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Mathematical Analysis

Universal p-adic series

Séries universelles p-adiques

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ABSTRACT

We establish the analogue of the original Fekete Theorem in the context of p-adic analysis.

RÉSUMÉ

On met en évidence l'existence de séries universelles à coefficients p-adiques en généralisant le théorème original de Fekete à $\mathbb{Q}_p$.

Version française abrégée

Le premier résultat sur les séries universelles est sans doute le théorème de Fekete [10] qui assure l'existence d'une série formelle $\sum_{n=1}^{+\infty} a_n x^n$ à coefficients réels vérifiant la propriété suivante : pour toute fonction g continue sur l'intervalle $[-1,1]$ avec $g(0)=0$, il existe une suite croissante d'entiers $(\lambda_n)_{n\geqslant 1}$ telle que

$$\sup_{x\in[-1,1]}\left|\sum_{n=1}^{\lambda_k} a_n x^n - g(x)\right| \to 0, \quad \text{lorsque } k\to+\infty.$$

Le phénomène d'universalité, aussi élégant que surprenant, a intéressé de nombreux mathématiciens au cours du vingtième siècle. On renvoie le lecteur par exemple aux références [12,9,3,8] ou aux excellentes synthèses de Grosse-Erdmann [5] et de Kahane [6]. Le sujet est encore en plein essor. Récemment Bayart, Grosse-Erdmann, Nestoridis et Papadimitropoulos ont développé une théorie abstraite des séries universelles qui, d'une part, unifie presque tous les résultats connus et, d'autre part, permet d'en obtenir de nouveaux [2]. Ils montrent que l'existence de séries universelles est équivalente à une condition d'approximation polynomiale bien construite. En se plaçant sur des espaces de Baire, ils obtiennent même des ensembles G_δ et denses de séries universelles. Cette théorie est écrite dans le cas des séries à coefficients dans $\mathbb{R}$ ou $\mathbb{C}$, ce qui englobe tous les cas connus. Dans cette note, on obtient l'analogue du théorème original de Fekete dans le corps $\mathbb{Q}_p$ des nombres p-adiques, où p est un nombre premier. On note $|.|_p$ la valeur absolue p-adique et on rappelle que $\mathbb{Q}_p$ est le complété de $\mathbb{Q}$ pour cette valeur absolue. L'ensemble $\mathbb{Z}_p$ des entiers p-adiques, défini par $\mathbb{Z}_p=\{x\in\mathbb{Q}_p\colon\ |x|_p\leqslant 1\}$, joue le rôle de l'intervalle $[-1,1]$. On sait que $\mathbb{Z}_p$ est un ensemble compact. On renvoie le lecteur à [4] ou [11] pour se familiariser avec l'analyse p-adique. On a alors l'énoncé suivant, en appelant topologie cartésienne de $\mathbb{Q}_p^{\mathbb{N}}$ la topologie définie par la distance $d(a,b)=\sum_{n=1}^{+\infty} 2^{-n}(|a_n-b_n|_p/(1+|a_n-b_n|_p))$, pour $a=(a_n)_{n\geqslant 1}, b=(b_n)_{n\geqslant 1}\in\mathbb{Q}_p^{\mathbb{N}}$ (l'espace métrique sous-jacent est alors complet).

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Théorème 0.1. *Il existe une suite $(a_n)_{n\geqslant 1}$ de $\mathbb{Q}_p^{\mathbb{N}}$ telle que, pour toute fonction g continue sur $\mathbb{Z}_p$ à valeurs dans $\mathbb{Q}_p$ vérifiant $g(0)=0$, il existe une suite croissante d'entiers $(\lambda_n)_{n\geqslant 1}$ telle que l'on ait*

$$\sup_{x\in\mathbb{Z}_p}\left|\sum_{n=1}^{\lambda_k}a_nx^n-g(x)\right|_p\to 0,\quad \textit{lorsque } k\to+\infty.$$

L'ensemble des telles suites $(a_n)_{n\geqslant 1}$ est G_δ et dense dans $\mathbb{Q}_p^{\mathbb{N}}$ muni de la topologie cartésienne de $\mathbb{Q}_p^{\mathbb{N}}$ et contient un sous-espace vectoriel dense privé de 0.

La preuve suit un schéma apparu dans [8] et maintenant assez classique pour la mise en évidence de phénomènes d'universalité. En particulier, on utilise de manière essentielle que $\mathbb{Q}$ est dense dans $\mathbb{Q}_p$ ainsi que la version ultramétrique du théorème classique d'approximation de Weierstrass (voir [7,11] ou [1], Théorème 1.4). Tout cela laisse à penser que la théorie abstraite des séries universelles reste valable si on remplace $\mathbb{R}$ ou $\mathbb{C}$ par un corps complet, qui contient un sous-ensemble dénombrable dense.

