Partial Differential Equations

Non-local PDEs with a state-dependent delay term presented by Stieltjes integral

EDP non-locales avec terme à retards dépendants de l'état exprimé par une intégrale de Stieltjes

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A R T I C L E   I N F O

A B S T R A C T

Parabolic partial differential equations with state-dependent delays (SDDs) are investigated. The delay term presented by Stieltjes integral simultaneously includes discrete and distributed SDDs. The singular Lebesgue–Stieltjes measure is also admissible. The conditions for the corresponding initial value problem to be well-posed are presented. The existence of a compact global attractor is proved.

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R É S U M É

On étudie des équations aux dérivées partielles avec des retards dépendants de l'état (RDE). Le terme comportant les retards est exprimé par une intégrale de Stieltjes incluant des RDE discrets et distribués. Une mesure de Lebesgue–Stieltjes singulière est aussi admissible. On présente des conditions pour que le problème de Cauchy soit bien posé. On montre l'existence d'un attracteur global compact.

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Nous étudions des équations aux dérivées partielles avec retards. Dans beaucoup de systèmes d'évolution présents dans les modèles obtenues à partir d'applications diverses les retards sont souvent *dépendants de l'état* (RDE). La théorie de ces équations, plus spécialement des équations différentielles ordinaires, se développe rapidement et, à ce jour, de nombreux résultats profonds ont été obtenus, cf. [22–24,15,14] par exemple, et aussi l'article [9] qui fait un tour d'horizon de la question. Les EDP avec retards dépendants de l'état ont été étudiées pour la première fois dans [16,10,17].

Dans ce travail, contrairement aux recherches précédentes, nous considérons un modèle où deux types différents de RDE sont représentés simultanément par une intégrale de Stieltjes, les retards discrets et les retards distribués. Une mesure singulière de Lebesgue–Stieltjes est aussi admissible dans cette représentation. De plus, toutes les hypothèses sur les retards (voir les hypothèses (A1)–(A5)) permettent de considérer des dynamiques où le long d'une solution le nombre de valeurs de RDE discrets peut changer, les retards discrets et/ou distribués peuvent s'annuler, disparaître et réapparaître de nouveau. Cette propriété nous permet d'étudier des modèles où certaines parties de l'espace de phase sont décrites par des RDE purement discrets, d'autres par des retards distribués et il y a des parties qui nécessitent un type général (combiné) de

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retards. Une solution peut passer dans différents sous-ensembles à des moments différents. Ceci signifie que, non seulement les valeurs des retards peuvent dépendre de l'état, mais même le *type* du retard *dépend de l'état*. Nous étudions les solutions faibles et leurs propriétés asymptotiques (on montre l'existence d'un attracteur). Les résultats peuvent être appliqués à l'équation de Nicholson de diffusion de la population avec RDE [19].

Nous considérons l'équation aux dérivées partielles non-locale (1) avec un terme de retard dépendant de l'état F exprimé par intégrale de Stieltjes (2) avec la condition initiale (3). Nous utilisons la décomposition de toute mesure de Lebesgue–Stieltjes associée $g(\theta, \varphi)$ en une somme de trois mesures (4) : discrète $g_d(\theta, \varphi)$, absolument continue $g_{ac}(\theta, \varphi)$ et singulière, $g_s(\theta, \varphi)$. Par C on note l'espace $C \equiv C([-r, 0]; L^2(\Omega))$. Le premier résultat présenté est le suivant :

Lemme 1. *Supposons que la fonction b est lipschitzienne et vérifie $|b(s)| \leqslant C_1|s| + C_2, \forall s \in \mathbb{R}$, où $C_i \geqslant 0$ et f est mesurable et bornée. Sous les hypothèses* (A1)–(A3)*, l'application non-linéaire $F : C \to L^2(\Omega)$, définie par* (2)*, est continue.*

Comme conséquence on obtient le théorème suivant :

