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A toric Positivstellensatz with applications to delay systems ☆

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ABSTRACT

The structure of positive polynomials on a torus is derived from recent results of real algebraic geometry. As an application, we propose some simple conditions for testing the hyperbolicity/stability of a generic class of linear systems of retarded type.

RÉSUMÉ

La structure des polynômes positifs sur un tore est déduite à l'aide de deux théorèmes récents de type Positivstellensatz. Comme application, on propose des conditions simples pour vérifier l'hyperbolicité/stabilité d'un système linéaire générique d'équations différentielles de type retardé.

1. Introduction

The last decade has witnessed a proliferation of applications of real algebraic geometry to non-linear optimization, see the volumes [4,17]. The present Note is aligned with this trend, by importing two basic facts about the existence of algebraic certificates for the non-emptiness of a semi-algebraic set, into the area of stability of linear systems of delay differential equations.

To recall a few basic algebraic facts, we start with a commutative algebra A with unit, over the rational field. A *quadratic module* $Q \subset A$ is a subset of A such that $Q + Q \subset Q, 1 \in Q$ and $a^2 Q \subset A$ for all $a \in A$. We denote by $Q(F; A)$ or simply $Q(F)$ the quadratic module generated in A by the set F. That is $Q(F; A)$ is the smallest subset of A which is closed under addition and multiplication by squares a^2, $a \in A$, containing F and the unit $1 \in A$. If F is finite, we say that the quadratic module is finitely generated. A quadratic module which is also closed under multiplication is called a *quadratic preordering*. The preordering generated by a set F will be denoted $PO(F; A)$ or simply $PO(F)$. Finally, for a set $F \subset A$ we write (F) for the ideal generated by F, while $MON(F)$ stands for the multiplicative monoid with unit generated by F. A quadratic module Q is called *archimedean* if the constant function 1 belongs to its algebraic interior, that is, for every $f \in Q$ there exists $\epsilon > 0$ such that $1 + tf \in Q$ for all $0 \leqslant t \leqslant \epsilon$.

Assume that $A = \mathbb{R}[x_1, \ldots, x_d]$ is the polynomial algebra. The positivity set $P(Q)$ of $Q \subset \mathbb{R}[x_1, \ldots, x_d]$ is the set of all points $x \in \mathbb{R}^d$ for which $q(x) \geqslant 0$, $q \in Q$. If Q is an archimedean quadratic module in $\mathbb{R}[x_1, \ldots, x_d]$, then there exists $\epsilon > 0$, such that $1 - \epsilon(x_1^2 + \cdots + x_d^2) \in Q$, that is the set $P(Q)$ is necessarily compact. For the aims of this note, two results stand aside. First is a Positivstellensatz due to Stengle, see [15].

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Theorem 1.1. *(See [18].) Let P_i, $i \in I$; Q_j, $j \in J$; R_k, $k \in K$, be finite sets of polynomials in $\mathbb{R}[x_1, \ldots, x_d]$. The set $\{x \in \mathbb{R}^d;\ P_i(x) \geqslant 0,\ \forall i \in I,\ Q_j(x) = 0,\ \forall j \in J,\ R_k(x) \neq 0,\ \forall k \in K\}$ is empty if and only if there exists $f \in PO(P_i)$, $g \in (Q_j)$ and $h \in MON(R_k)$, such that $f + h^2 = g$.*

Second is a partial refinement of the above general fact, with preorders replaced by quadratic modules.

Theorem 1.2. *(See [16].) Let $Q \subset \mathbb{R}[x_1, \ldots, x_d]$ be an archimedean quadratic module and assume that a polynomial f is positive on $P(Q)$. Then $f \in Q$.*

The latter theorem was recently generalized to algebras of non-polynomial functions on Euclidean space [9]. This extension will also be relevant to our study.