1. Introduction

According to [10] before 1914 Fekete proved that there exists a formal real power series $\sum_{n=1}^{+\infty}a_nx^n$ such that for every continuous function g on the set $\{x\in\mathbb{R}\colon |x|\leqslant 1\}$ that vanishes at 0, there exists an increasing sequence $(\lambda_n)_{n\geqslant 1}$ of positive integers such that

$$\sup_{x\in[-1,1]}\left|\sum_{n=1}^{\lambda_k}a_nx^n-g(x)\right|\to 0,\quad \text{as } k\to+\infty.$$

This result is the first example of universal series. Since then many authors dealt with the universality (see for instance [12,9,3,8]). We refer the reader to the excellent surveys of Grosse-Erdmann [5] and Kahane [6]. Recently Bayart, Grosse-Erdmann, Nestoridis and Papadimitropoulos developed an abstract framework for the theory of universal series, which covers most of the existing results and from which they deduce new nice statements [2]. They restricted their study to universal series with coefficients in $\mathbb{R}$ or $\mathbb{C}$. The purpose of this paper is to establish the analogue of the original Fekete Theorem in the context of p-adic analysis. Thus we will furnish the first example of universal series with coefficients which do not belong to $\mathbb{R}$ or $\mathbb{C}$.

In the following we use the standard notation of p-adic analysis. Let p be a prime number. We denote by $\mathbb{Q}_p$ the field of p-adic numbers endowed with the non-archimedean absolute value $|.|_p$. Let us recall that for each $x\in\mathbb{Q}_p$, we have $|x|_p=p^{-v_p(x)}$, where the integer $v_p(x)$ is the p-adic valuation extended to $\mathbb{Q}_p$. In other words, for each $x\in\mathbb{Q}_p$, $x\neq 0$, there exists an integer $n\in\mathbb{Z}$ such that $|x|_p=p^{-n}$ (we refer the reader to [4], Lemma 3.3.1). It is well known that $\mathbb{Q}_p$ is the completion of $\mathbb{Q}$ with respect to $|.|_p$. Thus $\mathbb{Q}_p$ is complete with respect to $|.|_p$ and there is an inclusion $\mathbb{Q}\hookrightarrow\mathbb{Q}_p$ whose image is dense in $\mathbb{Q}_p$ [4] or [11]. These properties are analogous to the ones obtained when we consider $\mathbb{R}$ as the completion of $\mathbb{Q}$ with respect to the usual absolute value. Moreover a famous theorem of Ostrowski asserts that every non-trivial absolute value on $\mathbb{Q}$ is equivalent (i.e. defines the same topology) to one of the absolute value $|.|_p$, where p is a prime number, or the usual absolute value [4]. It says that we have at our disposal all the absolute values on $\mathbb{Q}$. Let us introduce $\mathbb{Z}_p$ the ring of integers in $\mathbb{Q}_p$, i.e. $\mathbb{Z}_p=\{x\in\mathbb{Q}_p\colon |x|_p\leqslant 1\}$. We know that $\mathbb{Z}_p$ is a compact set [4] and let us consider $C(\mathbb{Z}_p,\mathbb{Q}_p)$ the space of all continuous functions on $\mathbb{Z}_p$. Our main result is the following:

Theorem 1.1. *There exists a sequence $(a_n)_{n\geqslant 1}$ in $\mathbb{Q}_p^{\mathbb{N}}$ such that, for every continuous function $g\in C(\mathbb{Z}_p,\mathbb{Q}_p)$ with $g(0)=0$, there exists an increasing sequence $(\lambda_n)_{n\geqslant 1}$ of positive integers such that*

$$\sup_{x\in\mathbb{Z}_p}\left|\sum_{n=1}^{\lambda_k}a_nx^n-g(x)\right|_p\to 0,\quad \textit{as } k\to+\infty.$$

The set of such sequences $(a_n)_{n\geqslant 1}$ is G_δ and dense in $\mathbb{Q}_p^{\mathbb{N}}$ endowed with its cartesian topology and it contains a dense vector subspace, apart from 0.

Let us recall that the cartesian topology here is defined by the metric $d(a,b)=\sum_{n=1}^{+\infty}2^{-n}(|a_n-b_n|_p/(1+|a_n-b_n|_p))$, for $a=(a_n)_{n\geqslant 1}, b=(b_n)_{n\geqslant 1}\in\mathbb{Q}_p^{\mathbb{N}}$.