Théorème 1. *Sous les hypothèses du Lemme 1, le problème de Cauchy* (1)*,* (3) *admet une solution faible (globale) pour tout $\varphi \in C$.*

Il vient alors le théorème suivant :

Théorème 2. *Supposons que la fonction b est lipschitzienne et bornée, et f est mesurable et bornée. Sous les hypothèses* (A1)–(A5)*, le problème* (1)*,* (3) *admet une unique solution faible pour tout $\varphi \in C$. La solution dépend continument des conditions initiales, c'est à dire $\|\varphi^n - \varphi\|_C \to 0$ implique $\|u_t^n - u_t\|_C \to 0$ pour tout $t \geqslant 0$.*

En particulier ceci signifie que le problème de Cauchy (1), (3) est `bien posé` dans l'espace C au sens de J. Hadamard. Ce fait nous permet de définir un *semigroupe d'évolution* $S_t : C \to C$ par $S_t\varphi \equiv u_t$, où u est l'unique solution faible (1), (3). On obtient alors le résultat principal :

Théorème 3. *Supposons la fonction $b : \mathbb{R} \to \mathbb{R}$ lipschitzienne et bornée et la fonction $f : \Omega - \Omega \to \mathbb{R}$ bornée et mesurable. Les hypothèses* (A1)–(A5) *sont satisfaites. Alors le système dynamique (S_t, C) possède un attracteur global compact dans tous les espaces $C_\delta \equiv C([-r, 0]; D(A^\delta)), \forall \delta \in [0, \frac{1}{2})$.*

1. Introduction

We investigate parabolic partial differential equations (PDEs) with delay. Studying of this type of equations is based on the well-developed theories of the ordinary differential equations (ODEs) with delays [8,6,1] and PDEs without delays [7,11,12]. Under certain assumptions both types of equations describe a kind of dynamical systems that are infinite-dimensional, see [2,20,5] and references therein; see also [21,3,4] and the monograph [26] that are close to our work.

In many evolution systems arising in applications the presented delays are frequently *state-dependent* (SDDs). The theory of such equations, especially the ODEs, is rapidly developing and many deep results have been obtained up to now (see e.g. [22–24,15,14] and also the survey paper [9] for details and references).

The PDEs with state-dependent delays were first studied in [16,10,17]. An alternative approach to the PDEs with discrete SDDs is proposed in [18]. Approaches to equations with discrete and distributed SDDs are different. Even in the case of ODEs, the discrete SDD essentially complicates the study since, in general, the corresponding nonlinearity is not locally Lipschitz continuous on open subsets of the space of continuous functions, and familiar results on existence, uniqueness, and dependence of solutions on initial data and parameters from, say [8,6] fail (see [25] for an example of the non-uniqueness and [9] for more details).

In this work, in contrast to previous investigations, we consider a model where two different types of SDDs (discrete and distributed) are presented simultaneously (by Stieltjes integral). The singular Lebesgue–Stieltjes measure is also admissible. Moreover, all the assumptions on the delay (see (A1)–(A5) below) allow the dynamics when along a solution the number and values of discrete SDDs may change, the whole discrete and/or distributed delays may vanish, disappear and appear again. This property allows us to study models where some subsets of the phase space are described by equations with purely discrete SDDs, and others by equations with purely distributed SDDs, and there are subsets which need the general (combined) type of the delay. A solution could be in different subsets at different time moments. This property particularly means that not only the values of the delays are state-dependent, but the *type* of the delay is *state-dependent* as well. We study mild solutions and their asymptotic properties (the existence of compact global attractors in suitable phase-space is proved). The results could be applied to the diffusive Nicholson's blowflies equation with SDDs.

