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## Stable spectrum for pseudo-Riemannian locally symmetric spaces

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## ABSTRACT

Let  $X = G/H$  be a reductive symmetric space with  $\text{rank } G/H = \text{rank } K/K \cap H$ , where  $K$  (resp.  $K \cap H$ ) is a maximal compact subgroup of  $G$  (resp. of  $H$ ). We investigate the discrete spectrum of certain Clifford–Klein forms  $\Gamma \backslash X$ , where  $\Gamma$  is a discrete subgroup of  $G$  acting properly discontinuously and freely on  $X$ : we construct an infinite set of joint eigenvalues for “intrinsic” differential operators on  $\Gamma \backslash X$ , and this set is stable under small deformations of  $\Gamma$  in  $G$ .

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## R É S U M É

Soit  $X = G/H$  un espace symétrique réductif vérifiant  $\text{rang } G/H = \text{rang } K/K \cap H$ , où  $K$  (resp.  $K \cap H$ ) est un sous-groupe compact maximal de  $G$  (resp. de  $H$ ). Nous étudions le spectre discret de certaines formes de Clifford–Klein  $\Gamma \backslash X$ , où  $\Gamma$  est un sous-groupe discret de  $G$  agissant librement et proprement sur  $X$ : nous construisons un ensemble infini de valeurs propres pour les opérateurs différentiels « intrinsèques » sur  $\Gamma \backslash X$ , et cet ensemble est stable par petites déformations de  $\Gamma$  dans  $G$ .

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## Version française abrégée

Soit  $X = G/H$  un espace symétrique, où  $G$  est un groupe de Lie réductif connexe non compact et  $H$  la composante neutre du groupe des points fixes de  $G$  par un certain automorphisme involutif  $\sigma$ . L'espace  $X$  est naturellement muni d'une métrique pseudo-riemannienne  $G$ -invariante. Une *forme de Clifford–Klein* de  $X$  est un quotient  $X_\Gamma = \Gamma \backslash X$  où  $\Gamma$  est un sous-groupe discret de  $G$  agissant librement et proprement sur  $X$ ; c'est une variété complète localement modelée sur  $X$ . Soit  $\mathbb{D}(X)$  l'algèbre des opérateurs différentiels  $G$ -invariants sur  $X$ . Tout élément  $D \in \mathbb{D}(X)$  (par exemple le laplacien) induit un opérateur différentiel  $D_\Gamma$  sur  $X_\Gamma$ . Le *spectre discret*  $\text{Spec}_d(X_\Gamma)$  de  $X_\Gamma$  est l'ensemble des morphismes d'algèbres  $\lambda : \mathbb{D}(X) \rightarrow \mathbb{C}$  pour lesquels il existe une fonction  $f \in L^2(X_\Gamma)$  non nulle vérifiant  $D_\Gamma f = \lambda(D)f$  pour tout  $D \in \mathbb{D}(X)$ , au sens des distributions. Soit  $K = G^\theta$  un sous-groupe compact maximal de  $G$ , où  $\theta$  est une involution de Cartan commutant avec  $\sigma$ . Notre résultat principal concerne les formes de Clifford–Klein de  $X$  qui sont *standard*, au sens où  $\Gamma$  est inclus dans un sous-groupe réductif de  $G$  agissant proprement sur  $X$ .

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**Théorème 0.1.** *Supposons  $\text{rang } G/H = \text{rang } K/K \cap H$ . Le spectre discret  $\text{Spec}_d(X_\Gamma)$  est infini pour toute forme de Clifford–Klein compacte standard  $X_\Gamma$  de  $X$ ; de plus, il existe une partie infinie de  $\text{Spec}_d(X_\Gamma)$  qui est stable par petites déformations de  $\Gamma$  dans  $G$ . Ceci reste vrai lorsque  $\Gamma$  est convexe cocompact dans un sous-groupe réductif de  $G$  de rang réel 1.*

Dans la situation du théorème 0.1, il existe un voisinage  $\mathcal{U} \subset \text{Hom}(\Gamma, G)$  de l'inclusion naturelle tel que pour tout  $\varphi \in \mathcal{U}$  le quotient  $X_{\varphi(\Gamma)} = \varphi(\Gamma) \backslash X$  soit une forme de Clifford–Klein de  $X$ , compacte si  $X_\Gamma$  l'est : cela résulte de [5] (propreté) et [8] (compacité). Le théorème 0.1 affirme que, quitte à réduire le voisinage  $\mathcal{U}$ , il existe un ensemble infini qui est inclus dans  $\text{Spec}_d(X_{\varphi(\Gamma)})$  pour tout  $\varphi \in \mathcal{U}$ . L'étude des petites déformations de formes de Clifford–Klein dans ce cadre général remonte à l'article [10].

