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Partial Differential Equations/Calculus of Variations

A class of existence results for the singular Liouville equation

Une classe de résultats d'existence pour l'équation de Liouville singulière

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ABSTRACT

We consider a class of elliptic PDEs on closed surfaces with exponential nonlinearities and Dirac deltas on the right-hand side. The study arises from abelian Chern–Simons theory in self-dual regime, or from the problem of prescribing the Gaussian curvature in presence of conical singularities. A general existence result is proved using global variational methods: the analytic problem is reduced to a topological problem concerning the contractibility of a model space, the so-called *space of formal barycenters*, characterizing the very low sublevels of a suitable functional.

RÉSUMÉ

Nous considérons une classe d'EDP elliptiques sur une surface compacte et sans bord, avec une nonlinéarité exponentielle et des masses de Dirac dans le membre de droite. Ce travail est motivé par l'étude d'équations de Chern–Simons abéliennes en régime auto-dual, ainsi que par le problème de la courbure gaussienne prescrite pour des surfaces avec singularités coniques. Nous démontrons un résultat général d'existence en utilisant des méthodes variationnels globales : le problème analytique est réduit à un problème topologique concernant la contractilité d'un espace modèle, l'espace des barycentres formels, qui caractérise les sous-niveaux très bas d'une fonctionnelle appropriée.

Version française abrégée

Dans ce travail nous étudions l'équation

$$-\Delta_g u = \rho\left(\frac{h(x)e^{2u}}{\int_\Sigma h(x)e^{2u}\,\mathrm{d}V_g} - 1\right) - 2\pi\sum_{i=1}^{m}\alpha_j(\delta_{p_j} - 1)$$

sur une surface compacte (Σ, g). ρ est un paramètre positif et $h : \Sigma \to \mathbb{R}$ est une fonction positive et régulière. Les p_j sont des points dans Σ et les a_j sont des nombres réels. L'étude de l'équation précedénte est motivée par la théorie des équations de Chern–Simons abéliennes en régime auto-dual ainsi si que par le problème de la courbure gaussienne prescrite pour des surface avec singularités coniques. En utilisant des inégalités du type Moser–Trudinger et des fonctions-test appropriées, le problème est réduit à l'étude de la topologie d'un ensemble approprié. Cet ensemble coïncide avec les barycentres formels

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sur la surface, où les multiplicités sont liées à des poids appropriés aux points p_j. Si cet ensemble est contractile nous pouvons démontrer l'existence de solutions sous des hypothèses génériques pour ρ.

1. Introduction

This article deals with the study of a class of equations having the form

$$-\Delta_g u = \rho\left(\frac{h(x)e^{2u}}{\int_\Sigma h(x)e^{2u}\,\mathrm{d}V_g} - 1\right) - 2\pi\sum_{i=1}^{m}\alpha_j(\delta_{p_j} - 1) \tag{1}$$

on a Riemannian 2-manifold (Σ, g). Here ρ is a positive parameter, $h : \Sigma \to \mathbb{R}$ a smooth positive function, the α_j's are real numbers, $p_j \in \Sigma$ are some fixed points and the subscripts $(\cdot)_g$ refer to the metric g.

This equation arises in different contexts and has therefore been object of intense study in the last decades. First of all, it naturally appears in the study of multivortices in the Electroweak Theory by Glashow–Salam–Weinberg [18], where u can be interpreted as the logarithm of the absolute value of the wave function and the points p_j's are the *vortices*, where the wave function vanishes. Secondly, this class of equations has proved to be relevant in many other physical frames, such as the study of the statistical mechanics of point vortices in the mean field limit [17,6,7] and the abelian Chern–Simons Theory, as discussed in [15,25]. Eq. (1) also admits a geometric interpretation, which is however much more evident if we restrict our attention to the *regular* case

$$-\Delta_g u = \rho\left(\frac{h(x)e^{2u}}{\int_\Sigma h(x)e^{2u}\,\mathrm{d}V_g} - 1\right). \tag{2}$$