2. The model with state-dependent delay and basic properties

Consider the following non-local partial differential equation with a state-dependent delay term F presented by Stieltjes integral

$$\frac{\partial}{\partial t} u(t,x) + Au(t,x) + du(t,x) = \big(F(u_t)\big)(x),$$
(1)

with

$$\big(F(u_t)\big)(x) \equiv \int\limits_{-r}^{0} \left\{ \int\limits_{\Omega} b\big(u(t+\theta, y)\big) f(x-y)\,\mathrm{d}y \right\} \cdot \mathrm{d}g(\theta, u_t), \quad x \in \Omega,$$
(2)

where A is a densely-defined self-adjoint positive linear operator with domain $D(A) \subset L^2(\Omega)$ and compact resolvent, which means that $A : D(A) \to L^2(\Omega)$ generates an analytic semigroup, $\Omega \subset \mathbb{R}^{n_0}$ is a smooth bounded domain, $f : \Omega - \Omega \to R$ is a bounded measurable function, $b : \mathbb{R} \to \mathbb{R}$ stands for a locally Lipschitz map, $d \in \mathbb{R}$, $d \geqslant 0$, and the function $g : [-r, 0] \times C([-r, 0]; L^2(\Omega)) \to [0, r] \subset \mathbb{R}_+$ denotes *a state-dependent delay*. Let $C \equiv C([-r, 0]; L^2(\Omega))$. Norms defined on $L^2(\Omega)$ and C are denoted by $\| \cdot \|$ and $\| \cdot \|_C$, respectively, and $\langle \cdot, \cdot \rangle$ stands for the inner product in $L^2(\Omega)$. As usual for delay equations, we denote $u_t \equiv u_t(\theta) \equiv u(t+\theta)$ for $\theta \in [-r, 0]$.

We consider Eq. (1) with the initial condition

$$u|_{[-r,0]} = \varphi \in C \equiv C\big([-r, 0]; L^2(\Omega)\big).$$
(3)

We assume the following:

(A1) For any $\varphi \in C$, the function $[-r, 0] \ni g(\cdot, \varphi) \to \mathbb{R}$ is of bounded variation on $[-r, 0]$. The variation $V_{-r}^0 g$ of g is uniformly bounded i.e. $\exists M_{Vg} > 0 : \forall \varphi \in C \Rightarrow V_{-r}^0 g(\varphi) \leqslant M_{Vg}$.

It is well known that any Lebesgue–Stieltjes measure (associated with g) may be split into a sum of three measures: discrete, absolutely continuous and singular ones. We will denote the corresponding splitting of g as follows

$$g(\theta, \varphi) = g_d(\theta, \varphi) + g_{ac}(\theta, \varphi) + g_s(\theta, \varphi),$$
(4)

where $g_d(\theta, \varphi)$ is a step-function, $g_{ac}(\theta, \varphi)$ is absolutely continuous and $g_s(\theta, \varphi)$ is singular continuous functions (see [13] for more details). We will also denote the continuous part by $g_c \equiv g_{ac} + g_s$.

Now we assume:

(A2) For any $\theta \in [-r, 0]$, the functions g_{ac} and g_s are continuous with respect to their second coordinates i.e. $\forall \theta \in [-r, 0]$, $\forall \varphi^n, \varphi \in C : \|\varphi^n - \varphi\|_C \to 0 \ (n \to +\infty) \Rightarrow g_{ac}(\theta, \varphi^n) \to g_{ac}(\theta, \varphi)$ and $g_s(\theta, \varphi^n) \to g_s(\theta, \varphi)$.

Remark 1. We notice that a discrete state-dependent delay does not satisfy assumption (A2). More precisely, we may consider the discrete SDD $\eta : C \to [0, r]$ which is presented by the step-function $g(\theta, \varphi) = 0$ for $\theta \in [-r, -\eta(\varphi)]$ and $g(\theta, \varphi) = 1$ for $\theta \in (-\eta(\varphi), 0]$. It is easy to see that for any sequence $\{\varphi^n\} \subset C$, such that $\eta(\varphi^n) \to \eta(\varphi)$ and $\eta(\varphi^n) > \eta(\varphi)$ one has for the value $\theta_0 = -\eta(\varphi)$ that $g(\theta_0, \varphi^n) \equiv 1 \neq 0 \equiv g(\theta_0, \varphi)$, i.e. (A2) does not hold.

