C. R. Acad. Sci. Paris, Ser. I 348 (2010) 1263–1266

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

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Complex Analysis/Algebraic Geometry

Microlocalization with growth conditions of holomorphic functions

Microlocalisation à croissance des fonctions holomorphes

Luca Prelli

Università degli Studi di Padova, Dipartimento di Matematica Pura ed Applicata, Via Trieste 63, 35121 Padova, Italy

ARTICLE INFO

Article history:
Received 23 March 2010
Accepted after revision 8 November 2010
Available online 23 November 2010

Presented by Jean-Michel Bony

ABSTRACT

In this Note we give some applications related to the microlocalization of subanalytic sheaves. We show the existence of a natural action of microlocal operators on tempered and formal microlocalization, and we give some applications to $\mathcal{D}$-modules. We show also the invariance under contact transformations of tempered and formal microlocalization.

RÉSUMÉ

Dans cette Note on donne des applications liées à la microlocalisation des faisceaux sous-analytiques. On voit l'existence d'une action naturelle des opérateurs microlocaux sur la microlocalisation tempérée et formelle, et on donne des applications aux $\mathcal{D}$-modules. On voit aussi l'invariance par transformations de contact de la microlocalisation tempérée et formelle.

1. Sheaves on subanalytic sites

We use the notations of [5,8]. Let X be a real analytic manifold and let k be a field. Let X_{sa} be the subanalytic site, whose objects are open subanalytic subsets of X and coverings are locally finite in X. Denote by $\rho : X \to X_{sa}$ the natural morphism of sites.

Let $\mathrm{Mod}(k_{X_{sa}})$ be the category of subanalytic sheaves and $\mathrm{Mod}_{\mathbb{R}\text{-c}}(k_X)$ the category of $\mathbb{R}$-constructible sheaves on X. Consider the functors ρ^{-1} and ρ_* of inverse image and direct image associated to ρ. The functor ρ^{-1} admits a left adjoint, denoted by $\rho_!$. The sheaf $\rho_! F$ is the sheaf associated to the presheaf $\mathrm{Op}_{sa}(X) \ni U \mapsto F(\overline{U})$. The functor ρ_* is fully faithful and exact on $\mathrm{Mod}_{\mathbb{R}\text{-c}}(k_X)$ and we identify $\mathrm{Mod}_{\mathbb{R}\text{-c}}(k_X)$ with its image in $\mathrm{Mod}(k_{X_{sa}})$ by ρ_*.

Let X, Y be two real analytic manifolds, and let $f : X \to Y$ be a real analytic map. The six Grothendieck operations $\mathcal{H}om$, $\otimes$, f_*, f^{-1}, $f_{!!}$, $f^!$ are well defined in the derived category of subanalytic sheaves.

2. Tempered and formal microlocalization

We use the notations of [3]. Let Δ be the diagonal of $X \times X$, denote by δ the diagonal embedding and let $q_1, q_2 : X \times X \to X$ be the projections. Set $p_i = q_i \circ p$, $i = 1, 2$. The normal deformation of the diagonal in $X \times X$ can be visualized by the following diagram

E-mail address: lprelli@math.unipd.it.

1631-073X/$ – see front matter
doi:10.1016/j.crma.2010.11.006

$$\begin{array}{ccccccc} TX & \xrightarrow{\sim} & T_\Delta(X\times X) & \xrightarrow{s} & \widetilde{X\times X} & \xleftarrow{i_\Omega} & \Omega \\ & & \downarrow{\scriptstyle \tau_X} & & \downarrow{\scriptstyle p} & \swarrow{\scriptstyle \widetilde{p}} & \\ & & \Delta & \xrightarrow{\delta} & X\times X & \underset{q_1}{\overset{q_2}{\rightrightarrows}} & X \end{array} \tag{1}$$

Definition 2.1. Let $F \in D^b_{\mathbb{R}\text{-c}}(k_X)$ and $G \in D^b(k_{X_{sa}})$. Denote by $^\wedge$ the Fourier–Sato transform for subanalytic sheaves of [9]. We set $\mu hom(F,G) := \rho^{-1}(s^{-1}R\mathcal{H}om((p_2^{-1}F)_\Omega, p_1^!G))^\wedge$.

This is isomorphic to $\rho^{-1}\mu^{sa}_\Delta R\mathcal{H}om(q_2^{-1}F, q_1^!G)$, where μ^{sa}_Δ is the microlocalization functor for subanalytic sheaves of [9].