Soit  $\mathfrak{j}_\mathbb{C}$  un sous-espace abélien semi-simple maximal de l'ensemble des points fixes de  $-\sigma$  dans l'algèbre de Lie complexifiée  $\mathfrak{g}_\mathbb{C}$  de  $G$ , soit  $\mathfrak{j}_\mathbb{C}^*$  son dual, et soit  $W$  le groupe de Weyl de  $\mathfrak{j}_\mathbb{C}$  dans  $\mathfrak{g}_\mathbb{C}$ . Le spectre discret de toute forme de Clifford–Klein de  $X$  s'identifie naturellement à une partie de  $\mathfrak{j}_\mathbb{C}^*/W$ . Sous les hypothèses du théorème 0.1 on peut supposer que  $\mathfrak{j}_\mathbb{C} = \mathfrak{j} \otimes_{\mathbb{R}} \mathbb{C}$  pour un certain sous-espace abélien maximal  $\mathfrak{j}$  de  $\sqrt{-1}\mathfrak{k}$ , où  $\mathfrak{k}$  est l'algèbre de Lie de  $K$ . Fixons un système  $\Sigma^+(\mathfrak{g}_\mathbb{C}, \mathfrak{j}_\mathbb{C})$  de racines positives de  $\mathfrak{j}_\mathbb{C}$  dans  $\mathfrak{g}_\mathbb{C}$ , ce qui définit une chambre de Weyl positive  $\mathfrak{j}_+^*$  de  $\mathfrak{j}_\mathbb{C}^*$ . Soient  $\rho \in \mathfrak{j}_+^*$  et  $\rho_c \in \mathfrak{j}_+^*$  les demi-sommes respectives des racines de  $\Sigma^+(\mathfrak{g}_\mathbb{C}, \mathfrak{j}_\mathbb{C})$  et  $\Sigma^+(\mathfrak{g}_\mathbb{C}, \mathfrak{j}_\mathbb{C}) \cap \Sigma(\mathfrak{k}_\mathbb{C}, \mathfrak{j}_\mathbb{C})$ , et soit  $A_+$  l'intersection de  $\mathfrak{j}_+^*$  avec le réseau de  $\mathfrak{j}$  engendré par les plus hauts poids des représentations irréductibles de  $K$  ayant des vecteurs  $(K \cap H)$ -invariants non nuls. Pour tout  $\lambda \in \mathfrak{j}_+^*$  nous notons  $d(\lambda)$  la « distance pondérée » naturelle de  $\lambda$  aux murs de  $\mathfrak{j}_+^*$  (voir le paragraphe 2). Avec ces notations, voici une version plus précise du théorème 0.1.

**Théorème 0.2.** *Sous les hypothèses du théorème 0.1, il existe une constante  $R > 0$  et un voisinage  $\mathcal{U} \subset \text{Hom}(\Gamma, G)$  de l'inclusion naturelle tels que  $\{\lambda \in 2\rho_c - \rho + A_+ : d(\lambda) \geq R\} \subset \text{Spec}_d(X_{\varphi(\Gamma)})$  pour tout  $\varphi \in \mathcal{U}$ .*

Nous donnons une liste d'espaces symétriques  $X$  auxquels nos théorèmes s'appliquent, et décrivons explicitement une partie infinie du spectre discret stable des formes de Clifford–Klein compactes standard de  $X = \text{SO}(2, 4)/\text{U}(1, 2)$  (en utilisant [11]) et de l'espace anti-de Sitter  $X = \text{AdS}^3 = \text{SO}(2, 2)/\text{SO}(1, 2)$  (voir (2)). Rappelons que les variétés anti-de Sitter (c'est-à-dire lorentziennes de courbure constante  $< 0$ ) compactes de dimension 3 sont les formes de Clifford–Klein compactes de  $\text{AdS}^3$ , à revêtement fini, isométrie et renormalisation près [7,13]. Nous démontrons un résultat analogue aux théorèmes 0.1 et 0.2 pour toutes ces formes de Clifford–Klein compactes, même celles qui ne sont pas standard.

