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Differential Geometry

A class of Poisson structures compatible with the canonical Poisson structure on the cotangent bundle

Une classe de structures de Poisson compatibles avec la structure de Poisson canonique sur le fibré cotangent

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A R T I C L E I N F O

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A B S T R A C T

Let M be a smooth manifold endowed with a Poisson tensor σ and a Riemannian metric g and let $J = \sigma_\# \circ \#^{-1}$ be the $(1,1)$ tensor field relating σ to g. It is well known that the complete lift of J defines a bivector field Π_J on T^*M which is a Poisson tensor compatible with canonical Poisson structure on T^*M if J is torsionless. We consider the Lie algebroid structure on T^*M associated to σ. It defines by duality a Poisson tensor Π_σ on TM. Denote by Π_σ^g the Poisson tensor on T^*M pull-back of Π_σ by the musical isomorphism associated to g. We show that the following three assertions are equivalent: (a) Π_σ^g is compatible with the canonical Poisson structure on T^*M, (b) $\Pi_\sigma^g = \Pi_J$, (c) σ is parallel with respect to the Levi-Civita connection of g.
We give also a large class of examples illustrating this situation.

R É S U M É

Soit M une variété différentiable munie d'un tenseur de Poisson σ et d'une métrique riemannienne g et soit $J = \sigma_\# \circ \#^{-1}$ le tenseur de type $(1,1)$ reliant σ à g. Il est connu que J se relève à T^*M et définit un champ de bivecteur Π_J qui est de Poisson et compatible avec la structure de Poisson canonique de T^*M si la torsion de J est nulle. On considère la structure d'algébroide de Lie sur T^*M associée à σ. Elle définit par dualité un tenseur de Poisson Π_σ sur TM. Notons Π_σ^g le tenseur de Poisson sur T^*M image de Π_σ par l'isomorphisme musical associé à g. Nous montrons que les trois assertions suivantes sont équivalentes: (a) Π_σ^g est compatible avec la structure de Poisson canonique sur T^*M, (b) $\Pi_\sigma^g = \Pi_J$, (c) σ est parallèle par rapport à la connexion de Levi-Civita de g.
Nous donnerons une large classe d'exemples illustrant cette situation.

Version française abrégée

La notion de compatibilité entre deux structures de Poisson a été introduite par Magri dans [7] pour étudier les système dits bi-hamiltoniens. Rappelons que deux tenseurs de Poisson π_1 et π_2 sont compatibles si $\pi_1 + \pi_2$ est encore un tenseur de Poisson. Le fibré cotangent T^*M muni de son tenseur de Poisson canonique noté Π_0, obtenu comme inverse de la forme

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symplectique de Liouville, est un exemple important de variété de Poisson. Il est connu (voir [4,8]) que tout tenseur de type $(1,1)$, $J: TM \to TM$, se relève à T^*M et définit un champ de bivecteur Π_J qui est de Poisson et compatible avec Π_0 si la torsion de Nijenhuis de J est nulle. Dans cette Note, on considère la situation suivante : (M,σ,g) est une variété différentiable munie d'un tenseur de Poisson σ et d'une métrique riemannienne g et $J = \sigma_\# \circ \#^{-1}$ le tenseur de type $(1,1)$ reliant σ à g. La structure d'algébroide de Lie sur T^*M associée à σ définit par dualité un tenseur de Poisson Π_σ sur TM. Notons Π_σ^g le tenseur de Poisson sur T^*M image de Π_σ par l'isomorphisme musical associé à g. Le but de cette Note est de démontrer le théorème suivant :

Théorème 0.1. *Sous les hypothèses ci-dessus, les assertions suivantes sont équivalentes* : (a) Π_σ^g *est compatible avec* Π_0, (b) $\Pi_\sigma^g = \Pi_J$, (c) σ *est parallèle par rapport à la connexion de Levi-Civita de* g.

