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Partial Differential Equations

On nondegeneracy of solutions to $SU(3)$ Toda system

Sur la nondégénérescence de solutions de $SU(3)$ système de Toda

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ABSTRACT

We prove that the solution to the following $SU(3)$ Toda system

$$\begin{cases} \Delta u + 2e^u - e^v = 0, \quad \Delta v - e^u + 2e^v = 0 \quad \text{in } \mathbb{R}^2, \\ \int_{\mathbb{R}^2} e^u < \infty, \quad \int_{\mathbb{R}^2} e^v < \infty, \end{cases}$$

is *nondegenerate*, i.e., the kernel of the associated linearized operator is exactly eight-dimensional.

RÉSUMÉ

On montre que pour toute solution de $SU(3)$ système de Toda suivant $\Delta u + 2e^u - e^v = 0$, $\Delta v - e^u + 2e^v = 0$ dans $\mathbb{R}^2$, $\int_{\mathbb{R}^2} e^u < \infty$, $\int_{\mathbb{R}^2} e^v < \infty$, le noyau de l'opérateur linéarisé associé est exactement de dimension huit, i.e., ce qu'on appelle la *nondégénérescence*.

Version française abrégée

On étudie la nondégénérescence des solutions de $SU(3)$ système de Toda suivant, Éq. (1), où $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$ est le Laplacien usuel de $\mathbb{R}^2$. Le système (1) est une généralisation naturelle de l'équation de Liouville, Éq. (2).

L'équation de Liouville et le système de Toda apparait dans de nombreux problèmes en physique et en géométrie. Dans la théorie de Chern–Simons, l'équation de Liouville est liée aux modèles abeliens, et le système de Toda est lié aux modèles non-abeliens. Voir les livres de Dunne [2], Yang [10] pour contexte en physique. Le modèle $SU(3)$ de Chern–Simons a été étudié dans beaucoup de travaux, voir par exemple Jost et Wang [3], Jost, Lin et Wang [5], Li et Li [7], Malchiodi et Ndiaye [8], Ohtsuka et Suzuki [9] et les références citées dedans.

En utilisant les résultats de géométrie algébrique, Jost et Wang [4] ont classifié toutes les solutions de (1). Plus précisement, en dimension deux, toutes les solutions de (1) s'écrivent sous la forme suivante des Éqs. (3) et (4). Ici la variable $z = x_1 + ix_2 \in \mathbb{C}$, les paramètres $a_1 > 0$, $a_2 > 0$, et $b = b_1 + ib_2$, $c = c_1 + ic_2$, $d = d_1 + id_2 \in \mathbb{C}$. On note que dans la représentation précédente, il existe huit paramètres $(a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2) \in \mathbb{R}^8$.

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Sur les études du comportement asymptotique des solutions explosives de $SU(3)$ système de Toda, une question essentielle est ce qu'on appelle la *nondégénérescence* de solution. Plus précisement, on a besoin de comprendre les éléments du noyau de l'opérateur linéarisé associé, i.e., le système linéaire, Éq. (5), où (u,v) est un couple solution de (1), donnée par (3) et (4). Certainement le noyau est au moins de dimension huit, car toute la différentiation de solution (u,v) par rapport à ces huit paramètres satisfont (5).

Le théorème suivant montre en effet qu'il n'y a plus d'autre éléments dans le noyau. Cela signifie que la solution (u,v) est *nondégénérée*.

