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Exact controllability of a cascade system of conservative equations

Contrôlabilité exacte d'un système en cascade d'équations conservatives

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ABSTRACT

We consider a cascade system of two conservative equations and prove the controllability of the full system when each equation is controllable, provided that the unitary group corresponding to the free evolution is time-periodic. Applications to systems of Schrödinger (resp. wave) equations are given. With the aid of Kannai transform we infer that a one-dimensional system of heat equations is null controllable even if the supports of the control function and of the coupling term do not intersect.

RÉSUMÉ

On considère un système en cascade de deux équations conservatives et l'on prouve la contrôlabilité du système complet lorsque chaque équation est contrôlable et que le groupe unitaire correspondant à l'évolution libre est périodique en temps. Ce résultat s'applique à des systèmes d'équations de Schrödinger ou des ondes. Utilisant la transformée de Kannai, on en déduit qu'un système en cascade d'équations de la chaleur est contrôlable à zéro en dimension un, même si les supports du contrôle et du couplage ne s'intersectent pas.

Version française abrégée

Le contrôle des systèmes d'EDP est un problème difficile. Tandis que le contrôle de systèmes d'équations paraboliques a reçu beaucoup d'attention ces dix dernières années (cf. [7] pour la contrôlabilité approchée, [2,6] pour la contrôlabilité à zéro simultanée avec un couplage linéaire et [3] pour un couplage cubique), le contrôle de systèmes d'équations d'ondes est encore assez peu connu. Les articles [4] et [10] se focalisent sur le contrôle « insensibilisant » de l'équation des ondes. Plus récemment, le contrôle d'un système de deux équations des ondes avec des termes de couplage présents dans les deux équations a été étudié dans [1].

Dans cette Note, nous étudions la contrôlabilité exacte d'un système abstrait de la forme

$$\dot{y} = Ay + B_0 u, \qquad \dot{q} = Aq + B_1 B_1^* y, \qquad q(0) = q^0, \qquad y(0) = y^0 \tag{1}$$

où A est un opérateur antiadjoint défini sur un espace de Hilbert H qui engendre un groupe d'isométries noté $(S(t))_{t\in\mathbb{R}}$, $B_0 \in \mathcal{L}(U, D(A)')$ est un opérateur de contrôle (éventuellement non-borné), et $B_1 \in \mathcal{L}(H)$. $D(A)'$ désigne l'espace dual de $D(A)$ avec l'espace pivot H. On fait les hypothèses suivantes :

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(H_1) Le groupe d'isométries $(S(t))_{t\in\mathbb{R}}$ est périodique, c'est-à-dire qu'il existe $T_p > 0$ tel que

$$S(t+T_p)v = S(t)v \quad \text{pour tous } t \in \mathbb{R},\ v \in H\ ;$$

(H_2) B_0 vérifie les propriétés d'admissibilité suivantes :

$$\int_0^{T_0} \left\| B_0^* S^*(t) v \right\|_U^2 \, dt \leqslant C \|v\|_H^2 \quad \text{pour tout } v \in D(A),$$

$$\int_0^{T_0} \left\| B_0^* S^*(t) \left[\int_0^t S^*(-\sigma) B_1 B_1^* S^*(\sigma) v \, d\sigma \right] \right\|_U^2 dt \leqslant C \|v\|_H^2 \quad \text{pour tout } v \in D(A)\ ;$$

(H_3) (Inégalités d'observabilité) Il existe deux temps $T_0 > 0$, $T_1 > 0$ et une constante $C > 0$ tels que les inégalités suivantes soient satisfaites

$$\int_0^{T_0} \left\| B_0^* S^*(t) v \right\|_U^2 \, dt \geqslant C \|v\|_H^2 \quad \text{pour tout } v \in D(A),$$

$$\int_0^{T_1} \left\| B_1^* S^*(t) v \right\|_H^2 \, dt \geqslant C \|v\|_H^2 \quad \text{pour tout } v \in H.$$

Le résultat principal de cette Note est le suivant :

Théorème 1. *Sous les hypothèses* (H_1), (H_2) *et* (H_3) *avec* $T_p \geqslant T_1$, *le système* (1) *est exactement contrôlable dans l'espace* $H \times H$ *en temps* $T \geqslant T_0 + T_p$.