On déduit aisément le corollaire suivant :

Corollaire 0.2. *Soit* (M,J,g) *une variété Kählerienne et soit* σ *le tenseur de Poisson associé à la forme fondamentale de* (M,J,g). *Alors* $\Pi_\sigma^g = \Pi_J$ *et* (Π_0, Π_σ^g) *sont compatibles.*

Dans la seconde partie de cette Note, nous allons caractériser les groupes de Lie G munis d'une structure de Poisson invariante à gauche σ et d'une métrique riemannienne invariante à gauche g tels que (Π_0, Π_σ^g) soient compatibles. Ceci permet d'avoir une large classe d'exemples.

1. Introduction and main results

The bi-Hamiltonian property, i.e., the property of being Hamiltonian with respect to two compatible Poisson structures is one of the most important mechanism of integrability in Hamiltonian mechanic. This notion was introduced by Magri in [7]. Recall that two Poisson tenors σ_1, σ_2 are called compatible if for any reals a,b, $a\sigma_1 + b\sigma_2$ is also a Poisson tensor. One of the most useful example of Poisson manifold in Hamiltonian mechanic is the cotangent T^*M with its canonical Poisson structure, denoted by Π_0, inverse of the Liouville symplectic form. It is well known (see [4,8]) that any $(1,1)$ tensor field on M gives rise to a bivector field Π_J on T^*M which is a Poisson tensor compatible with Π_0 if J is Nijenhuis torsionless. In this Note, we consider the following situation: (M,σ,g) is a differentiable manifold endowed with a Poisson tensor σ and a Riemannian metric g and $J = \sigma_\# \circ \#^{-1}$ is the $(1,1)$ tensor field relating σ to g. The Poisson tensor σ defines on T^*M a Lie algebroid structure and, by duality, a Poisson tensor Π_σ on TM, since a Lie algebroid structure on $\mathcal{A} \to M$ gives rise to a Poisson tensor on $\mathcal{A}^*$ (see [6] for instance). We denote by Π_σ^g the pull-back of Π_σ by the musical isomorphism $\# : T^*M \to TM$ associated to g. The purpose of this Note is to compare Π_σ^g to Π_J and find sufficient and necessary conditions on (σ,g) insuring (Π_0, Π_σ^g) to be compatible. As mentioned above, if J is torsionless then Π_J is a Poisson tensor compatible with Π_0. However, in general, $\Pi_J \neq \Pi_\sigma^g$ (see (5)). The main result of this Note clarifies the situation.

Theorem 1.1. *With the notations above, the following assertions are equivalent*:

(a) Π_σ^g *is compatible with* Π_0,
(b) $\Pi_\sigma^g = \Pi_J$,
(c) σ *is parallel with respect to the Levi-Civita connection of* g.

Note that if J is invertible, i.e., σ is the inverse of a symplectic form, then one of the conditions of Theorem 1.1 holds if and only if J is torsionless. An immediate consequence of Theorem 1.1 is that the cotangent bundle of a Kähler manifold carries a natural pair of compatible symplectic structures.

Corollary 1.2. *Let* (M,J,g) *be a Kähler manifold and let* σ *be the Poisson tensor associated to the fundamental form of* (M,J,g). *Then* $\Pi_\sigma^g = \Pi_J$ *and hence invertible and* (Π_0, Π_σ^g) *are compatible.*

The proof of Theorem 1.1 is based on Lemma 2.1 which gives a different interpretation of the complete lift of $(1,1)$ tensors on a manifold to its cotangent bundle and hence another proof of the result of [4,8] mentioned above. The formula (5) plays a crucial role in the proof. The Note is organized as follows: in Section 2, we prove Theorem 1.1. To give examples illustrating Theorem 1.1, we study in Section 3 the triple (G,σ,k), where G is a Lie group, σ a left invariant Poisson tensor on G and k a left invariant Riemannian metric on G such that $\nabla\sigma = 0$. Thanks to Theorem 1.1 and its proof, we express the sufficient and necessary conditions of $\nabla\sigma = 0$ at the level of the Lie algebra associated to G (see Proposition 3.1). Finally, we give a general method to build examples.

2. Proof of Theorem 1.1

Before to give a proof of Theorem 1.1, we recall the construction of the complete lift of $(1,1)$ tensors field on a manifold to its cotangent bundle and the associated bivector field. We show that this bivector field can be defined by using the Lie algebroid's terminology. For more details on Poisson structures and Lie algebroids one can see [3,6,9].