Indeed, if we consider a conformal change of metric of our manifold, say $\hat{g} = e^{2w}g$, the Gaussian curvature transforms according to the law

$$K_{\hat{g}} = e^{-2w}(-\Delta_g w + K_g), \tag{3}$$

and so, if K_g is constant, the problem of asking whether g can be conformal to a metric with Gaussian curvature a given function $K_{\hat{g}}$ is equivalent to (2). This is the famous *Kazdan–Warner problem* [16], also known as *Nirenberg problem* in the special case when (Σ, g) is just the standard sphere. While moving from Problem (2) to Problem (1), the extra terms can be viewed as singularities in the Gauss curvature corresponding to a *local conical structure*, as can be justified via an extension of the Gauss–Bonnet formula (see [26]). With a non-constant function $h(x)$, we are considering the problem of prescribing the Gaussian curvature through conformal metrics. This is a question dual to the *uniformization problem*, where on general surfaces one looks for metrics with constant Gaussian curvature (in higher dimension this is the *Yamabe problem*, see the survey [19]).

Mainly due to its natural geometric appeal, Eq. (2) has been widely studied in the last forty years. Because of its strong nonlinearity, it has been tackled by means of two classes of techniques both leading to quite sophisticated results: on the one hand, topological methods relying on the degree theory by Leray–Schauder (see [9,10,20], and also [11] for the singular case), on the other purely variational methods based on an improvement of the Moser–Trudinger inequality. Indeed (2) is the Euler–Lagrange equation associated to the C^1 functional

$$J_\rho(u) = \int_\Sigma |\nabla_g u|^2\,\mathrm{d}V_g + 2\rho\int_\Sigma u\,\mathrm{d}V_g - \rho\log\int_\Sigma h(x)e^{2u}\,\mathrm{d}V_g,$$

defined on the Sobolev space $H^1(\Sigma, g)$. The weak form of the *Moser–Trudinger inequality*

$$\log\int_\Sigma e^{2(u-\bar{u})}\,\mathrm{d}V_g \leqslant \frac{1}{4\pi}\int_\Sigma |\nabla_g u|^2\,\mathrm{d}V_g + C_{\Sigma,g} \quad u \in H^1(\Sigma, g) \tag{4}$$

guarantees that J_ρ is well-defined on $H^1(\Sigma, g)$ for any value of $\rho \in \mathbb{R}$. Moreover, J_ρ is lower semi-continuous with respect to the weak topology of that space and so, since (4) gives *coercivity* of J_ρ if $\rho < 4\pi$, we immediately get existence of *critical points* for this range of values and the corresponding solvability of (2). It is clear that such critical points are *global minima* for J_ρ. Such a direct variational approach does not apply to the case $\rho \geqslant 4\pi$ as can be seen by exhibiting explicit examples (i.e. standard bubbling functions). This is the reason why more sophisticated tools are needed in this regime. Such a variational structure can be recovered for the singular problem by performing a change of variables like

$$u \mapsto u + \sum_{j=1}^{m}\alpha_j\log d_g(x, p_j) \tag{5}$$

so that our singular equation can be rewritten as

$$-\Delta_g u + \rho = \rho \frac{\tilde{h}(x)e^{2u}}{\int_\Sigma \tilde{h}(x)e^{2u}\,\mathrm{d}V_g} \quad \text{on } \Sigma, \tag{6}$$

where $\tilde{h} \geqslant 0$ and $\tilde{h}(x) \simeq d_g(x, p_j)^{2\alpha_j}$ near p_j. To do this formally, consider for $j \in \{1, \dots, m\}$ the Green functions $G_{p_j} \in \mathcal{D}'(\Sigma)$ defined by the problem $\Delta_g G_{p_j} = 2\pi(\delta_{p_j} - 1)$: it is a standard fact [1] that such distributions exist and moreover that they are actually smooth away from the singularity, that is $G_{p_j} \in C^\infty(\Sigma \setminus p_j)$, $j \in \{1, \dots, m\}$. Hence, by performing the substitution $\tilde{u} := u - \sum_{j=1}^m \alpha_j G_{p_j}$ we really get Eq. (6) for $\tilde{u}$ with the modified datum $\tilde{h}(x) = h(x)e^{2\sum_{j=1}^m \alpha_j G_{p_j}}$. Due to the asymptotics of G_{p_j}, so to the fact that $G_{p_j} \simeq \log d_g(x, p_j)$ near p_j, we also find that $\tilde{h} \simeq d_g(x, p_j)^{2\alpha_j}$ near each p_j, as previously requested. Notice also that, as a result, (6) is nothing but the Euler–Lagrange equation for modified functional