Le Théorème 1 permet d'établir la contrôlabilité exacte de systèmes en cascade d'équations de Schrödinger (en dimension quelconque) ou des ondes (en dimension un), avec un contrôle et un terme de couplage localisés.

Théorème 2. *Soit* $N \geqslant 1$, *et soient* ω *et* $\mathcal{O}$ *deux ouverts non vides de* $\mathbb{T}^N = \mathbb{R}^N/(2\pi\mathbb{Z})^N$. *Le système*

$$iy_t + \Delta y = i1_\omega u, \qquad iq_t + \Delta q = i1_{\mathcal{O}} y \quad \textit{dans } (0,T) \times \mathbb{T}^N$$

est exactement contrôlable dans $L^2(\mathbb{T}^N) \times L^2(\mathbb{T}^N)$ *en temps* $T > 2\pi$.

Théorème 3. *Soit* $\Omega = (0,\pi)^N$ $(N \geqslant 1)$, *et soient* $\Gamma \subset \partial\Omega$ *un côté de* Ω *et* $a \in \mathbb{C}$ *une constante non nulle. Le système*

$$iy_t + \Delta y = 0, \qquad iq_t + \Delta q = iay \quad \textit{dans } (0,T) \times \Omega, \qquad \frac{\partial y}{\partial \nu} = 1_\Gamma u, \qquad \frac{\partial q}{\partial \nu} = 0 \quad \textit{sur } (0,T) \times \partial\Omega$$

est exactement contrôlable dans $L^2(\Omega) \times L^2(\Omega)$ *en temps* $T > 2\pi$.

Théorème 4. *Soient* ω *et* $\mathcal{O}$ *deux ouverts non vides de* $(0,1)$. *Le système*

$$y_{tt} - y_{xx} = 1_\omega u, \qquad q_{tt} - q_{xx} = 1_{\mathcal{O}} y \quad \textit{dans } (0,T) \times (0,1), \qquad y = q = 0 \quad \textit{sur } (0,T) \times \{0,1\}$$

est exactement contrôlable dans $H_0^1(0,1) \times L^2(0,1) \times (H^2(0,1) \cap H_0^1(0,1)) \times H_0^1(0,1)$ *en temps* $T \geqslant 4$.

Combinant le Théorème 4 à la transformée de Kannai, nous obtenons qu'un système en cascade d'équations de la chaleur est contrôlable à zéro en dimension un, même si les supports du contrôle et du couplage ne s'intersectent pas (cf. [2,6] pour la contrôlabilité à zéro et [7] pour la contrôlabilité approchée).

Théorème 5. *Soient* ω *et* $\mathcal{O}$ *deux ouverts non vides de* $(0,1)$. *Le système*

$$y_t - y_{xx} = 1_\omega u, \qquad q_t - q_{xx} = 1_{\mathcal{O}} y \quad \textit{dans } (0,T) \times (0,1), \qquad y = q = 0 \quad \textit{sur } (0,T) \times \{0,1\}$$

est contrôlable à 0 *dans* $[L^2(0,1)]^2$ *en tout temps* $T > 0$.

1. Introduction

The control of systems of PDEs is a challenging issue. While the control of coupled parabolic equations has received a lot of attention in the last decade (see [7] for the approximate controllability, [2] and [6] for the simultaneous null controllability with a linear coupling, and [3] for a cubic coupling), the control of systems of wave equations is still at its early stage. The papers [4] and [10] were concerned with the "insensitizing controls" of the wave equation. More recently, the control of a system of two wave equations with coupling terms present in both equations was considered in [1].