**Théorème 0.3.** *Le spectre discret de toute variété anti-de Sitter compacte de dimension 3 est infini, et contient une partie infinie qui est stable par petites déformations de la structure anti-de Sitter.*

Pour démontrer nos résultats, nous construisons des fonctions propres sur les formes de Clifford–Klein de  $X$  à partir de fonctions propres sur  $X$  construites par Flensted-Jensen [2]. Nous donnons des estimées asymptotiques uniformes de ces dernières, en fonction de la projection  $\nu : G \rightarrow \overline{\mathfrak{b}}_+$  associée à une décomposition  $G = KBH$  (voir le paragraphe 3). Nous relierons la projection  $\nu$  à la projection  $\mu : G \rightarrow \overline{\mathfrak{a}}_+$  associée à une décomposition de Cartan  $G = KAK$  où  $A \supset B$  (voir le paragraphe 3), et utilisons les estimées de [5] et [6] sur la restriction de  $\mu$  à  $\Gamma$  et à ses déformés. Les détails seront publiés ultérieurement.

## 1. A general program

Let  $X = G/H$  be a reductive symmetric space, where  $G$  is a connected noncompact reductive linear Lie group and  $H = (G^\sigma)_0$  the identity component of the set of fixed points of  $G$  under some involutive automorphism  $\sigma$ . We note that  $X$  naturally carries a  $G$ -invariant pseudo-Riemannian metric, which is induced by the Killing form of the Lie algebra  $\mathfrak{g}$  of  $G$  if  $G$  is semisimple. A Clifford–Klein form of  $X$  is a quotient  $X_\Gamma = \Gamma \backslash X$  where  $\Gamma$  is a discrete subgroup of  $G$  acting properly discontinuously and freely on  $X$ ; it is a complete manifold locally modeled on  $X$ . Any  $G$ -invariant differential operator  $D$  on  $X$  (such as the Laplacian) induces a differential operator  $D_\Gamma$  on  $X_\Gamma$ , and the map  $D \mapsto D_\Gamma$  is an injective  $\mathbb{C}$ -algebra homomorphism from the ring  $\mathbb{D}(X)$  of  $G$ -invariant differential operators on  $X$  into the ring of differential operators on  $X_\Gamma$ . We may think of its image as the set of “intrinsic” differential operators on  $X_\Gamma$ . The discrete spectrum  $\text{Spec}_d(X_\Gamma)$  of  $X_\Gamma$  is the set of  $\mathbb{C}$ -algebra homomorphisms  $\lambda : \mathbb{D}(X) \rightarrow \mathbb{C}$  such that the set  $L^2(X_\Gamma, \mathcal{M}_\lambda)$  of weak solutions  $f \in L^2(X_\Gamma)$  to the system

$$D_\Gamma f = \lambda(D)f \quad \text{for all } D \in \mathbb{D}(X) \quad (\mathcal{M}_\lambda)$$

is nontrivial. In this note we wish to initiate the following general program.

- A) Construct elements of  $L^2(X_\Gamma, \mathcal{M}_\lambda)$ , i.e. joint eigenfunctions on  $X_\Gamma$  corresponding to  $\text{Spec}_d(X_\Gamma)$ .
- B) Understand the behavior of  $\text{Spec}_d(X_\Gamma)$  under small deformations of  $\Gamma$  in  $G$ .

Problem B builds on the fact that for certain Clifford–Klein forms  $X_\Gamma$ , the proper discontinuity of the action of  $\Gamma$  on  $X$  is preserved under small deformations of  $\Gamma$ . The study of small deformations of Clifford–Klein forms in such a general setting was initiated in [10].

**2. Main results**

Let  $\theta$  be a Cartan involution of  $G$  commuting with  $\sigma$  and let  $K = G^\theta$  be the corresponding maximal compact subgroup of  $G$ , with Lie algebra  $\mathfrak{k}$ . In this note, we assume that

$$\text{rank } G/H = \text{rank } K/K \cap H, \tag{1}$$

where  $\text{rank } G/H$  (resp.  $\text{rank } K/K \cap H$ ) is the dimension of a maximal semisimple abelian subspace in the set of fixed points of  $-\delta\sigma$  in  $\mathfrak{g}$  (resp.  $\mathfrak{k}$ ). We investigate Problems A and B for an important class of Clifford–Klein forms  $X_\Gamma$ , namely those that are *standard*, in the sense that  $\Gamma$  is contained in some closed reductive subgroup  $L$  of  $G$  acting properly on  $X = G/H$ . Note that if such an  $X_\Gamma$  is compact, then  $\Gamma$  must be a uniform lattice in  $L$  and the action of  $L$  on  $X$  must be cocompact. Here is our main result.