Recall that the Nijenhuis torsion of a $(1,1)$ tensor field A is given by

$$N_A(X,Y) := [AX, AY] - A[AX, Y] - A[X, AY] + A^2[X,Y].$$

(i) Let $A : TM \to TM$ be a $(1,1)$ tensor field on a manifold M. We consider the 1-form θ_A on T^*M defined by $\langle \theta_A, Z_\alpha \rangle = \alpha(A(\pi_*(Z_\alpha)))$, where $\pi : T^*M \to M$ is the canonical projection. When $A = \mathrm{Id}_{TM}$, θ_A is the Liouville 1-form. We denote it by θ. For any $F \in C^\infty(T^*M)$, let X_F denote the Hamiltonian vector field associated to F with respect to $\omega = d\theta$. The relations

$$\{F, G\}_A = d\theta_A(X_F, X_G) = \omega(L_A X_F, X_G) \tag{1}$$

define both a $(1,1)$ tensor field L_A and a bivector field Π_A on T^*M. The tensor L_A is the *complete lift* of A and Π_A is its associated bivector field (see [4,8,5]).

(ii) A Lie algebroid $\mathcal{A}$ over a smooth manifold M is a vector bundle $p : \mathcal{A} \to M$ together with a Lie algebra structure $[\,,\,]'$ on the space of sections $\Gamma(\mathcal{A})$ and a bundle map $\# : \mathcal{A} \to TM$ called *anchor* such that, for any sections $a, b \in \Gamma(\mathcal{A})$ and for every smooth function $f \in C^\infty(M)$, we have the Leibniz identity

$$[a, fb]' = f[a,b]' + \#(a)(f)b.$$

An immediate consequence of this definition is that, for any $a, b \in \Gamma(\mathcal{A})$, $\#([a,b]') = [\#(a), \#(b)]$, where $[\,,\,]$ is the Lie bracket. It is known (see [6]) that any bracket on $\Gamma(\mathcal{A})$ and any anchor map satisfying the Leibniz rule define a bivector field on $\mathcal{A}^*$ called *dual* of $([\,,\,]', \#)$. Moreover, this bivector field is a Poisson bivector field iff $[\,,\,]'$ satisfies the Jacobi identity.

Two Lie algebroid structures $([\,,\,]_1, \#_1)$ and $([\,,\,]_2, \#_2)$ on a vector bundle $\mathcal{A} \to M$ are called compatible if $([\,,\,]_1 + [\,,\,]_2, \#_1 + \#_2)$ is a Lie algebroid structure on $\mathcal{A}$. It is obvious that the compatibility of two Lie algebroid structures is equivalent to the compatibility of their dual Poisson structures.

The canonical Lie algebroid structure on a smooth manifold M is the Lie algebroid structure on TM whose anchor map is the identity Id_{TM} and whose bracket is the Lie bracket. Its dual Poisson structure is the canonical Poisson structure on T^*M inverse of the symplectic Liouville form $d\theta$. We denote by Π_0 this Poisson structure. More generally, let $A : TM \to TM$ be a $(1,1)$ tensor field. We define a new bracket on the space of vector fields by

$$[X, Y]_A = [AX, Y] + [X, AY] - A[X, Y],$$

where $[\,,\,]$ is the Lie bracket. We have for any vector fields X and Y, and for any $f \in C^\infty(M)$, $[X, fY]_A = f[X,Y]_A + (A(X).f)Y$. Let $(x^1, \ldots, x^n)$ be local coordinates on M and $(x^1, \ldots, x^n, y_1, \ldots, y_n)$ the corresponding coordinates on T^*M. We have

$$A\partial_s = \sum_{i=1}^n A_s^i \partial_i \quad \text{and} \quad [\partial_s, \partial_t]_A = \sum_{i=1}^n \left(\partial_s A_t^i - \partial_t A_s^i\right)\partial_i.$$

Thus the bracket on $C^\infty(T^*M)$ associated to the dual bivector field of $([\,,\,]_A, A)$ is given by

$$\left\{x^i, x^j\right\}_A = 0, \qquad \left\{y_s, x^j\right\}_A = A_s^j \quad \text{and} \quad \{y_s, y_t\}_A = \sum_{i=1}^n \left(\partial_s A_t^i - \partial_t A_s^i\right) y_i. \tag{2}$$

The expression of the bracket (1) in local coordinates is given in [1] and one can see that this bracket is exactly the bracket defined by (2). So we have shown that the bivector field Π_A associated to L_A is the dual of $([\,,\,]_A, A)$.