Théorème 0.1. *Soit* (φ_1,φ_2) *solution de* (5)*. Supposons qu'il existe* $\tau\in(0,1)$ *tel que*

$$\left|\varphi_1(x)\right|\leqslant C\left(1+|x|\right)^{\tau},\qquad \left|\varphi_2(x)\right|\leqslant C\left(1+|x|\right)^{\tau}\quad \textit{pour tout } x\in\mathbb{R}^2.$$

Alors $\binom{\varphi_1}{\varphi_2}$ *est une combinaison linéaire de*

$$\begin{pmatrix}\partial_{a_1}u\\ \partial_{a_1}v\end{pmatrix},\quad \begin{pmatrix}\partial_{a_2}u\\ \partial_{a_2}v\end{pmatrix},\quad \begin{pmatrix}\partial_{b_1}u\\ \partial_{b_1}v\end{pmatrix},\quad \begin{pmatrix}\partial_{b_2}u\\ \partial_{b_2}v\end{pmatrix},\quad \begin{pmatrix}\partial_{c_1}u\\ \partial_{c_1}v\end{pmatrix},\quad \begin{pmatrix}\partial_{c_2}u\\ \partial_{c_2}v\end{pmatrix},\quad \begin{pmatrix}\partial_{d_1}u\\ \partial_{d_1}v\end{pmatrix},\quad \begin{pmatrix}\partial_{d_2}u\\ \partial_{d_2}v\end{pmatrix}.$$

Pour la démonstration de la nondégénérescence, on constate que pour l'équation de Liouville, le problème linéarisé devient $\Delta\varphi+e^u\varphi=0$ dans $\mathbb{R}^2$. À une transformation conforme près, on peut supposer que u est radiale. Ainsi on peut utiliser la séparation des variables pour obtenir la nondégénérescence de la solution u (voir Lemme 2.3 de Chen et Lin [1]). Ici on traite un système et on doit procéder différemment. Premièrement, on ne peut pas trouver transformation conforme pour que tout couple solution (u,v) soit radiale. Deuxièmement, même si (u,v) est radiale, le nouveau système est encore trop compliqué à étudier. Pour surmonter ces difficultés, on utilise les *invariants* du système (1). En appliquant ces invariants, on va obtenir des invariants du système linéarisé qui nous permettent de montrer le théorème.

D'ailleurs, nous pensons que notre méthode pourrait être utile pour traiter le $SU(N+1)$ système de Toda. Le problème majeur est de savoir comment obtenir des invariants d'ordre supérieur comme dans ce travail.

1. Introduction

Of concern is the nondegeneracy of solutions of the following two-dimensional $SU(3)$ Toda system

$$\begin{cases}\Delta u+2e^u-e^v=0,\quad \Delta v-e^u+2e^v=0\quad \text{in } \mathbb{R}^2,\\ \int_{\mathbb{R}^2} e^u<\infty,\quad \int_{\mathbb{R}^2} e^v<\infty,\end{cases}\tag{1}$$

where $\Delta=\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}$ is the usual Euclidean Laplacian in $\mathbb{R}^2$. System (1) is a natural generalization of the Liouville equation

$$\Delta u+e^u=0\quad \text{in } \mathbb{R}^2,\qquad \int_{\mathbb{R}^2} e^u<\infty.\tag{2}$$

The Liouville equation (2) and the Toda system (1) arise in many physical models and geometrical problems. In Chern–Simons theories, the Liouville equation is related to Abelian models, while the Toda system is related to non-Abelian models. We refer to the books by Dunne [2], Yang [10] for physical backgrounds. The $SU(3)$ Chern–Simons model has been studied in many papers. We refer to Jost and Wang [3], Jost, Lin and Wang [5], Li and Li [7], Malchiodi and Ndiaye [8], Ohtsuka and Suzuki [9] and the references therein.

Using algebraic geometry results, Jost and Wang [4] classified all solutions to (1). More precisely, when $N=2$, all solutions to (1) can be written as follows:

$$u(z)=\log\frac{4(a_1^2a_2^2+a_1^2|2z+c|^2+a_2^2|z^2+2bz+bc-d|^2)}{(a_1^2+a_2^2|z+b|^2+|z^2+cz+d|^2)^2},\tag{3}$$

$$v(z)=\log\frac{16a_1^2a_2^2(a_1^2+a_2^2|z+b|^2+|z^2+cz+d|^2)}{(a_1^2a_2^2+a_1^2|2z+c|^2+a_2^2|z^2+2bz+bc-d|^2)^2},\tag{4}$$

where the variable $z=x_1+ix_2\in\mathbb{C}$, the parameters $a_1>0$, $a_2>0$ are real numbers and $b=b_1+ib_2\in\mathbb{C}$, $c=c_1+ic_2\in\mathbb{C}$, $d=d_1+id_2\in\mathbb{C}$. Note that in the above representation there are eight parameters $(a_1,a_2,b_1,b_2,c_1,c_2,d_1,d_2)\in\mathbb{R}^8$.

