C. R. Acad. Sci. Paris, Ser. I 349 (2011) 115–117

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Combinatorics/Algebra

Are the hyperharmonics integral? A partial answer via the small intervals containing primes

Les hyperharmoniques sont-ils entiers? Une réponse partielle via les petits intervalles contenant des nombres premiers

Rachid Aït Amrane [a], Hacène Belbachir [b]

[a] *ESI/École nationale supérieure d'informatique, BP 68M, Oued Smar, 16309, El Harrach, Alger, Algeria*
[b] *USTHB, faculté de mathématiques, BP 32, El Alia, 16111 Bab Ezzouar, Alger, Algeria*

ARTICLE INFO

Article history:
Received 26 October 2010
Accepted 23 December 2010
Available online 5 January 2011

Presented by the Editorial Board

ABSTRACT

In a recent work, the authors have used Bertrand's postulate to give a partial answer to the conjecture of Mező which says that the hyperharmonic numbers – iterations of partial sums of harmonic numbers – are not integers. In this Note, using small intervals containing prime numbers, we prove that a great class of hyperharmonic numbers are not integers.

RÉSUMÉ

Dans un travail antérieur, les auteurs ont utilisé le postulat de Bertrand pour répondre, partiellement, à la conjecture de Mező selon laquelle les nombres hyperharmoniques – itérations de sommes partielles de nombres harmoniques – ne sont pas des entiers. Dans cette Note, nous montrons qu'une grande classe de nombres hyperharmoniques ne sont pas des entiers en utilisant les petits intervalles contenant des nombres premiers.

1. Introduction

In 1915, Taeisinger proved that the harmonic number $H_n := 1 + \frac{1}{2} + \cdots + \frac{1}{n}$, except H_1, is never an integer (see [10]). In [3, pp. 258–259], Conway and Guy introduced, for a positive integer r, the hyperharmonic numbers by the inductive relation:

$$H_n^{(1)} := H_n \quad \text{and} \quad H_n^{(r)} = \sum_{k=1}^{n} H_k^{(r-1)} \quad (r > 1),$$

where $H_n^{(r)}$ is called the nth hyperharmonic number of order r, this number can also be expressed in terms of binomial coefficients as follows, see [3],

$$H_n^{(r)} = \binom{n+r-1}{r-1}(H_{n+r-1} - H_{r-1}). \tag{1}$$

E-mail addresses: r_ait_amrane@esi.dz, raitamrane@gmail.com (R. Aït Amrane), hbelbachir@usthb.dz, hacenebelbachir@gmail.com (H. Belbachir).

1631-073X/$ – see front matter
doi:10.1016/j.crma.2010.12.015

I. Mező, in [7], proved that $H_n^{(r)}$, for $r=2$ and 3, except in case $H_1^{(r)}$, is never an integer. In his proof, I. Mező used the reduction modulo the prime number 2. For $n \geqslant 2$, he conjectured that $H_n^{(r)}$ is never an integer for $r \geqslant 4$. The authors, in [1], gave an answer to this conjecture in case $r = 2, 3$ and 4, and for any $r > 4$ when the integers $n+1, n+2, \dots, n+r-4$ are not primes. The principal tool used in the proof was Bertrand's postulate which was conjectured in 1845, and assures the existence of a prime number in the interval $]n, 2n-2[$ when $n \geqslant 4$. This conjecture, proved by Chebyshev in 1852, lead to much interest from many mathematicians, their interest was about finding the smallest intervals containing a prime number. More general results, see for example [9,5,8,2], have been proved, others conjectured (essentially using the Riemann hypothesis). In this work, we use some of these results to prove that another class of hyperharmonic numbers are not integers.

This new approach which is about reducing the interval in a multiplicative way, i.e. considering intervals of the form $]x, sx]$, leads us to better results than those we got in our preceding paper [1]. In Section 2, we give the main result establishing the non-integerness of hyperharmonic numbers for a large class of n and r. Section 3 is devoted to use results that give effective bounds of intervals that contain prime numbers, this allows us to establish the non-integerness of the hyperharmonic numbers $H_n^{(r)}$ when $5 \leqslant r \leqslant 25$.

2. Main result

The basic result of this section is a direct consequence of G. Giordano's theorem, see [4, Theorem 1]:

Theorem 2.1. *Let $s \in]1,2]$, then there exists a real number x_0 such that for any $x > x_0$, the interval $]x, sx]$ contains at least one prime number.*

In his proof, G. Giordano gave explicitly x_0. Here:

$$x_0 = \max\left(e^{22}, \frac{1}{s}\exp\left(-b + \sqrt[2]{b^2 - 4ac}\right)/2a\right),$$

where $a = 1 - \frac{1}{s}$, $b = -(1-\frac{1}{s})\ln s - 0.008(1+\frac{1}{s})$ and $c = 0.008 \ln s$.