Consider now a pair (M, τ), where $M = [A_0 A_1 \cdots A_m] \in \mathbb{R}^{n^2(m+1)}$ denotes a set of $(m+1)$ real $n \times n$ matrices and $\tau = [\tau_1 \cdots \tau_{n_d}] \in \mathbb{R}_+^{n_d}$ a delay vector for some positive integers n_d and m. For an indexed set $\mathcal{I} \subset \mathbb{N}^{n_d}$, we introduce a *delay system of retarded type* at (M, τ) by the following functional differential equation:

$$\dot{x}(t) = \sum_{k=0}^{m} A_k x(t - \gamma_k \cdot \tau), \tag{1}$$

under appropriate initial conditions (see, e.g. [2,11]). Here, $\gamma_k \cdot \tau$ denotes the standard inner product with $\gamma_k \in \mathcal{I}$ and $k = 0, \ldots, m$. Without any loss of generality, we assume that $\gamma_0 \cdot \tau = 0$, and that all the others are non-zero. Introduce now the following characteristic functions associated to (1):

$$f_\tau(s; M, \tau) := \det\left(sI_n - A_0 - \sum_{k=1}^{m} A_k e^{-s\gamma_k \cdot \tau}\right), \qquad f_z(s; M, z) := \det\left(sI_n - A_0 - \sum_{k=1}^{m} A_k z_1^{\gamma_1} \cdots z_{n_d}^{\gamma_{n_d}}\right), \tag{2}$$

where $z = [z_1, \ldots, z_m] \in \mathbb{T}^m$, the unit torus of dimension m. Next, for a matrix $N \in \mathbb{R}^{n^2}$, denote $\sigma(N)$ its spectrum. Furthermore, denote by $\sigma(M, \tau)$ and $\sigma(M, z)$ the spectra of the corresponding operators in (2). The link between f_τ and f_z becomes clear when $s = j\omega$ ($j = \sqrt{-1}$) for some $\omega \in \mathbb{R}$ and by identifying $z_k = e^{-j\omega\tau_k}$, for all $k = 1, \ldots, m$. The relation between the corresponding spectra and the stability of delay systems was discussed in [6,5,12,11], in simpler cases by using the idea that s and e^{-s} are independent algebraic variables. It is worth to mention that the link between quasipolynomials and polynomials over rings of operators was subject of recurring interest since the 70s, see for instance [7] for some fundamental results or [6] for some interpretations in the (exponential) stability case or, [1] for related structural properties.

With the notations above, the system (1) is called *hyperbolic* at (M, τ) [3] if $\sigma(M, \tau) \cap j\mathbb{R} = \emptyset$. Next, a *ray* $r^0 := r_{\tau^0}$ through τ^0 in the delay-parameter space $\mathbb{R}_+^{n_d}$ is the set $\{\delta\tau^0 \in \mathbb{R}_+^{n_d}\colon\ \delta \geqslant 0\}$ and the *hyperbolic cone* at $\tau := \tau^0$ [3] is defined by $H(r^0) := \{M \in \mathbb{R}^{n^2(m+1)}\colon\ \sigma(M, \tau) \cap j\mathbb{R} = \emptyset, \forall \tau \in r_{\tau^0}\}$, and the *hyperbolic cone* H by [3] $H := \bigcap_{r^0 \in \mathbb{R}_+^{n_d}} H(r^0)$. If the stability/hyperbolicity in the commensurate delays case has a complete solution by now (delay intervals explicitly computed by using an eigenvalue-based approach [2,11] or pseudo-delay techniques [14]), however the incommensurate delays case leads to more complicated problems, see, e.g. [19,2] for some discussions on its NP-hardness character.

In the sequel, we will use a *different interpretation* of the quasipolynomials in terms of positive polynomials [4,17] and this interpretation will allow taking advantage of the existing relaxation techniques [8] for deriving tractable solutions to the stability problem. More precisely, we will focus on the characterization of the *hyperbolic cones* in terms of sums-of-squares.

2. Main result

We denote by $z = (z_1, \ldots, z_d)$ the complex coordinates in $\mathbb{C}^d$, and write $z_k = x_k + iy_k$ with x_k, y_k real variables... The unit torus $\mathbb{T}^d = \{z \in \mathbb{C}^d; |z_i| = 1,\ 1 \leqslant i \leqslant d\}$ will be the support manifold for our computations. A real polynomial $p(x, y) = \mathbb{R}[x_1, \ldots, x_d, y_1, \ldots, y_d]$ can be written in complex form as $p(x, y) = P(z, \bar{z})$, where $P \in \mathbb{C}[z_1, \ldots, z_d, \bar{z}_1, \ldots, \bar{z}_d]$ and $\overline{P(z, \bar{z})} = P(z, \bar{z})$. Moreover, with the same notation, there exists $Q \in \mathbb{C}[z_1, \ldots, z_d]$, such that

$$p(x, y)^2 = \left|P(z, \bar{z})\right|^2 = \left|Q(z)\right|^2, \quad z \in \mathbb{T}^d.$$

Indeed, notice that $\bar{z}_k = \frac{1}{z_k}$ whenever $z_k \in \mathbb{T}$, hence a common denominator of the form $(z_1 \cdots z_d)^N$ in $P(z, \bar{z})$ is not seen after passing to the modulus.