In this Note, we investigate the exact controllability of an abstract system of the form

$$\dot{y} = Ay + B_0 u, \qquad \dot{q} = Aq + B_1 B_1^* y, \qquad q(0) = q^0, \qquad y(0) = y^0 \tag{2}$$

where A is a skew-adjoint operator on a Hilbert space H, B_0 is a (possibly unbounded) operator, and B_1 is a bounded operator on H. The result we obtain yields the exact controllability of cascade systems of Schrödinger equations (resp. wave equations) with localized distributed/boundary control and with a localized coupling term, under the key assumption that the group of isometries generated by A is time-periodic.

It was proved in [2,6] that a cascade system of heat equations is controllable if the control region and the coupling region intersect, and in [7] that the approximate controllability always holds. Using the above result for systems of wave equations and Kannai transform, we infer that a one-dimensional cascade system of two heat equations is null controllable, even if the control region and the coupling region do not intersect.

2. Main result

Let H denote a Hilbert space and let $A : D(A) \subset H \to H$ be a skew-adjoint operator (i.e. $A^* = -A$) which generates a group of isometries denoted by $(S(t))_{t\in\mathbb{R}}$. Let $B_0 \in \mathcal{L}(U, D(A)')$, where U is another Hilbert space and $D(A)'$ denotes the dual of $D(A)$ with the pivot space H, and let $B_1 \in \mathcal{L}(H)$. We are interested in the control properties of the system (2) where $(y,q) \in L^2(0,T; H\times H)$ is the state function, $(y^0, q^0) \in H \times H$ is the pair of initial data and $u \in L^2(0,T;U)$ is the control input. In (2) B_1^* stands for the adjoint of B_1. Note that $B_1 B_1^* \in \mathcal{L}(H)$, while $B_0^* \in \mathcal{L}(D(A), U)$. We shall make the following assumptions:

(H_1) The group of isometries $(S(t))_{t\in\mathbb{R}}$ is periodic; i.e., there exists some time $T_p > 0$ such that

$$S(t + T_p)v = S(t)v \quad \text{for all } t \in \mathbb{R},\ v \in H; \tag{3}$$

(H_2) B_0 fulfills the following admissibility properties:

$$\int_0^{T_0} \left\| B_0^* S^*(t) v \right\|_U^2 \, dt \leqslant C \|v\|_H^2 \quad \text{for all } v \in D(A), \tag{4}$$

$$\int_0^{T_0} \left\| B_0^* S^*(t) \left[\int_0^t S^*(-\sigma) B_1 B_1^* S^*(\sigma) v \, d\sigma \right] \right\|_U^2 dt \leqslant C \|v\|_H^2 \quad \text{for all } v \in D(A); \tag{5}$$

(H_3) (Observability inequalities) There exist two times $T_0 > 0$, $T_1 > 0$ and a constant $C > 0$ such that

$$\int_0^{T_0} \left\| B_0^* S^*(t) v \right\|_U^2 \, dt \geqslant C \|v\|_H^2 \quad \text{for all } v \in D(A), \tag{6}$$

$$\int_0^{T_1} \left\| B_1^* S^*(t) v \right\|_H^2 \, dt \geqslant C \|v\|_H^2 \quad \text{for all } v \in H. \tag{7}$$

The main result of this Note is the following:

Theorem 1. *Assume that (H_1), (H_2) and (H_3) are satisfied with $T_p \geqslant T_1$. Then system (2) is exactly controllable in the space $H \times H$ in time $T \geqslant T_0 + T_p$.*

Proof. C will denote a constant which may vary from line to line. Pick any time $T \geqslant T_0 + T_p$. (2) may be written in the form $\left[\begin{smallmatrix} y \\ q \end{smallmatrix}\right]' = \mathcal{A}\left[\begin{smallmatrix} y \\ q \end{smallmatrix}\right] + \mathcal{B}u$ with $\mathcal{A} = \left[\begin{smallmatrix} A & 0 \\ B_1 B_1^* & A \end{smallmatrix}\right]$, $\mathcal{B} = \left[\begin{smallmatrix} B_0 \\ 0 \end{smallmatrix}\right]$ and $\mathcal{D}(\mathcal{A}) = D(A) \times D(A) \subset H \times H$, $\mathcal{B} \in \mathcal{L}(U, D(\mathcal{A})')$. Note that $\mathcal{A}$ generates a strongly continuous group $(\mathcal{S}(t))_{t\in\mathbb{R}}$ on $H \times H$ given by

$$\mathcal{S}(t)\begin{bmatrix} y^0 \\ q^0 \end{bmatrix} = \begin{bmatrix} S(t)y^0 \\ S(t)q^0 + \int_0^t S(t-\sigma)B_1B_1^*S(\sigma)y^0\,\mathrm{d}\sigma \end{bmatrix}.$$