Now let X be a complex manifold, $X_\mathbb{R}$ the underlying real analytic manifold and $\overline{X}$ the complex conjugate manifold. One denotes by $\mathcal{D}_X$ the sheaf of finite order differential operators with holomorphic coefficients, $\mathcal{O}^t_X$ and $\mathcal{O}^{\mathrm{w}}_X$ the sheaves of tempered and Whitney holomorphic functions of [5].

Let us consider the normal deformation of the diagonal in $X \times X$ as in diagram (1). Let $F \in D^b_{\mathbb{R}\text{-c}}(\mathbb{C}_X)$, set $D'F = R\mathcal{H}om(F, \mathbb{C}_X)$ and let $(\cdot)^a$ denote the direct image for the antipodal map. We have the isomorphisms

$$\mu hom\big(F, \mathcal{O}^t_X\big) \simeq t\mu hom(F, \mathcal{O}_X) \quad \text{and} \quad \mu hom\big(F, \mathcal{O}^{\mathrm{w}}_X\big) \simeq \big(D'F \overset{\mathrm{w}}{\underset{\mu}{\otimes}} \mathcal{O}_X\big)^a,$$

where $t\mu hom(\cdot, \mathcal{O}_X)$ and $\cdot \overset{\mathrm{w}}{\underset{\mu}{\otimes}} \mathcal{O}_X$ denote tempered and formal microlocalization of [1] and [2] respectively.

3. Microlocal kernels

Let X be a complex manifold of complex dimension d_X. Consider the sheaves of rings over T^*X,

$$\mathcal{E}^\mathbb{R}_X = H^{d_X}\big(\mu_\Delta \mathcal{O}^{(0,d_X)}_{X\times X}\big) \quad \text{and} \quad \mathcal{E}^{\mathbb{R},f}_X \simeq \rho^{-1} H^{d_X} \mu^{sa}_\Delta \mathcal{O}^{t(0,d_X)}_{X\times X}$$

of microlocal and tempered microlocal operators. They contain a subring, denoted by $\mathcal{E}_X$ and called the ring of (finite-order) microdifferential operators. We will not recall all the properties of this sheaf and refer to [10] for a detailed study. The sheaves $\mathcal{E}^\mathbb{R}_X$ and $\mathcal{E}^{\mathbb{R},f}_X$ are concentrated in degree d_X and one has the ring inclusions $\mathcal{E}_X \subset \mathcal{E}^{\mathbb{R},f}_X \subset \mathcal{E}^\mathbb{R}_X$.

Let X, Y, Z be three manifolds. Let q_{ij} be the (i,j)-th projection defined on $X \times Y \times Z$ and let p_{ij} be the (i,j)-th projection defined on $T^*X \times T^*Y \times T^*Z$. Let p^a_{ij} be the composition of p_{ij} with the antipodal map a and let $\delta : X \times Y \times Z \to X \times Y \times Y \times Z$ be the diagonal embedding. Consider the following diagram:

$$\begin{array}{ccc} T^*(X\times Y)\times T^*(Y\times Z) & \xleftarrow{p^a_{12}\times p^a_{23}} & T^*X\times T^*Y\times T^*Z \\ \uparrow{\scriptstyle \delta_\pi} & & \wr\downarrow{\scriptstyle \mathrm{id}\times p_2\times a} \\ T^*(X\times Y)\times_Y T^*(Y\times Z) & \longleftarrow & T^*X\times T^*_{\Delta_Y}(Y\times Y)\times T^*Z \\ \downarrow{\scriptstyle {}^t\delta'} & \square & \downarrow \\ T^*(X\times Y\times Z) & \xleftarrow{{}^tq'_{13}} & T^*X\times Y\times T^*Z \\ & & \downarrow{\scriptstyle q_{13\pi}} \\ & & T^*X\times T^*Z \end{array} \qquad (\text{with } p^a_{13}: T^*X\times T^*Y\times T^*Z \to T^*X\times T^*Z) \tag{2}$$

The following proposition is the tempered analogue of Lemma 11.4.3 of [3]:

Proposition 3.1. *Let $K_1 \in D^b_{\mathbb{R}\text{-c}}(\mathbb{C}_{X\times Y})$ and $K_2 \in D^b_{\mathbb{R}\text{-c}}(\mathbb{C}_{Y\times Z})$. Suppose that q_{13} is proper on $\mathrm{supp}(q_{12}^{-1}K_1 \otimes q_{23}^{-1}K_2)$. Set $K_1 \circ K_2 = Rq_{13!}(q_{12}^{-1}K_1 \otimes q_{23}^{-1}K_2)$. There is a morphism*

$$Rp^a_{13!}\big(p^{a-1}_{12}\mu hom\big(K_1, \mathcal{O}^{t(0,d_Y)}_{X\times Y}\big) \otimes p^{a-1}_{23}\mu hom\big(K_2, \mathcal{O}^{t(0,d_Z)}_{Y\times Z}\big)\big)[d_Y] \to \mu hom\big(K_1 \circ K_2, \mathcal{O}^{t(0,d_Z)}_{X\times Z}\big). \tag{3}$$

Corollary 3.2.