**Theorem 2.1.** *Under the rank assumption (1), the discrete spectrum  $\text{Spec}_d(X_\Gamma)$  is infinite for any standard compact Clifford–Klein form  $X_\Gamma$  of  $X$ ; furthermore, there is an infinite subset of  $\text{Spec}_d(X_\Gamma)$  that is stable under small deformations of  $\Gamma$  in  $G$ . The same conclusion holds when  $\Gamma$  is convex cocompact in some reductive subgroup  $L$  of  $G$  with  $\text{rank}_{\mathbb{R}} L = 1$ .*

Underlying Theorem 2.1 is the existence, due to [5] (properness) and [8] (compactness), of a neighborhood  $\mathcal{U} \subset \text{Hom}(\Gamma, G)$  of the natural inclusion such that  $X_{\varphi(\Gamma)} = \varphi(\Gamma) \backslash X$  is a Clifford–Klein form of  $X$  for all  $\varphi \in \mathcal{U}$ , with  $X_{\varphi(\Gamma)}$  compact if  $X_\Gamma$  is. Theorem 2.1 states that, after possibly replacing  $\mathcal{U}$  by some smaller neighborhood, there is an infinite set that is contained in  $\text{Spec}_d(X_{\varphi(\Gamma)})$  for all  $\varphi \in \mathcal{U}$ .

Recall that a discrete subgroup  $\Gamma$  of a reductive group  $L$  with  $\text{rank}_{\mathbb{R}} L = 1$  is said to be *convex cocompact* if it acts cocompactly on the convex hull of its limit set in the Riemannian symmetric space of  $L$ , this limit set being nonempty. Convex cocompact groups include uniform lattices, but also discrete groups of infinite covolume such as Schottky groups.

In order to describe the *stable discrete spectrum* of  $X_\Gamma$  in Theorem 2.1, let us briefly recall the structure of  $\mathbb{D}(X)$  (see [4] for more details) and introduce some notation. Let  $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$  be the decomposition of  $\mathfrak{g}$  into eigenspaces of  $d\sigma$ , with respective eigenvalues  $+1$  and  $-1$ , and let  $\mathfrak{g}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} + \mathfrak{q}_{\mathbb{C}}$  be its complexification. Fix a maximal semisimple abelian subspace  $\mathfrak{j}_{\mathbb{C}}$  of  $\mathfrak{q}_{\mathbb{C}}$  and let  $W$  be the Weyl group of the restricted root system  $\Sigma(\mathfrak{g}_{\mathbb{C}}, \mathfrak{j}_{\mathbb{C}})$  of  $\mathfrak{j}_{\mathbb{C}}$  in  $\mathfrak{g}_{\mathbb{C}}$ . The  $\mathbb{C}$ -algebra  $\mathbb{D}(X)$  is naturally isomorphic to the subalgebra  $S(\mathfrak{j}_{\mathbb{C}})^W$  of  $W$ -fixed points in the symmetric algebra  $S(\mathfrak{j}_{\mathbb{C}})$ ; it is a polynomial ring in  $r := \dim_{\mathbb{C}} \mathfrak{j}_{\mathbb{C}} = \text{rank } G/H$  generators. In particular, the set of  $\mathbb{C}$ -algebra homomorphisms from  $\mathbb{D}(X)$  to  $\mathbb{C}$  naturally identifies with  $\mathfrak{j}_{\mathbb{C}}^*/W$ , where  $\mathfrak{j}_{\mathbb{C}}^*$  is the dual vector space of  $\mathfrak{j}_{\mathbb{C}}$ . For any Clifford–Klein form  $X_\Gamma$  of  $X$ , we see  $\text{Spec}_d(X_\Gamma)$  as a subset of  $\mathfrak{j}_{\mathbb{C}}^*/W$ . Under the rank hypothesis (1), we may assume that  $\mathfrak{j}_{\mathbb{C}}$  is the complexification of a maximal abelian subspace  $\mathfrak{j}$  of  $\sqrt{-1}\mathfrak{k}$ , on which all restricted roots  $\alpha \in \Sigma(\mathfrak{g}_{\mathbb{C}}, \mathfrak{j}_{\mathbb{C}})$  take real values. We endow  $\mathfrak{j}^*$  with a  $W$ -invariant inner product  $\langle \cdot, \cdot \rangle$ , fix a basis  $\Psi$  of  $\Sigma(\mathfrak{g}_{\mathbb{C}}, \mathfrak{j}_{\mathbb{C}})$ , defining a system  $\Sigma^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{j}_{\mathbb{C}})$  of positive roots, and let  $\mathfrak{j}_+^*$  be the corresponding positive Weyl chamber in  $\mathfrak{j}^*$ , defined by  $\langle \lambda, \alpha \rangle > 0$  for all  $\alpha \in \Psi$ . For  $\lambda \in \mathfrak{j}_+^*$ , we consider the natural “weighted distance” from  $\lambda$  to the walls of  $\mathfrak{j}_+^*$  given by  $d(\lambda) = \min_{\alpha \in \Psi} \langle \lambda, \alpha \rangle / \langle \alpha, \alpha \rangle$ . Let  $\rho$  (resp.  $\rho_{\mathbb{C}}$ ) be half the sum of roots in  $\Sigma^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{j}_{\mathbb{C}})$  (resp. in  $\Sigma^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{j}_{\mathbb{C}}) \cap \Sigma(\mathfrak{k}_{\mathbb{C}}, \mathfrak{j}_{\mathbb{C}})$ ), counted with multiplicities, and let  $\Lambda_+$  be the intersection of  $\mathfrak{j}_+^*$  with the lattice of  $\mathfrak{j}^*$  generated by all highest weights of irreducible representations of  $K$  with nonzero  $(K \cap H)$ -fixed vectors. With this notation, here is a more precise statement of Theorem 2.1.