In the study of the blow-up behaviors for solutions of $SU(3)$-Toda system, a crucial question is the *nondegeneracy* of solutions. More precisely, we need to study the elements in the kernel of the associated linearized operator, i.e. the following linear system

$$\begin{cases} \Delta\varphi_1 + 2e^u\varphi_1 - e^v\varphi_2 = 0 & \text{in } \mathbb{R}^2, \\ \Delta\varphi_2 - e^u\varphi_1 + 2e^v\varphi_2 = 0 & \text{in } \mathbb{R}^2. \end{cases} \tag{5}$$

Here (u, v) is a solution to (1) given by (3)–(4). Certainly the kernel is at least eight-dimensional, since any differentiation of (u, v) with respect to the eight parameters satisfies (5).

The following theorem shows that the kernel is exactly eight-dimensional, which means that the solution (u, v) is *nondegenerate*.

Theorem 1.1. *Let* (φ_1, φ_2) *satisfy* (5)*. Assume that*

$$|\varphi_1| \leqslant C\big(1+|x|\big)^\tau, \qquad |\varphi_2| \leqslant C\big(1+|x|\big)^\tau \quad \text{for } x \in \mathbb{R}^2 \text{ and some } \tau \in (0, 1).$$

Then $\binom{\varphi_1}{\varphi_2}$ *belongs to the following linear space*

$$\operatorname{span}\left\{\begin{pmatrix}\partial_{a_1}u\\ \partial_{a_1}v\end{pmatrix}, \begin{pmatrix}\partial_{a_2}u\\ \partial_{a_2}v\end{pmatrix}, \begin{pmatrix}\partial_{b_1}u\\ \partial_{b_1}v\end{pmatrix}, \begin{pmatrix}\partial_{b_2}u\\ \partial_{b_2}v\end{pmatrix}, \begin{pmatrix}\partial_{c_1}u\\ \partial_{c_1}v\end{pmatrix}, \begin{pmatrix}\partial_{c_2}u\\ \partial_{c_2}v\end{pmatrix}, \begin{pmatrix}\partial_{d_1}u\\ \partial_{d_1}v\end{pmatrix}, \begin{pmatrix}\partial_{d_2}u\\ \partial_{d_2}v\end{pmatrix}\right\}.$$

In the case of nondegeneracy of solutions of Liouville equation (2), the problem becomes to a single linear equation

$$\Delta\varphi + e^u\varphi = 0 \quad \text{in } \mathbb{R}^2. \tag{6}$$

Using conformal transformations, one can assume that u is radially symmetric. Then one can use the separation of variables to obtain the nondegeneracy result. See Lemma 2.3 of Chen and Lin [1]. Here we are dealing with a system. Firstly, we cannot find a conformal transformation to transform any solution (u, v) to radially symmetric solution. Second, even if (u, v) is radially symmetric, the new system is still too complicated to study. To overcome these difficulties, we employ the *invariants* of the system (1), see Section 2 below. Using the invariants of (1), we also obtain invariants for system (5) and thus prove Theorem 1.1.

For convenience, the language of complex variable is used in this note. We refer $\bar z = x_1 - ix_2$ to the usual conjugate of $z = x_1 + ix_2 \in \mathbb{C}$. In addition, $U_z := \partial_z U = \frac{1}{2}(\frac{\partial U}{\partial x_1} - i\frac{\partial U}{\partial x_2})$, $U_{\bar z} := \partial_{\bar z} U = \frac{1}{2}(\frac{\partial U}{\partial x_1} + i\frac{\partial U}{\partial x_2})$. Recall that $\Delta = 4\partial_{z\bar z}$. All functions and equations discussed in the sequel are defined in the whole plane $\mathbb{C}$. Notation C is a generic constant which may be different from line to line.