2. Auxiliaries results

First of all we need to prove the following approximation lemma:

Lemma 2.1. *Let $m \geqslant 0$ be a fixed natural number and let $\varepsilon > 0$ be a real number. For any continuous function $h \in C(\mathbb{Z}_p, \mathbb{Q}_p)$ with $h(0) = 0$, there exists a polynomial P with coefficients in $\mathbb{Q}_p$ such that*

$$\sup_{x \in \mathbb{Z}_p} \left| x^m P(x) - h(x) \right|_p < \varepsilon.$$

Proof. Since h is a continuous function that vanishes at 0, one find an integer $N \geqslant 1$ so that $|h(x)|_p < \varepsilon$ for $|x|_p < p^{-N}$. Let us consider the function $f(x)$ defined by $f(x) = h(x)/x^m$ for $p^{-N} \leqslant |x|_p \leqslant 1$ and $f(x) = p^{m(N+1)} h(x)$ for $|x|_p < p^{-N}$. Since the p-adic absolute value is discrete, the inequality $|x|_p < p^{-N}$ implies $|x|_p \leqslant p^{-(N+1)}$ and we easily derive that the function f is continuous on the compact set $\mathbb{Z}_p$. We apply the non-archimedean version of the classical Weierstrass Approximation Theorem (see [7,11] or [1], Theorem 1.4) to find a polynomial $P \in \mathbb{Q}_p[x]$ satisfying $\sup_{x \in \mathbb{Z}_p} |P(x) - f(x)|_p < \varepsilon$. Therefore, for $p^{-N} \leqslant |x|_p \leqslant 1$, we have

$$\left| x^m P(x) - h(x) \right|_p = |x|_p^m \left| P(x) - h(x)/x^m \right|_p < \varepsilon.$$

Moreover, the ultrametric inequality gives, for $|x|_p \leqslant p^{-(N+1)}$,

$$\left| x^m P(x) - h(x) \right|_p \leqslant \max\left(|x|_p^m \left| P(x) \right|_p, \left| h(x) \right|_p \right).$$

First it suffices to remember that $|h(x)|_p < \varepsilon$ for $|x|_p \leqslant p^{-(N+1)}$. On the other hand, one has, for $|x|_p \leqslant p^{-(N+1)}$,

$$|x|_p^m \left| P(x) \right|_p \leqslant p^{-m(N+1)} \max\left(\left| P(x) - p^{m(N+1)} h(x) \right|_p, \left| p^{m(N+1)} h(x) \right|_p \right) < \varepsilon.$$

This completes the proof. □

Next we denote by $\mathcal{U}$ the set of sequences $(a_n)_{n \geqslant 1}$ in $\mathbb{Q}_p^{\mathbb{N}}$ which satisfy the approximation property of Theorem 1.1. Let f_j, $j = 1, 2, \ldots,$ be an enumeration of all polynomials having coefficients with rational coordinates without constant term. For any integers $j, s, l,$ with $j \geqslant 1$, $s \geqslant 1$ and $l \geqslant 0$, we denote by $E(j, s, l)$ the set

$$E(j, s, l) = \left\{ (a_n)_{n \geqslant 1} \in \mathbb{Q}_p^{\mathbb{N}}\colon \sup_{x \in \mathbb{Z}_p} \left| \sum_{n=1}^{l} a_n x^n - f_j(x) \right|_p < p^{-s} \right\}.$$

Lemma 2.2. *For every integer $j \geqslant 1$, $s \geqslant 1$ and $l \geqslant 0$, the set $E(j, s, l)$ is open in $\mathbb{Q}_p^{\mathbb{N}}$ endowed with its cartesian topology.*

Proof. Let $a = (a_n)_{n \geqslant 1} \in E(j, s, l)$. Then we have

$$\sup_{x \in \mathbb{Z}_p} \left| \sum_{n=1}^{l} a_n x^n - f_j(x) \right|_p < p^{-s}.$$

We set $\varepsilon < 2^{-l}/p^s + 1$. Let us consider $b = (b_n)_{n \geqslant 1}$ so that $d(a, b) < \varepsilon$. We claim that $b \in E(j, s, l)$. Indeed we have, for $|x|_p \leqslant 1$,

$$\left| \sum_{i=1}^{l} b_i x^i - f_j(x) \right|_p \leqslant \max\left(\left| \sum_{i=1}^{l} (b_i - a_i) x^i \right|_p, \left| \sum_{i=1}^{l} a_i x^i - f_j(x) \right|_p \right) \leqslant \max\left(\max_{i=1,\ldots,l} |b_i - a_i|_p, \left| \sum_{i=1}^{l} a_i x^i - f_j(x) \right|_p \right)$$

and for $i = 1, \ldots, l$, we easily derive $|b_i - a_i|_p < \varepsilon/(2^{-i} - \varepsilon) \leqslant \varepsilon/(2^{-l} - \varepsilon) < p^{-s}$. □