**Theorem 2.2.** *In the setting of Theorem 2.1, there is a constant  $R > 0$  and a neighborhood  $\mathcal{U} \subset \text{Hom}(\Gamma, G)$  of the natural inclusion such that  $\{\lambda \in 2\rho_{\mathbb{C}} - \rho + \Lambda_+ : d(\lambda) \geq R\} \subset \text{Spec}_d(X_{\varphi(\Gamma)})$  for all  $\varphi \in \mathcal{U}$ .*

We explicitly construct eigenfunctions  $f \in L^2(X_{\varphi(\Gamma)}, \mathcal{M}_\lambda)$  for all  $\lambda \in 2\rho_{\mathbb{C}} - \rho + \Lambda_+$  with  $d(\lambda)$  large enough and all homomorphisms  $\varphi$  that are close enough to the natural inclusion of  $\Gamma$  in  $G$ . These eigenfunctions actually satisfy  $(\mathcal{M}_\lambda)$  as functions of class  $C^N$  where  $N$  is the maximal degree of  $r$  generators of  $\mathbb{D}(X)$ , and they belong to  $L^p(X_{\varphi(\Gamma)})$  for all  $1 \leq p \leq \infty$ .

Using [12, Cor. 3.3.7], we see that Theorems 2.1 and 2.2 apply in particular to the triples  $(G, H, L)$  of Table 1, where  $n, p, q \geq 1$ . In Example (vi), the group  $G_0$  may be  $\text{SO}(p, 2q)$ ,  $\text{SU}(p, q)$ ,  $\text{Sp}(p, q)$ ,  $\text{Sp}(n, \mathbb{R})$ ,  $\text{SO}^*(2n)$ , or certain exceptional groups, and  $\text{Diag}(G_0)$  denotes the diagonal of  $G_0 \times G_0$ .

Note that in Examples (vii) to (xiv), which are of the form  $(G, H, L) = (G_0 \times G_0, \text{Diag}(G_0), H_0 \times L_0)$ , if  $\Gamma_{H_0}$  (resp.  $\Gamma_{L_0}$ ) is a uniform lattice of  $H_0$  (resp. of  $L_0$ ), then the compact Clifford–Klein form  $(\Gamma_{H_0} \times \Gamma_{L_0}) \backslash G/H$  identifies with  $\Gamma_{L_0} \backslash G_0 / \Gamma_{H_0}$  and is locally modeled on  $G_0$ . In Examples (ii), (vii), (xi) and (xii), small nonstandard deformations of standard compact Clifford–Klein forms of  $X$  can be obtained using a *bending construction* due to Johnson and Millson (see [5]). In Example (vi), small nonstandard deformations also exist for  $G_0 = \text{SO}(1, 2n)$  or  $\text{SU}(1, n)$  (see [10]).

An infinite subset of the stable discrete spectrum for standard compact Clifford–Klein forms of  $X$  may be found explicitly for  $X = \text{SO}(2, 4)/\text{U}(1, 2)$  and for the 3-dimensional *anti-de Sitter space*  $X = \text{AdS}^3 = \text{SO}(2, 2)/\text{SO}(1, 2)$ . By [7] and [13], the 3-dimensional compact anti-de Sitter manifolds (i.e. the 3-dimensional compact Lorentz manifolds with constant negative curvature) are the compact Clifford–Klein forms of  $\text{AdS}^3$ , up to finite covering, isometry, and renormalization of the metric. Using [6], we prove that Theorems 2.1 and 2.2 actually hold true for *all* these compact Clifford–Klein forms, not only standard ones.

