lagrangians in Proposition 1.1 and $\partial\Phi_i$ stands for the subdifferential of saddle functions introduced by Rockafellar [6]. Here is our main result regarding Homogeneous boundary conditions.

Theorem 1.2. *Suppose $\Phi : V \times V^* \to \mathbb{R} \cup \{\infty\}$ is a saddle Nc-SD Lagrangian and $\Lambda : Dom(\Lambda) \subset V \to V^*$ is a symmetric operator that is also onto. Suppose one of the following conditions hold*:

(i) *The operator Λ is non-negative.*
(ii) *For each $p \in V^*$, the function $u \to \Phi(u, p)$ is Gâteaux differentiable and $\nabla_1\Phi(u, p) = -p$.*
(iii) *For each $u \in V$, the function $p \to \Phi(u, p)$ is Gâteaux differentiable and $\nabla_2\Phi(u, p) = u$. Then for every critical point u of $\Phi(u, \Lambda u)$ there exists $v \in V$ with $\Lambda u = \Lambda v$ and $(-\Lambda v, v) \in \partial\Phi(u, \Lambda u)$.*

Here is a useful corollary of Theorem 1.2 that provides a new variational principle for certain PDE's.

Corollary 1.3. *Let $\Lambda : Dom(\Lambda) \subset V \to V^*$ be a non-negative symmetric operator. If Λ is onto and $\varphi : V \to \mathbb{R}$ is convex and lower semicontinuous, then every critical point of $I(u) = \varphi^*(\Lambda u) - \varphi(u)$ is a solution of the equation $\Lambda u \in \partial\varphi(u)$.*

The above result was first established by the author via a direct computation [3]. It was then understood that all variational principles of this type fall under a unified principle as discussed in a series of papers [3–5]. One can indeed use this corollary (see [4]) to provide an existence result for system of super-linear transport equations with a small parameter ϵ,

$$\begin{cases} \epsilon a.\nabla u = \Delta v + |v|^{p-2}v, & x \in \Omega, \\ -\epsilon a.\nabla v = \Delta u + |u|^{q-2}, & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega, \end{cases} \tag{2}$$

by finding critical points of

$$I(u,v) = \frac{1}{p'}\int_\Omega |\epsilon a.\nabla u - \Delta v|^{p'}\,dx + \frac{1}{q'}\int_\Omega |\epsilon a.\nabla v + \Delta u|^{q'}\,dx - \frac{1}{p}\int_\Omega |v|^p\,dx - \frac{1}{q}\int_\Omega |u|^q\,dx$$

on $(W^{2,q'}(\Omega) \cap W_0^{1,p'}(\Omega)) \times (W^{2,p'}(\Omega) \cap W_0^{1,q'}(\Omega))$ where $p' = \frac{p}{p-1}$ and $q' = \frac{q}{q-1}$.

Here is another application of Theorem 1.2.

Corollary 1.4. *Let $\Lambda : Dom(\Lambda) \subset V \to V^*$ be a symmetric operator and $\varphi : V \to \mathbb{R}$ be convex, lower semicontinuous. If Λ is onto and u is a critical point of $I(w) = 2\varphi^*(\Lambda w) - \langle \Lambda w, w\rangle$ then there exists $v \in V$ with $\Lambda u = \Lambda v$ such that $\frac{v+u}{2}$ is a solution of $\Lambda w \in \partial\varphi(w)$.*

Remark 1.5. Note that the above corollary is indeed the well-known Clarke–Ekeland duality proposed for Hamiltonian systems associated with a convex Hamiltonian (see [1,2]).

Corollary 1.6. *Let $\Lambda : Dom(\Lambda) \subset V \to V^*$ be a symmetric operator and $\varphi : V \to \mathbb{R}$ be convex, lower semicontinuous. If u is a critical point of $I(w) = \langle \Lambda w, w\rangle - 2\varphi(w)$ then u is a solution of $\Lambda w \in \partial\varphi(w)$.*

This is nothing but the classical Euler–Lagrange functional associated to the inclusion $\Lambda u \in \partial\varphi(u)$. As seen, this theory allows us to have various functionals associated to certain inclusions that gives us the flexibility to choose the most appropriate one to study the corresponding inclusion.

2. Non-linear boundary conditions

We shall also deal with situations where the operator Λ is not symmetric provided one takes into account certain boundary terms. In fact, the operator Λ modulo the boundary operator $\beta := (\beta_1, \beta_2) : V \to Y \times Y^*$ (for some Banach spaces Y and Y^* that are in duality) corresponds to the "Green formula"

$$\langle \Lambda u, v\rangle_{V\times V^*} = \langle u, \Lambda v\rangle_{V\times V^*} + \langle \beta_1 u, \beta_2 v\rangle_{Y\times Y^*} - \langle \beta_1 v, \beta_2 u\rangle_{Y\times Y^*}.$$

We introduce the following notion:

Definition 2.1. Suppose the spaces V and V^* and also Y and Y^* are in separating duality. We say that a linear operator $\Lambda : Dom(\Lambda) \subset V \to V^*$ is symmetric modulo the linear boundary operator $\mathcal{B} = (\beta_1, \beta_2) : Dom(\Lambda) \to Y \times Y^*$ if the following properties are satisfied,

(1) The space $V_0 = Dom(\Lambda) \cap ker(\beta_1)$ is dense in V.
(2) The operator $\beta_1 : Dom(\Lambda) \subset V \to Y$ has a dense range.
(3) For every $u, v \in Dom(\Lambda)$ we have $\langle \Lambda u, v\rangle_{V\times V^*} = \langle u, \Lambda v\rangle_{V\times V^*} + \langle \beta_1 u, \beta_2 v\rangle_{Y\times Y^*} - \langle \beta_1 v, \beta_2 u\rangle_{Y\times Y^*}$.