Clearly $\mathcal{A}^* = \left[\begin{smallmatrix} A^* & B_1B_1^* \\ 0 & A^* \end{smallmatrix}\right]$ with domain $\mathcal{D}(\mathcal{A}^*) = \mathcal{D}(\mathcal{A})$, and $\mathcal{B}^* \in \mathcal{L}(\mathcal{D}(\mathcal{A}), U)$ is given by $\mathcal{B}^*\left[\begin{smallmatrix} y \\ q \end{smallmatrix}\right] = B_0^*y$. Furthermore, from (4)–(5) and the expression of $\mathcal{A}^*$ we have that $\int_0^{T_p} \|\mathcal{B}^*\mathcal{S}^*(t)\left[\begin{smallmatrix} z^0 \\ p^0 \end{smallmatrix}\right]\|_U^2\,\mathrm{d}t \leqslant C\|\left[\begin{smallmatrix} z^0 \\ p^0 \end{smallmatrix}\right]\|_{H\times H}^2$ for $\left[\begin{smallmatrix} z^0 \\ p^0 \end{smallmatrix}\right] \in \mathcal{D}(\mathcal{A}^*)$ so that $\mathcal{B}^*$ is admissible (in the usual sense) for $\mathcal{A}$. It is well known that the exact controllability of (2) in time $T_0 + T_p$ is equivalent to the following observability inequality

$$\int_0^{T_0+T_p} \|B_0^*z(t)\|_U^2\,\mathrm{d}t \geqslant C\left(\|z^0\|_H^2 + \|p^0\|_H^2\right) \tag{8}$$

where $\left[\begin{smallmatrix} z(t) \\ p(t) \end{smallmatrix}\right] = \mathcal{S}^*(t)\left[\begin{smallmatrix} z^0 \\ p^0 \end{smallmatrix}\right]$. From the expression of $\mathcal{A}^*$, we have that $p(t) = S^*(t)p^0 = S(-t)p^0$ and that

$$z(t) = S^*(t)z^0 + \int_0^t S^*(t-\sigma)B_1B_1^*S^*(\sigma)p^0\,\mathrm{d}\sigma. \tag{9}$$

By (H_1) we have that

$$z(t+T_p) = S(-t)\left(z^0 + \int_0^{t+T_p} S(\sigma)B_1B_1^*S(-\sigma)p^0\,\mathrm{d}\sigma\right) \tag{10}$$

so that

$$z(t+T_p) - z(t) = S(-t)\int_t^{t+T_p} S(\sigma)B_1B_1^*S(-\sigma)p^0\,\mathrm{d}\sigma = S(-t)\int_0^{T_p} S(\sigma)B_1B_1^*S(-\sigma)p^0\,\mathrm{d}\sigma.$$

(We used the periodicity of the map $\sigma \mapsto S(\sigma)B_1B_1^*S(-\sigma)p^0$.) Let us set $w^0 = \int_0^{T_p} S(\sigma)B_1B_1^*S(-\sigma)p^0\,\mathrm{d}\sigma$.

Claim 1. $\|w^0\|_H^2 \leqslant C\int_0^{T_0+T_p} \|B_0^*z(t)\|_U^2\,\mathrm{d}t$.

Indeed, since $z(t+T_p) - z(t) = S^*(t)w^0$, we have by (6)

$$\|w^0\|_H^2 \leqslant C\int_0^{T_0} \|B_0^*S^*(t)w^0\|_U^2\,\mathrm{d}t \leqslant C\int_0^{T_0} \|B_0^*[z(t+T_p) - z(t)]\|_U^2\,\mathrm{d}t \leqslant C\int_0^{T_0+T_p} \|B_0^*z(t)\|_U^2\,\mathrm{d}t$$

so that Claim 1 holds.