**Table 1**  
Triples  $(G, H, L)$ .

	$G$	$H$	$L$
(i)	$SO(2, 2n)$	$SO(1, 2n)$	$U(1, n)$
(ii)	$SO(2, 4n)$	$U(1, 2n)$	$SO(1, 4n)$
(iii)	$SO(4, 4n)$	$SO(3, 4n)$	$Sp(1, n)$
(iv)	$U(2, 2n)$	$U(1) \times U(1, 2n)$	$Sp(1, n)$
(v)	$SO(8, 8)$	$SO(7, 8)$	$Spin(1, 8)$
(vi)	$G_0 \times G_0$	$Diag(G_0)$	$G_0 \times \{1\}$
(vii)	$SO(2, 2n) \times SO(2, 2n)$	$Diag(SO(2, 2n))$	$SO(1, 2n) \times U(1, n)$
(viii)	$SO(4, 4n) \times SO(4, 4n)$	$Diag(SO(4, 4n))$	$SO(3, 4n) \times Sp(1, n)$
(ix)	$U(2, 2n) \times U(2, 2n)$	$Diag(U(2, 2n))$	$U(1, 2n) \times Sp(1, n)$
(x)	$SO(8, 8) \times SO(8, 8)$	$Diag(SO(8, 8))$	$SO(7, 8) \times Spin(1, 8)$
(xi)	$SO(4, 4) \times SO(4, 4)$	$Diag(SO(4, 4))$	$SO(4, 1) \times Spin(4, 3)$
(xii)	$SO(4, 3) \times SO(4, 3)$	$Diag(SO(4, 3))$	$SO(4, 1) \times G_{2(2)}$
(xiii)	$SO^*(8) \times SO^*(8)$	$Diag(SO^*(8))$	$U(3, 1) \times Spin(1, 6)$
(xiv)	$SO^*(8) \times SO^*(8)$	$Diag(SO^*(8))$	$(SO^*(6) \times SO^*(2)) \times Spin(1, 6)$

**Theorem 2.3.** *The discrete spectrum of any 3-dimensional compact anti-de Sitter manifold  $M$  is infinite. Explicitly, there is an integer  $n_0$  such that the discrete spectrum of the Laplacian  $\Delta_M$  on  $M$  satisfies*

$$Spec_d(\Delta_M) \supset \left\{ \frac{1}{2}n(n+1) : n \in \mathbb{N}, n \geq n_0 \right\}, \tag{2}$$

and (2) still holds after a small deformation of the anti-de Sitter structure of  $M$ .

Here we are using the normalization of the Lorentz metric by the Killing form. Theorem 2.3 holds more generally for any 3-dimensional anti-de Sitter manifold  $M$  satisfying some convex cocompactness property. We note that here  $\mathbb{D}(X)$  is generated by the Laplacian, so that  $Spec_d(M)$  identifies with  $Spec_d(\Delta_M) \subset \mathbb{C}$ . Since  $M$  is a Lorentz manifold,  $\Delta_M$  is a hyperbolic operator. We may compare (2) with the following easy computation:

$$Spec_d(\mathbb{P}^3(\mathbb{R}), \Delta_{\mathbb{P}^3(\mathbb{R})}) = \left\{ -\frac{1}{2}n(n+1) : n \in \mathbb{N} \right\}.$$

Theorems 2.1, 2.2, and 2.3 follow from a more general result that we prove for triples  $(G, H, \Gamma)$  that satisfy (1) and two conditions on the image of  $\Gamma$  by some Cartan projection of  $G$  (see Definition 3.3).

### 3. Ideas of proofs

For simplicity we assume that  $G$  is a real form of a simply connected complex Lie group  $G_{\mathbb{C}}$  with Lie algebra  $\mathfrak{g}_{\mathbb{C}}$ . Let  $(H^d, G^d, K^d)$  be the dual triple of  $(K, G, H)$ , i.e. the triple of connected reductive Lie groups with the same complexified Lie algebras and such that  $G^d/K^d$  is a Riemannian symmetric space. There is an injective homomorphism [2] from the set  $\mathcal{A}_K(X)$  of  $K$ -finite analytic functions on  $X = G/H$  into the set  $\mathcal{A}_{H^d}(G^d/K^d)$  of  $H^d$ -finite analytic functions on  $G^d/K^d$ . For  $\lambda \in 2\rho_{\mathbb{C}} - \rho + \Lambda_+$ , Flensted-Jensen [2] introduced the function  $\psi_{\lambda} \in \mathcal{A}_K(X)$  whose image in  $\mathcal{A}_{H^d}(G^d/K^d)$  is given by