Definition 2.2. We say that an operator Λ is non-negative modulo the boundary operator $\mathcal{B} = (\beta_1, \beta_2)$ if for every $u \in Dom(\Lambda)$ we have $\langle \Lambda u, u\rangle_{V\times V^*} + \langle \beta_1 u, \beta_2 u\rangle_{Y\times Y^*} \geqslant 0$.

Here is our main result when one considers certain boundary terms.

Theorem 2.3. *Let $\Lambda : Dom(\Lambda) \subset V \to V^*$ be a symmetric operator modulo the boundary operator $\beta := (\beta_1, \beta_2) : V \to Y \times Y^*$ such that $(\Lambda, \beta_2) : Dom(\Lambda) \subset V \to V^* \times Y^*$ is onto. Let $\Phi : V \times V^* \to \mathbb{R}$ and $\ell : Y \times Y^* \to \mathbb{R}$ be saddle non-convex self-dual Lagrangians that are Gâteaux differentiable with respect to their first variables. Suppose one of the following conditions holds*:

(i) *The operator Λ modulo the boundary operator $\beta := (\beta_1, \beta_2)$ is non-negative.*
(ii) *For each $(u, p) \in V \times V^*$ and each $(l, e) \in Y \times Y^*$ we have $\nabla_1\Phi(u, p) = -p$ and $\nabla_1\ell(l, e) = -e$.*
(iii) *For each $u \in V$ the function $p \to \Phi(u, p)$ is Gâteaux differentiable and $\nabla_2\Phi(u, p) = u$.*

Suppose u is a critical point of $I(u) = \Phi(u, \Lambda u) + \ell(\beta_1 u, \beta_2 u)$. Then there exists $v \in V$ with $\Lambda u = \Lambda v$ and $\beta_2 u = \beta_2 v$ such that (u, v) is a solution of the system

$$\begin{cases} (-\Lambda v, v) \in \partial\Phi(u, \Lambda u), \\ (-\beta_2 v, \beta_1 v) \in \partial\ell(\beta_1 u, \beta_2 u). \end{cases}$$

As an application we have the following principle:

Corollary 2.4. *Let $\Lambda : Dom(\Lambda) \subset V \to V^*$ be a non-negative symmetric operator modulo the boundary operator $\beta := (\beta_1, \beta_2) : V \to Y \times Y^*$ in such a way that $(\Lambda, \beta_2) : Dom(\Lambda) \to V^* \times Y^*$ is onto. Let $\varphi : V \to \mathbb{R}$ and $\psi : Y \to \mathbb{R}$ be convex, lower semicontinuous and also Gâteaux differentiable. Then every critical point of $I(u) = \varphi^*(\Lambda u) - \varphi(u) + \psi^*(\beta_2 u) - \psi(\beta_1 u)$ is a solution of the equation*

$$\begin{cases} \Lambda u = \nabla\varphi(u), \\ \beta_2 u = \nabla\psi(\beta_1 u). \end{cases} \tag{3}$$

Here is another application of Theorem 2.3.

Corollary 2.5. *Let $\Lambda : Dom(\Lambda) \subset V \to V^*$ be a symmetric operator modulo the boundary operator $\beta := (\beta_1, \beta_2) : V \to Y \times Y^*$ in such a way that $(\Lambda, \beta_2) : Dom(\Lambda) \to V^* \times Y^*$ is onto. Let $\varphi : V \to \mathbb{R}$ and $\psi : Y \to \mathbb{R}$ be convex, lower semicontinuous and also Gâteaux differentiable. If u is a critical point of*

$$I(w) = 2\varphi^*(\Lambda w) - \langle \Lambda w, w\rangle + 2\psi^*(\beta_2 w) - \langle \beta_2 w, \beta_1 w\rangle$$

then there exists $v \in V$ such that $\frac{v+u}{2}$ is a solution of (3).

Remark 2.6. The above corollary can be seen as a generalization of the Clarke–Ekeland duality when the operator Λ is not purely symmetric and one deals with boundary terms as well.

As in Corollaries 2.4 and 2.5, by considering different combinations of interior *Nc-SD* Lagrangians and boundary *Nc-SD* Lagrangians one can obtain different variational principles of Eq. (3). Here we state one more application of Theorem 2.3 and leave it to interested readers to generate more new principles by making use of this theorem and characterization of *Nc-SD* Lagrangians provided in [3].

Corollary 2.7. *Let $\Lambda : Dom(\Lambda) \subset V \to V^*$ be a symmetric operator modulo the boundary operator $\beta := (\beta_1, \beta_2) : V \to Y \times Y^*$ in such a way that $(\Lambda, \beta_2) : Dom(\Lambda) \to V^* \times Y^*$ is onto. Let $\varphi : V \to \mathbb{R}$ and $\psi : Y \to \mathbb{R}$ be convex lower semicontinuous and Gâteaux differentiable. If u is a critical point of*

$$I(w) = \langle \Lambda w, w\rangle - 2\varphi(w) + \psi(\beta_2 w) - \psi(\beta_1 w)$$

then u is a solution of (3).

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