$$J_{\rho,\underline{\alpha}}(u) = \int_\Sigma |\nabla_g u|^2\,\mathrm{d}V_g + 2\rho \int_\Sigma u\,\mathrm{d}V_g - \rho \log \int_\Sigma \tilde{h}(x)e^{2u}\,\mathrm{d}V_g, \quad u \in H^1(\Sigma, g) \tag{7}$$

(where $\underline{\alpha} = (\alpha_1, \dots, \alpha_m) \in \mathbb{N}^m$) and so we can study existence questions by global variational methods. Similar in spirit to (4) is the following Troyanov inequality (see [26]), valid for $\alpha > -1$ and $p \in \Sigma$:

$$\log \int_\Sigma d_g(x, p)^{2\alpha} e^{2(u-\bar{u})}\,\mathrm{d}V_g \leqslant \frac{1}{4\pi \min\{1, 1+\alpha\}} \int_\Sigma |\nabla_g u|^2\,\mathrm{d}V_g + C_{\alpha,\Sigma,g}, \quad u \in H^1(\Sigma, g) \tag{8}$$

that guarantees coercivity of $J_{\rho,\underline{\alpha}}$ for $\rho < 4\pi \min_{j=1,\dots,m}(1+\alpha_j)$. Again, it is seen by definition of suitable *singular* bubbling functions that this value is sharp and so the topology of low sublevels of the functionals needs to be studied with refined strategies.

2. Improved inequalities and statement of the main result

The core of the variational approach to Problem (2) is represented by an improvement of the Moser–Trudinger inequality first obtained by Chen and Li in [12]: the constant $1/(4\pi)$ can be improved whenever u is in some sense concentrated in well-separated regions on Σ.

Lemma 1. *Let l be a positive integer, let $\Omega_1, \dots, \Omega_{l+1}$ be disjoint subsets of Σ satisfying a separation condition* $\mathrm{dist}(\Omega_i, \Omega_j) > \delta_0$ *for any $i \neq j$ and some $\delta_0 > 0$ and consider any $\gamma_0 \in (0, \frac{1}{l+1})$. Then, for any $\tilde{\epsilon} > 0$, there exists a constant $C := C(\Sigma, g, l, \delta_0, \gamma_0, \tilde{\epsilon})$ such that*

$$\log \int_\Sigma e^{2(u-\bar{u})}\,\mathrm{d}V_g \leqslant \frac{1}{4(l+1)\pi - \tilde{\epsilon}} \int_\Sigma |\nabla_g u|^2\,\mathrm{d}V_g + C, \tag{9}$$

for all functions $u \in H^1(\Sigma)$ satisfying

$$\frac{\int_{\Omega_i} e^{2u}\,\mathrm{d}V_g}{\int_\Sigma e^{2u}\,\mathrm{d}V_g} \geqslant \gamma_0, \quad \forall i \in \{1, \dots, l+1\}. \tag{10}$$

This result gives important information on the structure of sublevels of J_ρ or, more precisely, on the concentration phenomena characterizing the functions belonging to sufficiently low sublevels. Thanks to a sort of covering argument presented in [14] it is possible to show that if for instance $\rho \in (4\pi, 8\pi)$ and u belongs to a sufficiently low sublevel of J_ρ, then it has to be *conformally concentrated* on a single region, and this is precisely what happens for the bubbling functions. This is the general case:

Proposition 1. *Assume $\rho \in (4k\pi, 4(k+1)\pi)$ for some $k \geqslant 1$. Then, for any $\epsilon > 0$ and $r > 0$ there exists a sufficiently large positive constant $L := L(\epsilon, r)$ such that for every $u \in H^1(\Sigma, g)$ with $J_\rho(u) \leqslant -L$ there are k points on Σ (say $p_{1,u}, \dots, p_{k,u}$) so that*

$$\frac{\int_{\Sigma \setminus \bigcup_{i=1}^k B_r(p_{i,u})} e^{2u}\,\mathrm{d}V_g}{\int_\Sigma e^{2u}\,\mathrm{d}V_g} < \epsilon. \tag{11}$$

By some hard work, this leads to the definition of a projection operator from the very low sublevels of J_ρ (say J_ρ^{-L}, $L \gg 1$) to the space of *formal barycenters*

$$\Sigma_k := \left\{ \sum_{i=1}^k t_i \delta_{p_i} \colon \sum_{i=1}^k t_i = 1,\ t_i \geqslant 0,\ p_i \in \Sigma\ \forall i \in \{1, \dots, k\} \right\} \tag{12}$$

and indeed this is proved to be a homotopy equivalence (see [14] and [22]). As a consequence, the non-contractibility of J_ρ^{-L} (for L big enough) is proved by studying the homology of the model space Σ_k.

The first step of our study was then a similar improved inequality that is based on both (4) and (8) and is proved by means of cut-off functions and a suitable spectral decomposition.

Lemma 2. *Let $n \in \mathbb{N}$ and let $I \subseteq \{1, \dots, m\}$ with $n + \operatorname{card}(I) > 0$, where $\operatorname{card}(I)$ denotes the cardinality of a set. Assume there exists $r > 0$, $\delta_0 > 0$ and pairwise distinct points $\{q_1, \dots, q_n\} \subseteq \Sigma \setminus \{p_1, \dots, p_m\}$ such that:*

- *For any couple $\{a, b\} \subseteq \{q_1, \dots, q_n \cup (\bigcup_{i\in I} p_i)\}$ with $a \neq b$ one has $\operatorname{dist}_g(B_r(a), B_r(b)) \geqslant \delta_0$;*
- *for any $a \in \{q_1, \dots, q_m\}$ one has $\operatorname{dist}_g(p_i, B_r(a)) \geqslant \delta_0$ for any $i \in \{1, \dots, m\} \setminus I$;*

and consider any $\gamma_0 \in (0, \frac{1}{n+\operatorname{card}(I)})$.

Then, for any $\tilde{\epsilon} > 0$ there exists a constant $C := C(\Sigma, g, n, I, r, \delta_0, \gamma_0, \tilde{\epsilon})$ such that

$$\log \int_\Sigma \tilde{h} e^{2(u-\bar{u})} \, dV_g \leqslant \frac{1}{4\pi (n + \sum_{i\in I}(1+\alpha_i) - \tilde{\epsilon})} \int_\Sigma |\nabla_g u|^2 \, dV_g + C, \tag{13}$$

for all functions $u \in H^1(\Sigma)$ satisfying

$$\frac{\int_{B_r(a)} \tilde{h} e^{2u} \, dV_g}{\int_\Sigma \tilde{h} e^{2u} \, dV_g} \geqslant \gamma_0, \quad \forall a \in \left\{ q_1, \dots, q_n \cup \left(\bigcup_{i\in I} p_i \right) \right\}. \tag{14}$$

Following the guide of the regular case, we are then led to claim the structure of the very low sublevels of the functional $J_{\rho,\underline{\alpha}}$.

Definition 1. Given a point $q \in \Sigma$ we define its weighted cardinality as follows:

$$\chi(q) = \begin{cases} 1 + \alpha_j & \text{if } q = p_j \text{ for some } j = 1, \dots, m; \\ 1 & \text{otherwise.} \end{cases}$$

The cardinality of any finite set of (pairwise distinct) points on Σ is obtained extending χ by additivity.

This enables us to easily describe selection rules to determine admissibility conditions for specific barycentric configurations in dependence on the values of the α_j's and ρ.