$$g^d K^d \mapsto \int_{K^d \cap H^d} e^{-(\lambda + \rho, \zeta((g^d)^{-1}\ell))} d\ell,$$

where  $G^d = K^d A^d N^d$  is an Iwasawa decomposition of  $G^d$  with  $A^d = \exp \mathfrak{a}$ , and  $\zeta : G^d \rightarrow \mathfrak{a}$  is given by  $g^d \in K^d e^{\zeta(g^d)} N^d$  for all  $g^d \in G^d$ . Assuming (1), he proved that  $\psi_{\lambda} \in L^2(X, \mathcal{M}_{\lambda})$  for  $d(\lambda)$  large enough. To establish Theorems 2.1, 2.2, and 2.3 we prove the following, where  $\bar{x}$  is the image of  $x \in X$  in  $X_{\varphi(\Gamma)}$ .

**Proposition 3.1.** *In the setting of Theorems 2.1 or 2.3, there is a constant  $R > 0$  and a neighborhood  $\mathcal{U} \subset \text{Hom}(\Gamma, G)$  of the natural inclusion such that for all  $\lambda \in 2\rho_{\mathbb{C}} - \rho + \Lambda_+$  with  $d(\lambda) \geq R$  and all  $\varphi \in \mathcal{U}$ , the eigenfunction*

$$\psi_{\lambda}^{\varphi(\Gamma)} : \bar{x} \mapsto \sum_{\gamma \in \Gamma} \psi_{\lambda}(\varphi(\gamma) \cdot x)$$

on  $X_{\varphi(\Gamma)}$  is well-defined, nonzero,  $L^p$  for all  $1 \leq p \leq \infty$ , and of class  $C^m$  whenever  $d(\lambda) \geq (m+1)R$ .

Let  $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$  be the Cartan decomposition associated with  $\theta$  and let  $\mathfrak{b}$  be a maximal abelian subspace of  $\mathfrak{p} \cap \mathfrak{q}$ . Fix a system  $\Sigma^+(\mathfrak{g}^{\sigma\theta}, \mathfrak{b})$  of positive restricted roots of  $\mathfrak{b}$  in the set of fixed points of  $d(\sigma\theta)$  in  $\mathfrak{g}$ , and let  $\bar{\mathfrak{b}}_+ \subset \mathfrak{b}$  be the corresponding closed positive Weyl chamber, so that the decomposition  $G = K \exp(\bar{\mathfrak{b}}_+)H$  holds. We define a map  $\nu : G \rightarrow \bar{\mathfrak{b}}_+$  by  $g \in Ke^{\nu(g)}H$  for all  $g \in G$ . Proposition 3.1 relies on the following uniform asymptotic estimates for  $\psi_{\lambda}$ , which we establish by building on the work of Flensted-Jensen [2] and Matsuki and Oshima [14]. Here  $\|\cdot\|$  denotes any fixed norm on  $\mathfrak{b}$ .

**Lemma 3.2.** Under the rank assumption (1), there is a constant  $\varepsilon > 0$  such that for any  $\lambda \in 2\rho_c - \rho + \Lambda_+$ ,

- (i)  $\psi_\lambda(eH) = 1$  and  $|\psi_\lambda(gH)| \leq \cosh(\varepsilon \|v(g)\|)^{-d(\lambda+\rho)}$  for all  $g \in G$ ,  
(ii) for any  $D \in \mathbb{D}(X)$ , the function  $g \mapsto D\psi_\lambda(gH) e^{\varepsilon d(\lambda+\rho)\|v(g)\|}$  is bounded on  $G$ .

To deduce Proposition 3.1 from Lemma 3.2, we consider a maximal abelian subspace  $\mathfrak{a}$  of  $\mathfrak{p}$  containing  $\mathfrak{b}$ . We fix a system  $\Sigma^+(\mathfrak{g}, \mathfrak{a})$  of positive restricted roots and let  $\overline{\mathfrak{a}_+} \subset \mathfrak{a}$  be the corresponding closed positive Weyl chamber, so that the Cartan decomposition  $G = K \exp(\overline{\mathfrak{a}_+})K$  holds. By the *properness criterion* of Benoist [1] and Kobayashi [9], the Cartan projection  $\mu : G \rightarrow \overline{\mathfrak{a}_+}$ , defined by  $g \in Ke^{\mu(g)}K$  for all  $g \in G$ , controls the properness of the action of any closed subgroup of  $G$  on  $G/H$ . We introduce the following two conditions, where  $\|\cdot\|$  denotes any norm on  $\mathfrak{a}$  extending that of  $\mathfrak{b}$ , inducing a distance  $\text{dist}_\mathfrak{a}$  on  $\mathfrak{a}$ , and  $\ell_F : \Gamma \rightarrow \mathbb{N}$  is the word length with respect to  $F$ .