We believe that our method may be used to deal with the general case $SU(N+1)$. The major problem is how to obtain higher-order invariants as in Section 2.

2. Invariants of (1)

In this section, we derive some invariants for (1). For more discussions, we refer to Section 5.5 of the book by Leznov and Saveliev [6]. For any solution (u, v) of (1), let us define the following transformation in whole $\mathbb{C}$,

$$U(z, \bar z) = \frac{2u}{3} + \frac{v}{3} - \log 4, \qquad V(z, \bar z) = \frac{u}{3} + \frac{2v}{3} - \log 4. \tag{7}$$

Then the Toda system (1) can be rewritten as

$$\begin{cases} U_{z\bar z} + e^{2U-V} = 0, \quad V_{z\bar z} + e^{2V-U} = 0 \quad \text{in } \mathbb{C}, \\ \displaystyle\int_{\mathbb{R}^2} e^{2U-V} < \infty, \quad \int_{\mathbb{R}^2} e^{2V-U} < \infty. \end{cases} \tag{8}$$

We prove now some preliminary lemmas.

Lemma 2.1. *We have, in whole* $\mathbb{C}$*, that* $U_{zz} + V_{zz} - U_z^2 - V_z^2 + U_zV_z \equiv 0$, $U_{\bar z\bar z} + V_{\bar z\bar z} - U_{\bar z}^2 - V_{\bar z}^2 + U_{\bar z}V_{\bar z} \equiv 0$.

Proof. The proof is a straightforward calculation. We only prove the first identity because the second one can be dealt with similarly.

Let $f(z, \bar z) = U_{zz} + V_{zz} - U_z^2 - V_z^2 + U_zV_z$.

A direct computation and (8) show that in whole $\mathbb{C}$,

$$U_{zz\bar z} = -e^{2U-V}(2U_z - V_z), \qquad V_{zz\bar z} = -e^{2V-U}(2V_z - U_z),$$
$$\big(-U_z^2\big)_{\bar z} = 2U_z e^{2U-V}, \qquad \big(-V_z^2\big)_{\bar z} = 2V_z e^{2V-U}, \qquad (U_zV_z)_{\bar z} = -e^{2U-V}V_z - e^{2V-U}U_z.$$

Thus it holds that $f_{\bar z} \equiv 0$ in $\mathbb{C}$. Since f is smooth and goes to 0 at infinity by (3), (4) and (7), we have that, by Liouville's theorem, $f \equiv 0$ in $\mathbb{C}$. The first identity is then concluded.

Simply exchanging $\bar z$ and z in the above proof leads to the second identity. The proof is then complete. □

Lemma 2.2. *We have*

$$U_{zzz} - 3U_zU_{zz} + U_z^3 \equiv 0, \qquad V_{zzz} - 3V_zV_{zz} + V_z^3 \equiv 0, \tag{9}$$
$$U_{\bar z\bar z\bar z} - 3U_{\bar z}U_{\bar z\bar z} + U_{\bar z}^3 \equiv 0, \qquad V_{\bar z\bar z\bar z} - 3V_{\bar z}V_{\bar z\bar z} + V_{\bar z}^3 \equiv 0. \tag{10}$$

Proof. Since the proofs of (9) and (10) are similar, we will only check the former. For convenience, we denote that

$$f_1(z,\bar z) = U_{zzz} - 3U_zU_{zz} + U_z^3, \qquad f_2(z,\bar z) = V_{zzz} - 3V_zV_{zz} + V_z^3.$$