(A3) The step-function $g_d(\theta, \varphi)$ is continuous with respect to its second coordinate in the sense that discontinuities of $g_d(\theta, \varphi)$ at points $\{\theta_k\} \subset [-r, 0]$ satisfy the property: there are continuous functions $\eta_k : C \to [0, r]$ and $h_k : C \to R$ such that $\theta_k = -\eta_k(\varphi)$ and $h_k(\varphi)$ is the jump of g_d at point $\theta_k = -\eta_k(\varphi)$ i.e. $h_k(\varphi) \equiv g_d(\theta_k + 0, \varphi) - g_d(\theta_k - 0, \varphi)$.
Taking into account that g_d may, in general, have infinite number of points of discontinuity $\{\theta_k\}$, we assume that the series $\sum_k h_k(\varphi)$ converges absolutely and uniformly on any bounded subsets of C.

Remark 2. Assumption (A3) means that for any $\chi \in C$ one has $\Phi_d(\chi) \equiv \int_{-r}^{0} \chi(\theta)\,\mathrm{d}g_d(\theta, \varphi) = \sum_k \chi(\theta_k) \cdot h_k(\varphi) = \sum_k \chi(-\eta_k(\varphi)) \cdot h_k(\varphi)$.

Lemma 1. *Assume the function b is a Lipschitz map, satisfying $|b(s)| \leqslant C_1|s| + C_2, \forall s \in \mathbb{R}$ with $C_i \geqslant 0$ and f is measurable and bounded. Under assumptions* (A1)–(A3)*, the nonlinear mapping $F : C \to L^2(\Omega)$, defined by (2), is continuous.*

Remark 3. We emphasize that nonlinear map F is not Lipschitz in the presence of discrete SDDs (i.e. when $g \neq g_c$). The proof of Lemma 1 is based on the properties of the uniformly convergent series and the first Helly's theorem [13, p. 359].

3. Mild solutions and their properties

In our study we use the standard

Definition 1. A function $u \in C([-r, T]; L^2(\Omega))$ is called a mild solution on $[-r, T]$ of the initial value problem (1), (3) if it satisfies (3) and $u(t) = e^{-At}\varphi(0) + \int_0^t e^{-A(t-s)}\{F(u_s) - d \cdot u(s)\}\,\mathrm{d}s$, $t \in [0, T]$.

Theorem 1. *Under assumptions of Lemma* 1*, initial value problem* (1)*,* (3) *possesses a (global) mild solution for any* $\varphi \in C$.

The existence of a mild solution is a consequence of the continuity of $F : C \to L^2(\Omega)$, given by Lemma 1, which gives us the possibility to use the standard method based on the Schauder fixed point theorem (see e.g. [26, Theorem 2.1, p. 46]). The solution is also global (defined for all $t \geqslant -r$), see e.g. [26, Theorem 2.3].

To get the uniqueness of mild solutions we need the following additional assumptions:

(A4) The total variation of function $g_c \equiv g_{ac} + g_s$ satisfies the `Lipschitz condition`:

$$V_{-r}^0 \big[g_c(\cdot, \varphi) - g_c(\cdot, \psi) \big] \leqslant L_{V g_c} \|\varphi - \psi\|_C.$$

(A5) Discrete generating function g_d satisfies the following `uniform` condition:

$\exists \eta_{ign} > 0$ such that `all` η_k and h_k "`ignore`" values of $\varphi(\theta)$ for $\theta \in (-\eta_{ign}, 0]$, i.e.

$$\exists \eta_{ign} > 0: \forall \varphi^1, \varphi^2 \in C: \forall \theta \in [-r, -\eta_{ign}] \quad \Rightarrow \quad \varphi^1(\theta) = \varphi^2(\theta) \quad \Rightarrow \quad \eta_k(\varphi^1) = \eta_k(\varphi^2), h_k(\varphi^1) = h_k(\varphi^2).$$

Remark 4. Assumption (A5) is the natural generalization to the case of multiple discrete state-dependent delays of the ignoring condition introduced in [18]. For more details and examples see [18].