(i) *Morphism* (3) *induces the ring structure on $\mathcal{E}^{\mathbb{R},f}_X$.*

(ii) *There is a morphism $\mathcal{E}^{\mathbb{R},f}_X \otimes \mu hom(F, \mathcal{O}^t_X) \to \mu hom(F, \mathcal{O}^t_X)$ which endows $H^k\mu hom(F, \mathcal{O}^t_X)$ with a structure of $\mathcal{E}^{\mathbb{R},f}_X$-module for each $k \in \mathbb{Z}$ and $F \in D^b_{\mathbb{R}\text{-c}}(\mathbb{C}_X)$.*

Let X be a complex manifold. There is a natural morphism induced by the multiplication between tempered functions and functions vanishing up to infinity

$$\rho^{-1}R\mathcal{H}om\big(F,\big(\mathcal{O}^t_X\big)_S\big)\otimes\rho^{-1}R\mathcal{H}om\big(D'(F\otimes G)_S,\mathcal{O}^{\mathrm{w}}_X\big)\to\rho^{-1}R\mathcal{H}om\big(D'G_S,\mathcal{O}^{\mathrm{w}}_X\big),$$

for $F,G\in D^b_{\mathbb{R}\text{-c}}(\mathbb{C}_X)$ and S closed subanalytic, the analogue of Proposition 10.6 of [4]. From this morphism we obtain a morphism

$$\rho^{-1}R\mathcal{H}om\big(\big(p_2^{-1}F\big)_\Omega,p_1^{-1}\mathcal{O}^t_X\big)\otimes\rho^{-1}R\mathcal{H}om\big(\big(p_2^{-1}D'(F\otimes G)\big)_\Omega,p_1^{-1}\mathcal{O}^{\mathrm{w}}_X\big)\to\rho^{-1}R\mathcal{H}om\big(\big(p_2^{-1}D'G\big)_\Omega,p_1^{-1}\mathcal{O}^{\mathrm{w}}_X\big)$$

which is the key point for the construction of (4) below.

Let us consider the diagram (2) with $Z=\{\text{point}\}$. Set $p_X:T^*X\times T^*Y\to T^*X$, $p_Y:T^*X\times T^*Y\to T^*Y$, $q_X:X\times Y\to X$, $q_Y:X\times Y\to Y$.

Proposition 3.3. *Let $G\in D^b_{\mathbb{R}\text{-c}}(\mathbb{C}_X)$ and $K\in D^b_{\mathbb{R}\text{-c}}(\mathbb{C}_{X\times Y})$ such that* $\operatorname{supp}(q_X^{-1}G)\cap\operatorname{supp}(K)$ *is proper over Y. There is a morphism*

$$Rp^a_{X!}\big(\mu hom\big(K,\mathcal{O}^{t(0,d_Y)}_{X\times Y}\big)^a[d_Y]\otimes p_Y^{a\,-1}\mu hom\big(D'(K\circ G),\mathcal{O}^{\mathrm{w}}_Y\big)\big)\to\mu hom\big(D'G,\mathcal{O}^{\mathrm{w}}_X\big).\tag{4}$$

Corollary 3.4. *There is a morphism $\mathcal{E}^{\mathbb{R},f}_X\otimes\mu hom(F,\mathcal{O}^{\mathrm{w}}_X)\to\mu hom(F,\mathcal{O}^{\mathrm{w}}_X)$ which endows $H^k\mu hom(F,\mathcal{O}^{\mathrm{w}}_X)$ with a structure of $\mathcal{E}^{\mathbb{R},f}_X$-module for each $k\in\mathbb{Z}$ and $F\in D^b_{\mathbb{R}\text{-c}}(\mathbb{C}_X)$.*

Let X,Y be two complex analytic manifolds of the same complex dimension n and let $\Omega_X\subset T^*X$, $\Omega_Y\subset T^*Y$ be two open subanalytic subsets. Let χ be a contact transformation from Ω_X to Ω_Y and let Λ be the Lagrangian manifold associated to its graph. We follow the notations of [3], Chapter VII.