Theorem 2.1. *Let $q \in \mathbb{C}[z_1, \ldots, z_d]$. Then $q(z) \neq 0$ for all $z \in \mathbb{T}^d$ if and only if there are complex polynomials $p_1, \ldots, p_k, r_1, \ldots, r_\ell \in \mathbb{C}[z_1, \ldots, z_d]$ such that*

$$1 + \left|p_1(z)\right|^2 + \cdots + \left|p_k(z)\right|^2 = \left|q(z)\right|^2\left(\left|r_1(z)\right|^2 + \cdots + \left|r_\ell(z)\right|^2\right), \quad z \in \mathbb{T}^d.$$

Proof. The condition is clearly sufficient. To prove the necessity one invokes Stengle's Theorem 1.1, applied to the real polynomial $|q(z)|^2$. Specifically, we seek a certificate for the emptiness of the set of common zeros of the ideal $(1-|z_1|^2, \dots, 1-|z_d|^2, |q(z)|^2)$, with the trivial conditions $1 \neq 0$ and $0 \geqslant 0$. Assume $q(z) \neq 0$ for $z \in \mathbb{T}^d$. There are then real polynomials $f_i(x, y)$ and $g(x, y)$, such that

$$1 + f_1(x, y)^2 + \cdots + f_m(x, y)^2 = g(x, y)\left|q(z)\right|^2 + h,$$

where $h \in (1-|z_1|^2, \dots, 1-|z_d|^2)$. Since $g(x, y) > 0$ for $x + iy \in \mathbb{T}^d$, we infer from Theorem 1.2 that there are real polynomials $r_j(x, y)$ such that

$$g = r_1^2 + \cdots + r_n^2 + h_1,$$

with $h_1 \in (1-|z_1|^2, \dots, 1-|z_d|^2)$. Since we are working on the torus, we can pull out a common factor $|z_1 \cdots z_d|^{2\ell}$ so that all real squares are becoming hermitian squares. The proof is complete. □

In practice it is important to have degree bounds in the above decompositions. Partial results in this directions are contained in [13]. When working with delay systems, a more natural setting is offered by elements of the algebra generated by the coordinates and a finite number of exponential functions, see [10].

3. Application to time-delay systems

Introduce now the polynomial $f_{a,\tau}$ defined by $f_{a,\tau}(\omega) = f_\tau(j\omega; M, \tau)$ with ω real, allowing thus an evaluation of f_τ on $j\mathbb{R}$. Consider $R_0 = \sum_{m=0}^{n_d} \|A_k\|_2$ representing a rough upper bound for the real-part of the rightmost characteristic root of f_τ (see, e.g., [11]) and $R = \sqrt{R_0^2 + n_d}$. Now, the certificate contained in Theorem 2.1 to the above framework leads, thanks to the following result, to an effective characterization of the hyperbolic cone in terms of sums-of-squares.

Proposition 3.1. *Consider a set of matrices $M \in \mathbb{R}^{n^2(m+1)}$. Then the following statements are equivalent*:

(i) $M \in H$.
(ii) $0 \notin \sigma(\sum_{k=0}^m A_k)$ *and* $\sigma(M, z) \cap j\mathbb{R}^* = \emptyset$, *for all* $z \in \mathbb{T}^m$.
(iii) *For all* $\tau \geqslant 0$, *the polynomial* $f_{a,\tau}$ *does not vanish on* $[-R, R]$.

The equivalence between (i) and (ii) was derived in [3]. An algorithm for checking the property (ii) in the case when the n_d delays are commensurate was reported by [12] (see [11] for further comments). The main feature of (ii) is that the root at the origin of f_z is not necessarily a root of f_τ. Finally, (iii) ⇒ (ii) as a consequence of our Positivstellensatz and (i) ⇒ (iii) from the definition of the hyperbolic cone H.

A variety of positivity certificates for non-polynomial functions, as in the preceding specific framework, as well as an SDP relaxation algorithm aimed at their validation is contained in [10,9].

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