Now, here is our main result:

Theorem 2.2. *For any $s \in]1,2[$, there is a prime number P_0 such that for any integers r and n with $5 \leqslant r \leqslant (2-s)P_0 + 2$ and $n \geqslant P_0$, the hyperharmonic number $H_n^{(r)}$ is never an integer.*

Proof. Let $s \in]1,2[$, by Theorem 2.1, there is x_0 such that for any $x > x_0$ there exists a prime number Q verifying $x < Q \leqslant sx$. Denote by P_0 the smallest prime number strictly greater than x_0. Let r, n be such that $5 \leqslant r \leqslant (2-s)P_0 + 2$ and $n \geqslant P_0$, and assume $H_n^{(r)} \in \mathbb{N}$. We have

$$\begin{aligned} H_n^{(r)} &= \frac{(n+1)(n+2)\cdots(n+r-1)}{(r-1)!}\left(\frac{1}{r} + \frac{1}{r+1} + \cdots + \frac{1}{r+n-1}\right) \\ &= \frac{(n+1)(n+2)\cdots(n+r-1)}{(r-1)!}\left(H_n + \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+r-1} - H_{r-1}\right). \end{aligned}$$

Put

$$E_{r,n} := (r-1)!H_n^{(r)} + (r-1)!\binom{n+r-1}{r-1}H_{r-1} - (n+1)(n+2)\cdots(n+r-1)(H_{n+r-1} - H_n).$$

Therefore

$$E_{r,n} = (n+1)(n+2)\cdots(n+r-1)\left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right).$$

Since $H_n^{(r)}$ is integral, $E_{r,n}$ also is. Let P be the greatest prime $\leqslant n$, then $P \geqslant P_0$ and multiplying the last equality by $\frac{n!}{P}$ we get:

$$\frac{n!}{P}E_{r,n} = \frac{(n+r-1)!}{P}\left(1 + \cdots + \frac{1}{P} + \cdots + \frac{1}{n}\right).$$

Hence

$$\frac{n!}{P}E_{r,n} - \frac{(n+r-1)!}{P}\left(1 + \frac{1}{2} + \cdots + \frac{1}{P-1} + \frac{1}{P+1} + \cdots + \frac{1}{n}\right) = \frac{(n+r-1)!}{P^2}.$$

The left-hand side of the equality is an integer. Assume that the right-hand side of the equality is an integer, then P must divides one of the factors $(n+1), (n+2), \ldots, (n+r-1)$, hence there exists an integer j such that $1 \leqslant j \leqslant r-1$ and $n+j = 2P$ (more precisely $j \geqslant 4$). By the hypothesis we have

$$\begin{aligned} r \leqslant (2-s)P_0 + 2 \quad &\Rightarrow \quad j \leqslant r-1 \leqslant (2-s)P_0 + 1 \leqslant (2-s)P + 1 \\ &\Rightarrow \quad sP \leqslant 2P - j + 1 = n+1 \\ &\Rightarrow \quad sP \leqslant n+1 \end{aligned}$$

hence the existence of a prime Q such that $x_0 < P < Q \leqslant sP < n+1$, and this contradicts the fact that P is the greatest integer less than or equal to n. □

3. Effective bounds

Observe that the real number x_0 given in Theorem 2.1 is very large, a thing which explains the particular use of results that give effective bounds of intervals that contain prime numbers.

The basic result for this section is the following theorem (see [6]):

Theorem 3.1. *For any integer $n \geqslant 25$, there is a prime number Q such that $n < Q < (1 + \frac{1}{5})n$.*

We deduce the following first result:

Theorem 3.2. *The hyperharmonic number $H_n^{(r)}$, $n \geqslant 2$ and $5 \leqslant r \leqslant 25$, is not an integer.*

Proof. For $n = 2, 3$ and 4, see [1].

For $5 \leqslant r \leqslant 25$ and $n \leqslant 28$, a computation with Maple allows to check that $H_n^{(r)}$ is not an integer.

For $5 \leqslant r \leqslant 25$ and $n \geqslant 29$, use Theorem 2.2 with $s = \frac{6}{5}$ and $P_0 = 29$. □

The same procedure allows to prove the following theorem:

Theorem 3.3.

(i) *The hyperharmonic number $H_n^{(r)}$ is not integral when r, n are integers such that $5 \leqslant r \leqslant 2.010.761$ and $n \geqslant 2.010.881$.*
(ii) *The hyperharmonic number $H_n^{(r)}$ is not integral for any integers r, n such that $5 \leqslant r \leqslant 10.726.904.664$ and $n \geqslant 10.726.905.041$.*

Proof. The results in this theorem use the following:

(i) For any $n \geqslant 2.010.760$, there is a prime Q such that $n < Q < (1 + \frac{1}{16597})n$, see [9, Theorem 12].
(ii) For any $n \geqslant 10.726.905.041$, there is a prime number Q such that $n < Q \leqslant (1 + \frac{1}{28.313.999})n$, see [8, Theorem 3]. □

Remark 1. In the same context, we can use the table given in [8, p. 11] to show that another infinity of hyperharmonic numbers are not integers.

Remark 2. It appears that we can obtain a remarkable improvement using explicit bounds assuming the Riemann Hypothesis. For example, Theorem 1 of [8] says that for $x \geqslant 2$ the interval $]x - \frac{8}{5}\sqrt{x}\log x, x]$ contains a prime number.

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