Lemma 2.3. *For every integer $j \geqslant 1$ and $s \geqslant 1$, the set $\bigcup_{l \geqslant 0} E(j, s, l)$ is open and dense in $\mathbb{Q}_p^{\mathbb{N}}$ endowed with its cartesian topology.*

Proof. Let $\varepsilon > 0$ and $b = (b_n)_{n \geqslant 1} \in \mathbb{Q}_p^{\mathbb{N}}$. We seek $l \geqslant 0$ and $a = (a_n)_{n \geqslant 1} \in \mathbb{Q}_p^{\mathbb{N}}$ such that $a \in E(j, s, l)$ and $d(a, b) < \varepsilon$. Thus let us choose a natural number k so that $\sum_{i \geqslant k+1} 2^{-i} < \varepsilon$. We set $h(x) = f_j(x) - \sum_{i=1}^{k} b_i x^i$. By applying Lemma 2.1 with $m = k + 1$ we find a polynomial $P(x) = \sum_{i=0}^{N} c_i x^i \in \mathbb{Q}_p[x]$ satisfying

$$\sup_{x \in \mathbb{Z}_p} \left| x^{k+1} P(x) + \sum_{i=1}^{k} b_i x^i - f_j(x) \right|_p < p^{-s}.$$

Therefore we set $l = N + k + 1$ and $a_i = b_i$, for $i = 1, \ldots, k$ and $a_i = c_{i-k-1}$ for $i = k + 1, \ldots, N + k + 1$. This completes the proof. □

3. Proof of the main result

The proof follows now classical arguments introduced by [8]. Observe that the set of all polynomials with coefficients from $\mathbb{Q}$ without constant term is countable and dense in the space of all functions $f \in C(\mathbb{Z}_p, \mathbb{Q}_p)$ that vanish at 0 endowed with the norm $\|f\| = \sup_{\mathbb{Z}_p} |f|_p$ according to the ultrametric Weierstrass Approximation Theorem (see [7,11] or [1], Theorem 1.4). Now the set $\mathcal{U}$ can be written as follows

$$\mathcal{U} = \bigcap_{j\geqslant 1} \bigcap_{s\geqslant 1} \bigcup_{l\geqslant 0} E(j,s,l).$$

The proof is similar to the proof of Lemma 2.2 in [8]. Combining Lemma 2.2 with Lemma 2.3 and Baire's Theorem in the metrizable complete space $\mathbb{Q}_p^{\mathbb{N}}$ we deduce that the set $\mathcal{U}$ is a G_δ and dense set. To obtain a dense vector subspace apart from 0, it suffices to notice that one can choose the integer $l \geqslant N + k + 1$ in the proof of Lemma 2.3 to belong to any increasing sequence of $\mathbb{N}$ and to follow very closely the proof of Theorem 1, (4) $\Rightarrow$ (5) of [2].

Remark 1. In fact we can construct a universal p-adic series with coefficients in $\mathbb{Q}$. To see this, let us consider a universal series $\sum_{n\geqslant 1} a_n x^n$ with $a_n \in \mathbb{Q}_p$, $n = 1, 2, \ldots$ Since $\mathbb{Q}$ is dense in $\mathbb{Q}_p$, for every integer $n \geqslant 1$ one can find b_n in $\mathbb{Q}$ so that $|b_n - a_n|_p < p^{-n}$. Observe that $|(b_n - a_n)x^n|_p \to 0$, as $n \to +\infty$, for $x \in \mathbb{Z}_p$. Therefore $w(x) = \sum_{n\geqslant 1} (b_n - a_n)x^n$ is a convergent power series on $\mathbb{Z}_p$ and the function $x \mapsto w(x)$ is continuous on $\mathbb{Z}_p$ (see [4], Lemma 4.4.1). It is also easy to check that the series $\sum_{n\geqslant 1} b_n x^n = w(x) + \sum_{n\geqslant 1} a_n x^n$ satisfies the universal approximation property of Theorem 1.1.

Remark 2. A careful examination of the abstract theory of universal series of [2] shows that it remains valid replacing $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ by a complete field which contains a countable and dense subset.

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