Claim 2. $\|p^0\|_H^2 \leqslant C\|w^0\|_H^2$.

Indeed, if we introduce the bounded operator $\Gamma : H \to H$ defined by $\Gamma(p^0) = w^0$, then $(\Gamma(p^0), p^0)_H = \int_0^{T_p} \|B_1^*S^*(\sigma)p^0\|_H^2\,\mathrm{d}\sigma \geqslant C\|p^0\|_H^2$ by (7), since $T_p \geqslant T_1$. It follows from Lax–Milgram Theorem that Γ is invertible, and Claim 2 follows. From Claim 2 and Claim 3, we have that

$$\|p^0\|_H^2 \leqslant C\int_0^{T_0+T_p} \|B_0^*z(t)\|_U^2\,\mathrm{d}t. \tag{11}$$

Claim 3. $\|z^0\|_H^2 \leqslant C\int_0^{T_0+T_p} \|B_0^*z(t)\|_U^2\,\mathrm{d}t$.

To estimate z^0, we use (5), (6), (9) and (11) to get

$$\|z^0\|_H^2 \leqslant C\int_0^{T_0}\|B_0^*S^*(t)z^0\|_U^2\,dt \leqslant C\left(\int_0^{T_0}\|B_0^*z(t)\|_U^2\,dt + \int_0^{T_0}\left\|B_0^*S^*(t)\int_0^t S^*(-\sigma)B_1B_1^*S^*(\sigma)p^0\,d\sigma\right\|_U^2 dt\right)$$
$$\leqslant C\int_0^{T_0+T_p}\|B_0^*z(t)\|_U^2\,dt. \tag{12}$$

Then (8) follows from (11) and (12). The proof of Theorem 1 is complete. □

3. Applications

3.1. Cascade systems of Schrödinger equations

For any $N\in\mathbb{N}^*$ let $\mathbb{T}^N=\mathbb{R}^N/(2\pi\mathbb{Z})^N\sim[0,2\pi)^N$ denote the N-dimensional torus. For any $s\in\mathbb{R}$ let $H^s(\mathbb{T}^N)$ denote the Sobolev space of (periodic) functions in H^s, i.e. the completion of $C^\infty(\mathbb{T}^N)$ for the norm $\|u\|_s=(\sum_{k\in\mathbb{Z}^N}\langle k\rangle^{2s}|\hat u_k|^2)^{\frac12}$ where $u(x)=\sum_{k\in\mathbb{Z}^N}\hat u_k e^{ik\cdot x}$ and $\langle k\rangle=(1+|k|^2)^{\frac12}$ for $k\in\mathbb{Z}^N$. We are interested in the control properties of the following cascade system of Schrödinger equations:

$$iy_t+\Delta y=i1_\omega u,\qquad iq_t+\Delta q=i\big|a(x)\big|^2y\quad\text{in }(0,T)\times\mathbb{T}^N \tag{13}$$

where $\omega\subset\mathbb{T}^N$ is a nonempty open set, $a\in L^\infty(\mathbb{T}^N,\mathbb{R})$ with $|a(x)|>\varepsilon$ a.e. on some ball $B_\varepsilon(x_0)$ ($x_0\in\mathbb{T}^N$ and $\varepsilon>0$ being arbitrary), and $u\in L^2(0,T;L^2(\mathbb{T}^N,\mathbb{C}))$. Using Jaffard–Komornik result and Theorem 1, we obtain (with $T_p=2\pi$)

Theorem 2. (13) *is exactly controllable in $L^2(\mathbb{T}^N)\times L^2(\mathbb{T}^N)$ in time $T>2\pi$.*