Theorem 2. *Assume the function* b *is a Lipschitz and bounded map,* f *is measurable and bounded. Under assumptions* (A1)–(A5)*, initial value problem* (1)*,* (3) *possesses a unique mild solution for any* $\varphi \in C$*. The solution is continuous with respect to initial data, i.e.* $\|\varphi^n - \varphi\|_C \to 0$ *implies* $\|u_t^n - u_t\|_C \to 0$ *for any* $t \geqslant 0$.

The proof of Theorem 2 is based on the Gronwall lemma, mean value theorem for the Stieltjes integral, properties of g_d due to condition (A5), and the Lebesgue–Fatou lemma [27, p. 32].

In the standard way we define an `evolution semigroup` $S_t : C \to C$ by the rule $S_t \varphi \equiv u_t$, where u is the unique mild solution of (1), (3).

Remark 5. The continuity of S_t with respect to time variable follows from Definition 1 (the solution is a continuous function $u \in C([-r, T]; L^2(\Omega))$). This and the continuity of S_t with respect to initial function (see Theorem 2) particularly mean that, under assumptions (A1)–(A5), the initial value problem (1), (3) is `well-posed` in the space C in the sense of J. Hadamard [7].

The last remark means that the pair (S_t, C) forms a `dynamical system` (for the definition see e.g. [2,20,5]). Following the line of argument given in [18, Theorem 2] we show that the dynamical system (S_t, C) generated by the initial value problem (1), (3) possesses a compact global attractor (for more details on attractors see, for example, [2,20,5]). More precisely, we have the following result:

Theorem 3. *Assume the function* $b : \mathbb{R} \to \mathbb{R}$ *is a Lipschitz and bounded map and* $f : \Omega - \Omega \to \mathbb{R}$ *is a bounded and measurable function. Let assumptions* (A1)–(A5) *be satisfied. Then the dynamical system* (S_t, C) *has a compact global attractor which is a compact set in all spaces* $C_\delta \equiv C([-r, 0]; D(A^\delta)), \forall \delta \in [0, \frac{1}{2})$.

The proof is based on the classical theorem on the existence of a compact global attractor for a dissipative and asymptotically compact semigroup [2,20,5] and technique developed in [18, Theorem 2].

As an application we can consider the diffusive Nicholson's blowflies equation (see e.g. [19]) with state-dependent delays, i.e. Eq. (1) where $-A$ is the Laplace operator with the Dirichlet boundary conditions, $\Omega \subset \mathbb{R}^{n_0}$ is a bounded domain with a smooth boundary, the nonlinear (birth) function b is given by $b(w) = p \cdot w e^{-w}$. Hence under assumptions (A1)–(A5), we conclude that the initial value problem (1) and (3) is well-posed in C and the dynamical system (S_t, C) has a compact global attractor.

References

[1] N.V. Azbelev, V.P. Maksimov, L.F. Rakhmatullina, Introduction to the Theory of Functional Differential Equations, Nauka, Moscow, 1991.
[2] A.V. Babin, M.I. Vishik, Attractors of Evolutionary Equations, North-Holland, Amsterdam, 1992.
[3] I.D. Chueshov, On a certain system of equations with delay, occuring in aeroelasticity, J. Soviet Math. 58 (1992) 385–390.
[4] I.D. Chueshov, A.V. Rezounenko, Global attractors for a class of retarded quasilinear partial differential equations, C. R. Acad. Sci. Paris, Ser. I 321 (1995) 607–612.
[5] I.D. Chueshov, Introduction to the Theory of Infinite-Dimensional Dissipative Systems, Acta, Kharkov, 1999 (in Russian); English transl.: Acta, Kharkov, 2002, http://www.emis.de/monographs/Chueshov.
[6] O. Diekmann, S. van Gils, S. Verduyn Lunel, H.-O. Walther, Delay Equations: Functional, Complex, and Nonlinear Analysis, Springer-Verlag, New York, 1995.
[7] J. Hadamard, Le problème de Cauchy et les èquations aux derivees partielles linéaires hyperboliques, Hermann, Paris, 1932.