Definition 2. Suppose all the parameters $\rho, \alpha_1, \dots, \alpha_m$ are fixed. We define the corresponding space of formal barycenters as follows

$$\Sigma_{\rho,\underline{\alpha}} = \left\{ \sum_{q_j \in J} t_j \delta_{q_j} \colon \sum_{q_j\in J} t_j = 1,\ t_j \geqslant 0,\ q_j \in \Sigma\ \pi \chi(J) < \rho \right\}. \tag{15}$$

We expect that $\Sigma_{\rho,\underline{\alpha}}$ is homotopy equivalent to the very low sublevels of the functional $J_{\rho,\underline{\alpha}}$: we are indeed able to define a non-trivial projection operator $\Pi : J_{\rho,\underline{\alpha}}^{-L} \to \Sigma_{\rho,\underline{\alpha}}$ (for some appropriate choice of L) and an immersion $\Phi : \Sigma_{\rho,\underline{\alpha}} \to J_{\rho,\underline{\alpha}}^{-L}$ so that the composition $\Pi \cdot \Phi : \Sigma_{\rho,\underline{\alpha}} \hookleftarrow$ is (homotopy) equivalent to the identity on the same space. Although this fact does not imply the homotopic equivalence, it is however sufficient for our purposes. For the sake of clarity, assume from now onwards that the parameter L is fixed so that Π may be defined and the gauge of Φ (which is, to be precise, a one-parameter family of immersion operators $\Phi_\lambda, \lambda \in \mathbb{R}_{>0}$) is fixed so that $\Phi = \Phi_{\hat{\lambda}}$ takes values in $J_{\rho,\underline{\alpha}}^{-2L}$. Moreover, let us write φ_σ instead of $\Phi(\sigma)$ for $\sigma \in \Sigma_{\rho,\underline{\alpha}}$.

Remark 1.

- The definition of the immersion Φ was first done in [23] and, naively, associates to a given measure of $\Sigma_{\rho,\underline{\alpha}}$ a corresponding generalized multipole bubbling functions. This is a delicate point because some sort of smart interpolation between the regular and the singular bubbling functions is needed.
- In order to define such Π we follow the line of [14] but need to deal with the very sophisticated topological structure of $\Sigma_{\rho,\underline{\alpha}}$ as a stratified set.

These are the tools we need to prove our main result, which is essentially an existence theorem for non-critical values of ρ (depending on $\underline{\alpha}$), see the denominators in (13).

Definition 3. We say that $\overline{\rho} \in \mathbb{R}_{>0}$ is a singular value for Problem (1) if it is representable as follows:

$$\overline{\rho} = 4\pi n + 4\pi \sum_{i \in I} (1 + \alpha_i) \tag{16}$$

for some $n \in \mathbb{N}$ and $I \subseteq \{1, \ldots, m\}$ (possibly empty) satisfying $n + \mathrm{card}(I) > 0$. The set of singular values will be denoted by $\mathfrak{S} = \mathfrak{S}(\underline{\alpha})$.

Theorem 1. *Suppose that the parameters $\underline{\alpha} \in (-1, 0)^m$ and $\rho \in \mathbb{R}_{>0} \setminus \mathfrak{S}$ are such that the set $\Sigma_{\rho,\underline{\alpha}}$ is not contractible with respect to the topology of $C^1(\Sigma, g)^*$. Then Problem (1) admits a solution $u \in H^1(\Sigma, g)$. Moreover, we have that $u = v + \sum_{j=1}^m \alpha_j G_{p_j}$ with $G_{p.}$ the Green functions defined above and $v \in C^{\gamma}(\Sigma, g)$ for any $\gamma \in [0, \gamma_0)$ with $\gamma_0 \in (0, 1)$ solving Eq. (6).*

3. Outline of the proof

Our plan is to use a general min–max scheme in the form of a suitable *topological cone* construction.