**Definition 3.3.** Let  $c, C > 0$ . A subgroup  $\Gamma$  of  $G$  with finite generating subset  $F$  is said to satisfy

- the *angle condition* with constants  $(c, C)$  if  $\text{dist}_\mathfrak{a}(\mu(\gamma), \mu(H)) \geq c\|\mu(\gamma)\| - C$  for all  $\gamma \in \Gamma$ ,
- the *QI condition* with constants  $(c, C)$  if  $\|\mu(\gamma)\| \geq c\ell_F(\gamma) - C$  for all  $\gamma \in \Gamma$ .

In the setting of Theorem 2.1 or 2.3, there are constants  $c, C > 0$  and a neighborhood  $\mathcal{U} \subset \text{Hom}(\Gamma, G)$  of the natural inclusion such that for all  $\varphi \in \mathcal{U}$ , the group  $\varphi(\Gamma)$  satisfies both the angle condition with constants  $(c, C)$  (by [5] and [6]) and the QI condition with constants  $(c, C)$  (as was first proved by Guichard [3]). Proposition 3.1 is a consequence of this fact, of Lemma 3.2, and of the following inequality.

**Lemma 3.4.** There is a constant  $C_0 > 0$  such that  $\|v(g)\| \geq C_0 \text{dist}_\mathfrak{a}(\mu(g), \mu(H))$  for all  $g \in G$ .

Detailed proofs will appear elsewhere.

## References

- [1] Y. Benoist, Actions propres sur les espaces homogènes réductifs, *Ann. Math.* 144 (1996) 315–347.
- [2] M. Flensted-Jensen, Discrete series for semisimple symmetric spaces, *Ann. Math.* 111 (1980) 253–311.
- [3] O. Guichard, Groupes plongés quasi-isométriquement dans un groupe de Lie, *Math. Ann.* 330 (2004) 331–351.
- [4] S. Helgason, *Groups and Geometric Analysis. Integral Geometry, Invariant Differential Operators, and Spherical Functions*, Mathematical Surveys and Monographs, vol. 83, American Mathematical Society, Providence, RI, 2000.
- [5] F. Kassel, Deformation of proper actions on reductive homogeneous spaces, arXiv:0911.4247.
- [6] F. Kassel, Quotients compacts d’espaces homogènes réels ou  $p$ -adiques, PhD thesis, Université Paris-Sud 11, November 2009, see <http://www.math.u-psud.fr/~kassel/>.
- [7] B. Klingler, Complétude des variétés lorentziennes à courbure constante, *Math. Ann.* 306 (1996) 353–370.
- [8] T. Kobayashi, Proper action on a homogeneous space of reductive type, *Math. Ann.* 285 (1989) 249–263.
- [9] T. Kobayashi, Criterion for proper actions on homogeneous spaces of reductive groups, *J. Lie Theory* 6 (1996) 147–163.
- [10] T. Kobayashi, Deformation of compact Clifford–Klein forms of indefinite-Riemannian homogeneous manifolds, *Math. Ann.* 310 (1998) 394–408.
- [11] T. Kobayashi, Hidden symmetries and spectrum of the Laplacian on an indefinite Riemannian manifold, in: *Spectral Analysis in Geometry and Number Theory*, in: *Contemp. Math.*, vol. 484, American Mathematical Society, Providence, RI, 2009, pp. 73–87.
- [12] T. Kobayashi, T. Yoshino, Compact Clifford–Klein forms of symmetric spaces – revisited, *Pure Appl. Math. Quart.* 1 (2005) 591–653.
- [13] R.S. Kulkarni, F. Raymond, 3-Dimensional Lorentz space-forms and Seifert fiber spaces, *J. Differential Geom.* 21 (1985) 231–268.
- [14] T. Matsuki, T. Oshima, A description of discrete series for semisimple symmetric spaces, in: *Group Representations and Systems of Differential Equations*, in: *Adv. Stud. Pure Math.*, vol. 4, North-Holland, Amsterdam, 1984, pp. 331–390.