It is shown in [4,8] that if A is Nijenhuis torsionless then Π_A is a Poisson tensor and it is compatible with Π_0. In the following lemma, we recover this result by using the Lie algebroid's point of view:

Lemma 2.1.

(i) *Let $A : TM \to TM$ be a $(1,1)$ tensor field. Then $[\,,\,]_A$ satisfies the Jacobi identity iff A is Nijenhuis torsionless and, in this case, $([\,,\,]_A, A)$ is a Lie algebroid structure on TM compatible with the canonical Lie algebroid structure and hence its dual Poisson tensor Π_A is compatible with Π_0.*

(ii) *Conversely, if $([\,,\,]', A)$ is a Lie algebroid structure on TM compatible with its canonical Lie algebroid structure, then $[\,,\,]' = [\,,\,]_A$. Moreover, A is Nijenhuis torsionless.*

Proof. (i) If $[\,,\,]_A$ satisfies the Jacobi identity, then $([\,,\,]_A, A)$ is a Lie algebroid structure on TM, which implies $A[X,Y]_A = [AX, AY]$, thus A is a Nijenhuis tensor. The converse is a consequence of the following straightforward formula:

$$\oint_{X,Y,Z} \big\{\big[X,[Y,Z]_A\big]_A - \big[N_A(X,Y),Z\big] - N_A\big([X,Y],Z\big)\big\} = 0,$$

where $\oint$ denotes the cyclic sum over the vector fields X, Y and Z. Now it is straightforward to check that if $([\,,\,]_A, A)$ is a Lie algebroid structure on TM then it is compatible with the canonical structure.

(ii) Assume that $([\,,\,]', A)$ is compatible with $([\,,\,], \mathrm{Id}_{TM})$. Then $([\,,\,]+[\,,\,]', \mathrm{Id}_{TM}+A)$ is a Lie algebroid structure on TM. Hence $\mathrm{Id}_{TM}+A$ is a Lie algebra homomorphism, i.e., for any vector fields X and Y,

$$(\mathrm{Id}_{TM}+A)\big([X,Y]+[X,Y]'\big) = \big[(\mathrm{Id}_{TM}+A)X, (\mathrm{Id}_{TM}+A)Y\big].$$

Since $A[X,Y]' = [AX, AY]$, we deduce that the equation above is equivalent to $[\,,\,]' = [\,,\,]_A$. Moreover, the relation $A[X,Y]_A = [AX, AY]$ is equivalent to $N_A \equiv 0$. $\square$

Remark 1. We can deduce form Lemma 2.1 that a Lie algebroid structure $([\,,\,]', A)$ is compatible with the canonical one iff $[\,,\,]' = [\,,\,]_A$. When A is invertible this is equivalent to $N_A = 0$.

Let (M, σ, g) be a smooth manifold endowed with a Poisson tensor σ and a Riemannian metric g. Denote by ∇ the Levi-Civita connection associated to g and by $\# : T^*M \to TM$ the musical isomorphism associated to g. The Poisson tensor σ defines a Lie algebroid structure on T^*M. Its anchor is the map $\sigma_\# : T^*M \to TM$ given by $\beta(\sigma_\#(\alpha)) = \sigma(\alpha, \beta)$ and its bracket is the Koszul bracket of differential 1-forms given by

$$[\alpha,\beta]_\sigma = \mathcal{L}_{\sigma_\#(\alpha)}\beta - \mathcal{L}_{\sigma_\#(\beta)}\alpha - d\big(\sigma(\alpha,\beta)\big). \tag{3}$$

Denote by Π the Poisson tensor on TM dual of this Lie algebroid structure. Let us push by $\#$ this Lie algebroid structure to TM. Thus we get a Lie algebroid $(TM, [\,,\,]_\sigma^g, J)$ where

$$[X,Y]_\sigma^g = \#\big[\#^{-1}(X), \#^{-1}(Y)\big]_\sigma \quad \text{and} \quad J = \sigma_\# \circ \#^{-1}. \tag{4}$$

The following proposition is obvious:

Proposition 2.2.