We *claim* that $f_{1,\bar z} \equiv 0$ and $f_{2,\bar z} \equiv 0$. In fact, a direct calculation gives that

$$U_{zzz\bar z} = -\left(e^{2U-V}\right)_{zz} = -\left[e^{2U-V}(2U_z - V_z)\right]_z = e^{2U-V}\left(-4U_z^2 + 4U_zV_z - V_z^2 - 2U_{zz} + V_{zz}\right),$$
$$-3(U_zU_{zz})_{\bar z} = -3U_{z\bar z}U_{zz} - 3U_zU_{zz\bar z} = 3e^{2U-V}U_{zz} + 3e^{2U-V}U_z(2U_z - V_z) = e^{2U-V}\left(3U_{zz} + 6U_z^2 - 3U_zV_z\right),$$

and $(U_z^3)_{\bar z} = 3U_z^2U_{z\bar z} = e^{2U-V}(-3U_z^2)$.

So we have

$$f_{1,\bar z} = e^{2U-V}\left(U_{zz} + V_{zz} - U_z^2 - V_z^2 + U_zV_z\right).$$

Then Lemma 2.1 implies that $f_{1,\bar z} \equiv 0$. Similarly we also have $f_{2,\bar z} \equiv 0$. The claim is proved.

Since $f_1 \to 0$ and $f_2 \to 0$ as $|z| \to \infty$, again by Liouville's theorem, we get (9). This concludes the proof. □

3. Proof of the main theorem

In what follows, we discuss the kernel of the corresponding linearized operator of (8), which is equivalent to (5). Let ϕ, ψ be functions satisfy

$$\phi_{z\bar z} + e^{2U-V}(2\phi - \psi) = 0, \qquad \psi_{z\bar z} + e^{2V-U}(2\psi - \phi) = 0. \tag{11}$$

We prove the following proposition, which gives the proof of Theorem 1.1.

Proposition 3.1. *Let* (ϕ,ψ) *satisfy* (11)*. Assume that*

$$|\phi| \leqslant C\left(1+|z|\right)^{\tau}, \qquad |\psi| \leqslant C\left(1+|z|\right)^{\tau} \quad \text{for some } \tau \in (0,1). \tag{12}$$

Then $\binom{\phi}{\psi}$ *belongs to the following linear space*

$$\operatorname{span}\left\{\begin{pmatrix}\partial_{a_1}U\\ \partial_{a_1}V\end{pmatrix}, \begin{pmatrix}\partial_{a_2}U\\ \partial_{a_2}V\end{pmatrix}, \begin{pmatrix}\partial_{b_1}U\\ \partial_{b_1}V\end{pmatrix}, \begin{pmatrix}\partial_{b_2}U\\ \partial_{b_2}V\end{pmatrix}, \begin{pmatrix}\partial_{c_1}U\\ \partial_{c_1}V\end{pmatrix}, \begin{pmatrix}\partial_{c_2}U\\ \partial_{c_2}V\end{pmatrix}, \begin{pmatrix}\partial_{d_1}U\\ \partial_{d_1}V\end{pmatrix}, \begin{pmatrix}\partial_{d_2}U\\ \partial_{d_2}V\end{pmatrix}\right\}.$$

Remark 3.2. Under the assumption (12), we know that all the derivatives of ϕ and ψ approach to 0 as $|z|$ goes to ∞. Indeed, if we define that, for $x \in \mathbb{R}^2$,

$$\tilde\phi(x) = \frac{1}{8\pi}\int_{\mathbb{R}^2} \log|x-y|e^{2U(y)-V(y)}\left[2\phi(y) - \psi(y)\right]\mathrm{d}y,$$

then $|\tilde\phi| \leqslant C\log(1+|x|)$ and $\Delta(\phi - \tilde\phi) = 0$. Therefore, $\phi = \tilde\phi + C$. The potential theory implies that ϕ's derivatives vanish at infinity. So do the derivatives of ψ.