Let $\Omega=(0,\pi)^N$, and let us consider now the system

$$iy_t+\Delta y=0,\qquad iq_t+\Delta q=iay\quad\text{in }(0,T)\times\Omega,\qquad \frac{\partial y}{\partial\nu}=1_\Gamma u,\qquad \frac{\partial q}{\partial\nu}=0\quad\text{on }(0,T)\times\partial\Omega \tag{14}$$

where $\Gamma\subset\partial\Omega$ is a side of Ω, $a\in\mathbb{C}$ with $a\neq0$, and $u\in L^2(0,T;L^2(\partial\Omega,\mathbb{C}))$. Using [9, Prop. 2.4 and Cor. 3.3] and Theorem 1, we obtain (still with $T_p=2\pi$)

Theorem 3. (14) *is exactly controllable in $L^2(\Omega)\times L^2(\Omega)$ in time $T>2\pi$.*

Notice that (4) is classical (see e.g. [9]), while (5) follows from (4), for $\int_0^t S^*(-\sigma)B_1B_1^*S^*(\sigma)v\,d\sigma=|a|^2tv$.

3.2. Cascade system of wave equations

We are interested in the control properties of the cascade system of wave equations

$$y_{tt}-y_{xx}=1_\omega u,\qquad q_{tt}-q_{xx}=\big|a(x)\big|^2y\quad\text{in }(0,T)\times(0,1),\qquad y=q=0\quad\text{on }(0,T)\times\{0,1\} \tag{15}$$

where ω is a nonempty open set in $(0,1)$, and $a\in L^\infty(0,1)$ satisfies $|a(x)|>\varepsilon$ a.e. on $\mathcal{O}$, a nonempty open subset of $(0,1)$, for some $\varepsilon>0$. Let

$$\mathcal{H}:=H_0^1(0,1)\times L^2(0,1)\times\big(H^2(0,1)\cap H_0^1(0,1)\big)\times H_0^1(0,1).$$

Then the following result holds:

Theorem 4. *Pick any time $T\geqslant4$. Then* (15) *is exactly controllable in $\mathcal{H}$ in time T.*

Proof. Let (z,q) solve the system

$$z_{tt}-z_{xx}=1_\omega v,\qquad q_{tt}-q_{xx}=\big|a(x)\big|^2z_t\quad\text{in }(0,T)\times(0,1),\qquad z=q=0\quad\text{on }(0,T)\times\{0,1\}. \tag{16}$$

We aim to apply Theorem 1 to system (16) with state (z,z_t,q,q_t) and control v. Let $A=\left[\begin{smallmatrix}0 & I\\ \partial_x^2 & 0\end{smallmatrix}\right]$ with domain $D(A)=[H^2(0,1)\cap H_0^1(0,1)]\times H_0^1(0,1)\subset H_0^1(0,1)\times L^2(0,1)=H$. Let $U=L^2(0,1)$ and $B_0v=\left[\begin{smallmatrix}0\\ 1_\omega v\end{smallmatrix}\right]$ and $B_1\left[\begin{smallmatrix}z^0\\ z^1\end{smallmatrix}\right]=\left[\begin{smallmatrix}0\\ a(x)z^1\end{smallmatrix}\right]$. Note that $B_1^*=B_1$. Each solution of the first equation in (16) with $v\equiv0$ may be written $z(t,x)=\sum_{k\geqslant1}[c_k^+e^{ik\pi t}+c_k^-e^{-ik\pi t}]\sin(k\pi x)$ so that $z(t+2,x)=z(t,x)$, i.e. (H_1) is satisfied with $T_p=2$. (H_2) is obvious as B_0 is bounded, while (H_3) clearly holds for some $T_0 \leqslant 2$ and $T_1 \leqslant 2$. We infer from Theorem 1 that (16) is exactly controllable in time T in $[H_0^1(0,1) \times L^2(0,1)]^2$. It follows from [5, Theorem 1.4] that system (16) is exactly controllable in the space $[(H^2(0,1) \cap H_0^1(0,1)) \times H_0^1(0,1)]^2$ with control inputs $v \in H_0^1(0,T;L^2(0,1))$. If (z,q,v) satisfies (16), then $(y,q,u) := (z_t,q,v_t)$ satisfies (15). Furthermore, since $v \in H_0^1(0,T;L^2(0,1))$, then $u \in L^2(0,T;L^2(0,1))$, and $(y(0),y_t(0)) = (z^1, \partial_x^2 z^0)$ ranges over $H_0^1(0,1) \times L^2(0,1)$ when (z^0,z^1) ranges over $[H^2(0,1) \cap H_0^1(0,1)] \times H_0^1(0,1)$. The same holds true for $(y(T),y_t(T))$. This proves the exact controllability of (15) in $\mathcal{H}$ in time T. $\square$