(i) *The Poisson tensor Π_σ^g on T^*M pull-back of Π by $\#$ is the dual of $(TM, [\,,\,]_\sigma^g, J)$.*
(ii) *Π_σ^g and Π_0 are compatible iff $([\,,\,]_\sigma^g, J)$ is compatible with the canonical Lie algebroid structure of TM.*

Thus we get on T^*M a Poisson bivector field Π_σ^g dual of $(TM, [\,,\,]_\sigma^g, J)$ and a bivector field Π_J dual of $(TM, [\,,\,]_J, J)$. Let us compare these bivector fields. Indeed, a straightforward computation using (3), (4) and the properties of ∇ gives

$$g\big([X,Y]_\sigma^g - [X,Y]_J, Z\big) = -g\big((\nabla_X J)Y, Z\big) + g\big((\nabla_Y J)X, Z\big) - g\big((\nabla_Z J)Y, X\big). \tag{5}$$

Proof of Theorem 1.1. According to Proposition 2.2, the Poisson tensor Π_σ^g and Π_0 are compatible iff $([\,,\,]_\sigma^g, J)$ is compatible with the canonical Lie algebroid structure of TM. According to Remark 1, $([\,,\,]_\sigma^g, J)$ is compatible with the canonical Lie algebroid structure of TM if and only if, for any vector fields X and Y, $[X,Y]_\sigma^g = [X,Y]_J$. This shows that (a) is equivalent to (b).

Now, if $\nabla\sigma = 0$ then from (5) we deduce that (b) holds. Conversely, suppose that $\Pi_\sigma^g = \Pi_J$. Denote by $\Lambda(X,Y,Z) = g((\nabla_X J)Y, Z)$. Note first that from the fact that J is skew-symmetric with respect to g, one can deduce easily that

$$\Lambda(X,Y,Z) + \Lambda(X,Z,Y) = 0. \tag{6}$$

Now, from (5) we deduce that, for any vector fields X, Y, Z,

$$\Lambda(X,Y,Z) + \Lambda(Z,Y,X) = \Lambda(Y,X,Z) \quad \text{and} \quad \Lambda(Y,Z,X) + \Lambda(X,Z,Y) = \Lambda(Z,Y,X).$$

By adding these two equalities and by using (6), we get $2\Lambda(Y,Z,X) = 0$ which implies $\nabla\sigma = 0$. $\square$

3. Lie groups with left-invariant Riemannian metric and parallel left-invariant Poisson structure

Let (G,σ,k) be a Lie group endowed with a left invariant Poisson tensor σ and a left invariant Riemannian metric k. Let $\mathfrak{g}$ be the Lie algebra of G identified with T_eG. The values of k and σ at the identity define, respectively, a scalar product $\langle\,,\rangle$ on $\mathfrak{g}$ and $r\in\mathfrak{g}\wedge\mathfrak{g}$ a solution of the classical Yang–Baxter equation. Denote by $r_\#:\mathfrak{g}^*\to\mathfrak{g}$ be the value of $\sigma_\#$ at the identity, by $\#:\mathfrak{g}^*\to\mathfrak{g}$ the isomorphism musical at the identity and let $J=r_\#\circ\#^{-1}$. The restriction of the Koszul bracket to $\mathfrak{g}^*$ defines a Lie algebra structure on $\mathfrak{g}^*$ by $[\alpha,\beta]_r:=\mathrm{ad}^*_{r_\#(\beta)}\alpha-\mathrm{ad}^*_{r_\#(\alpha)}\beta$, and $r_\#$ is a Lie algebra morphism. According to Theorem 1.1, $\nabla\sigma=0$ iff, for any $u,v\in\mathfrak{g}$,

$$[u,v]_r^\#=[u,v]_J, \tag{7}$$

where $[u,v]_r^\#=\#[\#^{-1}(u),\#^{-1}(v)]_r$ and $[u,v]_J=[Ju,v]+[u,Jv]-J[u,v]$. So we get:

Proposition 3.1. *Let $\mathfrak{g}$ be a Lie algebra endowed with a scalar product and a solution of the classical Yang–Baxter equation r. Then $(\mathfrak{g},\langle\,,\rangle,r)$ satisfies Eq.* (7) *iff the following conditions hold*:

(i) $(\mathrm{Im}\,r_\#)^\perp$ *is a Lie subalgebra of* $\mathfrak{g}$.
(ii) *For any* $u,v\in\mathrm{Im}\,r_\#$, $[Ju,Jv]-J[Ju,v]-J[u,Jv]+J^2[u,v]=0$.
(iii) *For any* $u\in\mathrm{Im}\,r_\#$ *and for any* $v,w\in(\mathrm{Im}\,r_\#)^\perp$, $\langle\mathrm{ad}_u v,w\rangle+\langle v,\mathrm{ad}_u w\rangle=0$.
(iv) *For any* $u\in(\mathrm{Im}\,r_\#)^\perp$ *and for any* $v,w\in\mathrm{Im}\,r_\#$, $\langle\mathrm{ad}_u\circ J(v),w\rangle-\langle v,\mathrm{ad}_u\circ J(w)\rangle=0$.
(v) *For any* $u\in(\mathrm{Im}\,r_\#)^\perp$ *and for any* $v,w\in\mathrm{Im}\,r_\#$, $\langle(\mathrm{ad}_u\circ J-J\circ\mathrm{ad}_u)v,w\rangle=0$.

We will now give a large class of $(\mathfrak{g},\langle\,,\rangle,r)$ satisfying the conditions of Proposition 3.1. Let $(\mathfrak{g}_1,\langle\,,\rangle_1,J)$ be a Kähler Lie algebra, i.e., $\mathfrak{g}_1$ is a Lie algebra, $\langle\,,\rangle_1$ a scalar product on $\mathfrak{g}_1$ and $J:\mathfrak{g}_1\to\mathfrak{g}_1$ is an endomorphism such that $J^2=-\mathrm{Id}_{\mathfrak{g}_1}$, for any $u,v\in\mathfrak{g}_1$,

$$\langle Ju,Jv\rangle=\langle u,v\rangle\quad\text{and}\quad[Ju,Jv]-J[Ju,v]-J[u,Jv]-[u,v]=0$$

and the fundamental form $\omega(u,v)=\langle Ju,v\rangle$ satisfies $\omega(u,[v,w])+\omega(v,[w,u])+\omega(w,[u,v])=0$, for any $u,v,w\in\mathfrak{g}_1$ (see [2] for examples of Kähler Lie algebra). Let $(\mathfrak{g}_2,\langle\,,\rangle_2)$ be a Lie algebra with a scalar product and suppose that there exists a representation $\rho:\mathfrak{g}_1\to\mathrm{End}(\mathfrak{g}_2)$ such that for any $u\in\mathfrak{g}_1$, $\rho(u)$ is skew-symmetric with respect to $\langle\,,\rangle_2$. Define on $\mathfrak{g}=\mathfrak{g}_1\oplus\mathfrak{g}_2$ the Lie bracket

$$[u_1+u_2,v_1+v_2]=[u_1,u_2]+[v_1,v_2]+\rho(u_1)(v_2)-\rho(v_1)(u_2),$$

the scalar product $\langle\,,\rangle=\langle\,,\rangle_1+\langle\,,\rangle_2$ and $r_\#:\mathfrak{g}^*\to\mathfrak{g}$ by $r_\#=i\circ\omega_\#\circ i^*$, where $\omega_\#:\mathfrak{g}_1^*\to\mathfrak{g}_1$ is the isomorphism defined by the symplectic form ω, $i:\mathfrak{g}_1\to\mathfrak{g}$ the canonical injection and $i^*:\mathfrak{g}^*\to\mathfrak{g}_1^*$ its dual. The endomorphism $r_\#$ defines $r\in\mathfrak{g}\wedge\mathfrak{g}$ which is a solution of the Yang–Baxter equation. One can see easily that the triple $(\mathfrak{g},\langle\,,\rangle,r)$ satisfies the conditions of Proposition 3.1.

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