Lemma 3.3. *Under the assumption of Proposition* 3.1*, it holds that*

$$\phi_{zz} + \psi_{zz} - 2U_z\phi_z - 2V_z\psi_z + U_z\psi_z + V_z\phi_z \equiv 0, \qquad \phi_{\bar z\bar z} + \psi_{\bar z\bar z} - 2U_{\bar z}\phi_{\bar z} - 2V_{\bar z}\psi_{\bar z} + U_{\bar z}\psi_{\bar z} + V_{\bar z}\phi_{\bar z} \equiv 0.$$

Proof. According to Remark 3.2, the proof can be done by argument similar to those in the proof of Lemma 2.1. □

Lemma 3.4. *Under the assumption of Proposition* 3.1*, we have*

$$\phi_{zzz} - 3\phi_{zz}U_z - 3\phi_zU_{zz} + 3U_z^2\phi_z \equiv 0, \qquad \phi_{\bar z\bar z\bar z} - 3\phi_{\bar z\bar z}U_{\bar z} - 3\phi_{\bar z}U_{\bar z\bar z} + 3U_{\bar z}^2\phi_{\bar z} \equiv 0,$$
$$\psi_{zzz} - 3\psi_{zz}V_z - 3\psi_zV_{zz} + 3V_z^2\psi_z \equiv 0, \qquad \psi_{\bar z\bar z\bar z} - 3\psi_{\bar z\bar z}V_{\bar z} - 3\psi_{\bar z}V_{\bar z\bar z} + 3V_{\bar z}^2\psi_{\bar z} \equiv 0.$$

Proof. We only check the first one since the others are similar. By direct computation, we get, using (8),

$$\begin{aligned}\phi_{zzz\bar z} &= -\big[e^{2U-V}(2\phi-\psi)\big]_{zz} = -\big[e^{2U-V}(2U_z-V_z)(2\phi-\psi)\big]_z - \big[e^{2U-V}(2\phi_z-\psi_z)\big]_z\\ &= -e^{2U-V}(2U_z-V_z)^2(2\phi-\psi) - e^{2U-V}(2U_{zz}-V_{zz})(2\phi-\psi)\\ &\quad - e^{2U-V}2(2U_z-V_z)(2\phi_z-\psi_z) - e^{2U-V}(2\phi_{zz}-\psi_{zz})\\ &= e^{2U-V}\big(-8U_z^2\phi+4U_z^2\psi+8U_zV_z\phi-4U_zV_z\psi-2V_z^2\phi+V_z^2\psi-4U_{zz}\phi\\ &\quad +2U_{zz}\psi+2V_{zz}\phi-V_{zz}\psi-8U_z\phi_z+4U_z\psi_z+4V_z\phi_z-2V_z\psi_z-2\phi_{zz}+\psi_{zz}\big),\\ -3(\phi_{zz}U_z)_{\bar z} &= 3\big[e^{2U-V}(2\phi-\psi)\big]_zU_z+3e^{2U-V}\phi_{zz}\\ &= e^{2U-V}\big(12U_z^2\phi-6U_z^2\psi-6U_zV_z\phi+3U_zV_z\psi+6U_z\phi_z-3U_z\psi_z+3\phi_{zz}\big),\\ -3(\phi_zU_{zz})_{\bar z} &= e^{2U-V}(6U_{zz}\phi-3U_{zz}\psi+6U_z\phi_z-3V_z\phi_z),\\ 3\big(U_z^2\phi_z\big)_{\bar z} &= e^{2U-V}\big(-6U_z\phi_z-6U_z^2\phi+3U_z^2\psi\big).\end{aligned}$$

So it holds that

$$\begin{aligned}\big(\phi_{zzz}-3\phi_{zz}U_z-3\phi_zU_{zz}+3U_z^2\phi_z\big)_{\bar z} &= e^{2U-V}\big[\big(U_{zz}+V_{zz}-U_z^2-V_z^2+U_zV_z\big)(2\phi-\psi)\big]\\ &\quad + e^{2U-V}(\phi_{zz}+\psi_{zz}-2U_z\phi_z-2V_z\psi_z+U_z\psi_z+V_z\phi_z).\end{aligned}$$