1. **Minimax scheme.** We define the topological cone over $\Sigma_{\rho,\underline{\alpha}}$ as follows:

$$\Theta_{\rho,\underline{\alpha}} = \left(\Sigma_{\rho,\underline{\alpha}} \times [0, 1]\right) \big/ \left(\Sigma_{\rho,\underline{\alpha}} \times \{1\}\right), \tag{17}$$

where we are identifying all the points in $\Sigma_{\rho,\underline{\alpha}} \times \{1\}$. Consequently, we consider the family of continuous maps

$$\mathcal{H}_{\rho,\underline{\alpha}} = \left\{\mathfrak{h} : \Theta_{\rho,\underline{\alpha}} \to H^1(\Sigma, g)\colon\ \mathfrak{h}(\sigma) = \varphi_\sigma \text{ for every } \sigma \in \Sigma_{\rho,\underline{\alpha}}\right\}, \tag{18}$$

and then the number

$$\overline{\mathcal{H}}_{\rho,\underline{\alpha}} = \inf_{\mathfrak{h} \in \mathcal{H}} \sup_{\sigma \in \Theta} J_{\rho,\underline{\alpha}}\left(\mathfrak{h}(\sigma)\right). \tag{19}$$

We claim that under the assumption of Theorem 1 one has $\overline{\mathcal{H}}_{\rho,\underline{\alpha}} \geqslant -L$. It is worth proving first that the class $\mathcal{H}_{\rho,\underline{\alpha}}$ is not empty. To this aim, notice that the map $\underline{\mathfrak{h}}(\sigma, t) = (1-t)\varphi_\sigma$, $(\sigma, t) \in \Sigma_{\rho,\underline{\alpha}}$ belongs to $\mathcal{H}_{\rho,\underline{\alpha}}$.
Concerning the lower bound on the min–max value, we just need to argue by contradiction. If $\overline{\mathcal{H}}_{\rho\underline{\alpha}} < -L$, then there should be a map $\mathfrak{h}$ such that its image $\mathfrak{h}(\Theta_{\rho,\underline{\alpha}})$ (which is a topological cone in $H^1(\Sigma, g)$) would be in $J^{-L}_{\rho,\underline{\alpha}}$. As a consequence, the composite map

$$t \to \Pi\left(\mathfrak{h}(\sigma, t)\right), \quad \sigma \in \Sigma_{\rho,\underline{\alpha}} \tag{20}$$

would be a homotopy equivalence between $\Pi(\mathfrak{h}(0, \sigma)) = \Pi \cdot \Phi(\sigma)$ and a constant map. On the other hand, we know that the function $\Pi \cdot \Phi(\sigma)$ is homotopic to the identity in $\Sigma_{\rho,\underline{\alpha}}$ and hence, by composition the space $\Sigma_{\rho,\underline{\alpha}}$ would be contractible, a contradiction. Hence we deduce $\overline{\mathcal{H}}_{\rho\underline{\alpha}} \geqslant -L$.
2. **Existence on a dense set.** We then show that the scheme outlined in the previous step leads to existence for a dense set of ρ's (in a suitable neighborhood of a fixed value). This relies on a monotonicity trick by Struwe [24] and exploited also in [13].
3. **Conclusion via blow-up analysis.** To conclude, we need to build a sequence of solutions for approximating values of ρ and extract a converging sub-sequence via blow-up analysis. This may be done thanks to the results in [4], generalized in [3] specifically for the case of negative parameters, and extending previous estimates in [5,21,20].

4. Final remark

Theorem 1 reduces an analytic problem (existence for (1)) to a topological problem (contractibility of $\Sigma_{\rho,\underline{\alpha}}$). As a result, we would hope such a space to be non-contractible for *generic* values of the parameters ρ and $\underline{\alpha}$. Indeed, this seems to be true. More precisely, we conjecture that the cases of contractibility of $\Sigma_{\rho,\underline{\alpha}}$ can be classified so that, very surprisingly, such topological question may be reduced to test a simple algebraic relation involving ρ and $\underline{\alpha}$. An outline of our conjecture would first require a description of the stratifies structure of $\Sigma_{\rho,\underline{\alpha}}$ and is for brevity postponed to another work. It would be also interesting to find some general connection with the topological argument developed in [2].

Full details about the proof of Theorem 1 are contained in the forthcoming paper [8].

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