[8] J.K. Hale, S.M. Verduyn Lunel, Theory of Functional Differential Equations, Springer-Verlag, New York, 1993.

[9] F. Hartung, T. Krisztin, H.-O. Walther, J. Wu, Functional differential equations with state-dependent delays: theory and applications, in: A. Canada, P. Drabek, A. Fonda (Eds.), Handbook of Differential Equations: Ordinary Differential Equations, vol. 3, Elsevier B.V., 2006.

[10] E. Hernandez, A. Prokopczyk, L. Ladeira, A note on partial functional differential equations with state-dependent delay, Nonlinear Anal. Real World Appl. 7 (4) (2006) 510–519.

[11] J.L. Lions, Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires, Dunod, Paris, 1969.

[12] J.L. Lions, E. Magenes, Problèmes aux Limites Non Homogénes et applications, Dunod, Paris, 1968.

[13] A.N. Kolmogorov, S.V. Fomin, Elements of Theory of Functions and Functional Analysis, Nauka, Moscow, 1968.

[14] T. Krisztin, A local unstable manifold for differential equations with state-dependent delay, Discrete Contin. Dyn. Syst. 9 (2003) 933–1028.

[15] J. Mallet-Paret, R.D. Nussbaum, P. Paraskevopoulos, Periodic solutions for functional-differential equations with multiple state-dependent time lags, Topol. Methods Nonlinear Anal. 3 (1) (1994) 101–162.

[16] A.V. Rezounenko, J. Wu, A non-local PDE model for population dynamics with state-selective delay: local theory and global attractors, J. Comput. Appl. Math. 190 (1–2) (2006) 99–113.

[17] A.V. Rezounenko, Partial differential equations with discrete and distributed state-dependent delays, J. Math. Anal. Appl. 326 (2) (2007) 1031–1045, see also detailed preprint, http://arxiv.org/pdf/math.DS/0503470, March 22, 2005.

[18] A.V. Rezounenko, Differential equations with discrete state-dependent delay: uniqueness and well-posedness in the space of continuous functions, Nonlinear Anal. 70 (2009) 3978–3986.

[19] J.W.-H. So, Y. Yang, Dirichlet problem for the diffusive Nicholson's blowflies equation, J. Differential Equations 150 (2) (1998) 317–348.

[20] R. Temam, Infinite Dimensional Dynamical Systems in Mechanics and Physics, Springer, Berlin/Heidelberg/New York, 1988.

[21] C.C. Travis, G.F. Webb, Existence and stability for partial functional differential equations, Trans. Amer. Math. Soc. 200 (1974) 395–418.

[22] H.-O. Walther, Stable periodic motion of a system with state dependent delay, Differential Integral Equations 15 (2002) 923–944.

[23] H.-O. Walther, The solution manifold and C^1-smoothness for differential equations with state-dependent delay, J. Differential Equations 195 (1) (2003) 46–65.

[24] H.-O. Walther, On a model for soft landing with state-dependent delay, J. Dynam. Differential Equations 19 (3) (2007) 593–622.

[25] E. Winston, Uniqueness of the zero solution for differential equations with state-dependence, J. Differential Equations 7 (1970) 395–405.

[26] J. Wu, Theory and Applications of Partial Functional Differential Equations, Springer-Verlag, New York, 1996.

[27] K. Yosida, Functional Analysis, Springer-Verlag, New York, 1965.