Then Lemmas 2.1, 3.3 and Remark 3.2 yield that

$$\phi_{zzz}-3\phi_{zz}U_z-3\phi_zU_{zz}+3U_z^2\phi_z\equiv 0.$$

The proof is completed. □

Proof of Proposition 3.1. Let $\phi_1=e^{-U}\phi$. Since we have easily that

$$\phi_{zzz}-3\phi_{zz}U_z-3\phi_zU_{zz}+3U_z^2\phi_z = e^U\big[\phi_{1,zzz}+\big(U_{zzz}-3U_zU_{zz}+U_z^3\big)\phi_1\big],$$

using Lemmas 2.2 and 3.4, we have $\phi_{1,zzz}\equiv 0$. Similarly, it also holds that $\phi_{1,\bar z\bar z\bar z}\equiv 0$. This implies that

$$\phi_1=\sum_{k,\ell=0}^{2}\alpha_{k\ell}z^k\bar z^\ell \quad (\text{with all } \alpha_{k\ell}\in\mathbb{C}). \tag{13}$$

Since ϕ_1 is real, it must hold that

$$\alpha_{00},\alpha_{11},\alpha_{22}\in\mathbb{R} \quad\text{and}\quad \alpha_{01}=\bar\alpha_{10},\qquad \alpha_{02}=\bar\alpha_{20},\qquad \alpha_{12}=\bar\alpha_{21}.$$

On the other hand, denote that $\psi_1=e^{-V}\psi$. Similarly we can also obtain that

$$\psi_1=\sum_{k,\ell=0}^{2}\beta_{k\ell}z^k\bar z^\ell \quad (\text{with all } \beta_{k\ell}\in\mathbb{C}), \tag{14}$$

where $\beta_{k\ell}$ satisfy $\beta_{00},\beta_{11},\beta_{22}\in\mathbb{R}$ and $\beta_{01}=\bar\beta_{10}$, $\beta_{02}=\bar\beta_{20}$, $\beta_{12}=\bar\beta_{21}$.

Rewriting (11) in the term of ϕ_1 and ψ_1, we have

$$\phi_{1,z\bar z}+U_{\bar z}\phi_{1,z}+U_z\phi_{1,\bar z}+\big(e^{2U-V}+U_zU_{\bar z}\big)\phi_1-e^U\psi_1=0, \tag{15}$$

$$\psi_{1,z\bar z}+V_{\bar z}\psi_{1,z}+V_z\psi_{1,\bar z}+\big(e^{2V-U}+V_zV_{\bar z}\big)\psi_1-e^V\phi_1=0. \tag{16}$$

Substituting (13), (14) into (15) and using *Mathematica*, we find that

$$\beta_{00}=\frac{\alpha_{11}a_1^2+\alpha_{00}a_2^2-\alpha_{10}a_2^2b-\alpha_{01}a_2^2\bar b+\alpha_{11}a_2^2|b|^2+\alpha_{00}|c|^2-\alpha_{10}\bar c d-\alpha_{01}c\bar d+\alpha_{11}|d|^2}{2^{2/3}\sqrt[3]{a_1^2a_2^2}},$$

$$\beta_{11}=\frac{2\sqrt[3]{2}(\alpha_{22}a_1^2+\alpha_{00}+a_2^2\alpha_{22}|b|^2-\alpha_{20}d-\alpha_{02}\bar d+\alpha_{22}|d|^2)}{\sqrt[3]{a_1^2a_2^2}},$$

$$\beta_{22}=\frac{\alpha_{22}a_2^2+\alpha_{11}-\alpha_{21}c-\alpha_{12}\bar c+\alpha_{22}|c|^2}{2^{2/3}\sqrt[3]{a_1^2a_2^2}},$$