3.3. Cascade system of heat equations

Let ω be a nonempty open set in $(0,1)$, and let $a \in L^\infty(0,1)$ satisfy $|a(x)| > \varepsilon$ a.e. on $\mathcal{O}$, a nonempty open subset of $(0,1)$, for some $\varepsilon > 0$. From Theorem 4 and Kannai transform we shall derive a null controllability result for the following system of coupled heat equations:

$$y_t - y_{xx} = 1_\omega h, \qquad q_t - q_{xx} = \left|a(x)\right|^2 y \quad \text{in } (0,T) \times (0,1), \tag{17}$$

$$y(t,x) = q(t,x) = 0 \quad \text{on } (0,T) \times \{0,1\}, \qquad y(0,.) = y^0, \qquad q(0,.) = q^0 \quad \text{in } (0,1). \tag{18}$$

Theorem 5. *Pick any $T > 0$. Then for every $(y^0,q^0) \in [L^2(0,1)]^2$ there exists $h \in L^2(0,T,L^2(0,1))$ such that the corresponding solution to* (17)–(18) *satisfies $y(T,.) = q(T,.) = 0$.*

Proof. Considering a control h vanishing for say $0 < t < T/100$ and using the classical smoothing effect of the heat equation, we may as well assume that $(y^0,q^0) \in H_0^1(0,1) \times [H^2(0,1) \cap H_0^1(0,1)]$. From Theorem 4 we know that for any time $S \geqslant 4$ one may choose a control $u \in L^2(0,S,L^2(0,1))$ such that the solution (ψ,φ) to

$$\psi_{ss} - \psi_{xx} = 1_\omega u, \qquad \varphi_{ss} - \varphi_{xx} = \left|a(x)\right|^2 \psi \quad \text{in } (0,S) \times (0,1), \tag{19}$$

$$\psi(s,x) = \varphi(s,x) = 0 \quad \text{on } (0,S) \times \{0,1\}, \qquad \psi(0,.) = y^0, \qquad \psi_s(0,.) = 0, \qquad \varphi(0,.) = q^0, \qquad \varphi_s(0,.) = 0 \tag{20}$$

satisfies $\psi(S,.) = \psi_s(S,.) = \varphi(S,.) = \varphi_s(S,.) = 0$. Pick any $T > 0$ and a "null control" $\gamma \in C^\infty([0,T])$ for the fundamental solution of the heat equation in the space domain $(-S,S)$ (see [8]); that is, the solution to

$$\begin{cases} \rho_t - \rho_{ss} = 0 \quad \text{in } (0,T) \times (-S,S), \\ \rho(t,-S) = 0, \quad \rho(t,S) = \gamma(t) \quad \text{for } t \in (0,T), \quad \rho(0,.) = \delta_{s=0}, \end{cases} \tag{21}$$

satisfies $\rho(T,.) = 0$. For any $f : [0,S] \times [0,1] \to \mathbb{R}$, let $\tilde f(s,x) = f(s,x)$ if $s \geqslant 0$; $\tilde f(s,x) = f(-s,x)$ if $s < 0$. Define the Kannai transform of ψ and φ as $y(t,x) = \int_{-S}^{S} \tilde\psi(s,x)\rho(t,s)\,\mathrm{d}s$; $q(t,x) = \int_{-S}^{S} \tilde\varphi(s,x)\rho(t,s)\,\mathrm{d}s$. Then the pair (y,q) is a solution of (17)–(18) for $h(t,x) = \int_{-S}^{S} \tilde u(s,x)\rho(t,s)\,\mathrm{d}s$ and $y(T,.) = q(T,.) = 0$. $\square$

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