$$\beta_{01}=\frac{\sqrt[3]{2}(\alpha_{12}a_1^2-\alpha_{02}a_2^2\bar b+\alpha_{12}a_2^2|b|^2+\alpha_{00}c-\alpha_{10}d-\alpha_{02}c\bar d+\alpha_{12}|d|^2)}{\sqrt[3]{a_1^2a_2^2}},$$
$$\beta_{02}=-\frac{\alpha_{02}a_2^2-\alpha_{12}ba_2^2-\alpha_{01}c+\alpha_{02}|c|^2+\alpha_{11}d-\alpha_{12}\bar c d}{2^{2/3}\sqrt[3]{a_1^2a_2^2}},$$
$$\beta_{12}=\frac{\sqrt[3]{2}(\alpha_{22}ba_2^2+\alpha_{01}-\alpha_{02}\bar c-\alpha_{21}d+\alpha_{22}\bar c d)}{\sqrt[3]{a_1^2a_2^2}},$$
$$\beta_{10}=\bar\beta_{01},\qquad \beta_{20}=\bar\beta_{02},\qquad \beta_{21}=\bar\beta_{12}.$$

Recall that the parameters a_1, a_2, b, c, d appeared in the above equalities come from (u,v) given by (3)–(4). Finally we insert all the above quantities into (16) again and thus obtain another relation

$$\begin{aligned}\alpha_{22}&=\frac{-\alpha_{11}a_1^2+\alpha_{21}ca_1^2+\alpha_{12}\bar c a_1^2-\alpha_{00}a_2^2+\alpha_{10}a_2^2b+\alpha_{01}a_2^2\bar b-\alpha_{11}a_2^2|b|^2}{a_1^2a_2^2+|c|^2a_1^2+a_2^2|b|^2|c|^2-a_2^2\bar b\bar c d-a_2^2bc\bar d+a_2^2|d|^2}\\&\quad+\frac{-\alpha_{20}a_2^2bc-\alpha_{02}a_2^2\bar b\bar c+\alpha_{21}a_2^2|b|^2c+\alpha_{12}a_2^2|b|^2\bar c+\alpha_{20}a_2^2d+\alpha_{02}a_2^2\bar d}{a_1^2a_2^2+|c|^2a_1^2+a_2^2|b|^2|c|^2-a_2^2\bar b\bar c d-a_2^2bc\bar d+a_2^2|d|^2}\\&\quad+\frac{-\alpha_{21}a_2^2\bar b d-\alpha_{12}a_2^2b\bar d}{a_1^2a_2^2+|c|^2a_1^2+a_2^2|b|^2|c|^2-a_2^2\bar b\bar c d-a_2^2bc\bar d+a_2^2|d|^2},\end{aligned}$$

from which we know that ϕ_1 and ψ_1 actually depend on eight real parameters rather than formally nine. Therefore, the dimension of the space $\{(\phi,\psi)\}$ is exactly eight. Since it is known that

$$\begin{pmatrix}\partial_{a_1}U\\\partial_{a_1}V\end{pmatrix},\quad\begin{pmatrix}\partial_{a_2}U\\\partial_{a_2}V\end{pmatrix},\quad\begin{pmatrix}\partial_{b_1}U\\\partial_{b_1}V\end{pmatrix},\quad\begin{pmatrix}\partial_{b_2}U\\\partial_{b_2}V\end{pmatrix},\quad\begin{pmatrix}\partial_{c_1}U\\\partial_{c_1}V\end{pmatrix},\quad\begin{pmatrix}\partial_{c_2}U\\\partial_{c_2}V\end{pmatrix},\quad\begin{pmatrix}\partial_{d_1}U\\\partial_{d_1}V\end{pmatrix},\quad\begin{pmatrix}\partial_{d_2}U\\\partial_{d_2}V\end{pmatrix}$$

are linearly independent and satisfy (11), we then complete the proof of Proposition 3.1. □

Finally let $\varphi_1=2\phi-\psi$ and $\varphi_2=2\psi-\phi$, where ϕ, ψ satisfy (11). It is easy to check that φ_1, φ_2 satisfy (5). Thus Theorem 1.1 is equivalent to Proposition 3.1, so it is concluded.

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