It is well known that then there exists $K\in D^b_{\mathbb{C}\text{-}c}(X\times Y,(p_X,p^a_Y))$ simple with shift 0 along Λ with $SS(K)\subseteq\Lambda$ and $s\in\mu hom(K,\mathcal{O}^{t(0,n)}_{X\times Y})_{(p_X,p^a_Y)}$ such that the correspondence $\mathcal{E}_{X,p_X}\ni P\mapsto Q\in\mathcal{E}_{Y,p_Y}$ such that $Ps=sQ$ is an isomorphism of rings. For the construction of such an s see [10], Chapter I.

Theorem 3.5. *For each $F\in D^b_{\mathbb{R}\text{-c}}(Y,p_Y)$ and $G\in D^b_{\mathbb{R}\text{-c}}(X,p_X)$ the morphisms associated to s induced by* (3) *and* (4)

$$\varphi_s:\mu hom\big(F,\mathcal{O}^t_Y\big)_{p_Y}[n]\to\mu hom\big(\Phi^\mu_K F,\mathcal{O}^t_X\big)_{p_X},\qquad\psi_s:\mu hom\big(D'\big(\Phi^\mu_K G\big),\mathcal{O}^{\mathrm{w}}_Y\big)_{p_Y}[n]\to\mu hom\big(D'G,\mathcal{O}^{\mathrm{w}}_X\big)_{p_X}$$

are isomorphisms. Here Φ^μ_K denotes the microlocal integral transform associated to K.

A consequence of the action of $\mathcal{E}_X$ on tempered and formal microlocalization is the following.

Corollary 3.6. *Let $F\in D^b_{\mathbb{R}\text{-c}}(\mathbb{C}_X)$ and let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module. Assume that $SS(F)\cap\operatorname{Char}(\mathcal{M})\subseteq T^*_XX$. Then for $\lambda=t,\mathrm{w}$ we have the isomorphism*

$$R\mathcal{H}om_{\mathcal{D}_X}\big(\mathcal{M},D'F\otimes\mathcal{O}_X\big)\xrightarrow{\sim}R\mathcal{H}om_{\mathcal{D}_X}\big(\mathcal{M},\rho^{-1}R\mathcal{H}om\big(F,\mathcal{O}^\lambda_X\big)\big).\tag{5}$$

Let $\mathcal{M}$ be a $\mathcal{D}_X$-module and let $\lambda=t,\mathrm{w}$. One sets for short $\mathrm{Sol}^\lambda(\mathcal{M}):=R\mathcal{H}om_{\rho_!\mathcal{D}_X}(\rho_!\mathcal{M},\mathcal{O}^\lambda_X)$. Let $SS(F)$ be the micro-support for subanalytic sheaves of [6,7]. As a consequence of Corollary 3.6 we have

Corollary 3.7. *Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module. Then $SS(\mathrm{Sol}^\lambda(\mathcal{M}))=\operatorname{Char}(\mathcal{M})$.*

Let $f:X\to Y$ be a morphism of complex manifolds. Using Corollary 3.7, the fact that $f^!F\simeq f^{-1}F[2\dim_{\mathbb{C}}X-2\dim_{\mathbb{C}}Y]$ if f is non-characteristic for $SS(F)$, $F\in D^b(\mathbb{C}_{X_{sa}})$, and inverse image formulas for holomorphic functions with growth conditions we obtain an analogue with growth conditions of the Cauchy–Kowalevskaya–Kashiwara theorem. Here $\underline{f}^{-1}$ denotes the inverse image for $\mathcal{D}$-modules.

Theorem 3.8. *Let $\mathcal{M}$ be a coherent $\mathcal{D}_Y$-module, and suppose that f is non-characteristic for $\mathcal{M}$. Then there is a natural isomorphism $f^{-1}R\mathcal{H}om_{\rho_!\mathcal{D}_Y}(\rho_!\mathcal{M},\mathcal{O}^\lambda_Y)\simeq R\mathcal{H}om_{\rho_!\mathcal{D}_X}(\rho_!\underline{f}^{-1}\mathcal{M},\mathcal{O}^\lambda_X)$ for $\lambda=t,\mathrm{w}$.*

References

[1] E. Andronikof, Microlocalisation tempérée, Mém. Soc. Math. Fr. 57 (1994).
[2] V. Colin, Formal microlocalization, C. R. Acad. Sci. Paris, Ser. I 327 (1998) 289–293.
[3] M. Kashiwara, P. Schapira, Sheaves on Manifolds, Grundlehren Math., vol. 292, Springer-Verlag, Berlin, 1990.
[4] M. Kashiwara, P. Schapira, Moderate and formal cohomology associated with constructible sheaves, Mém. Soc. Math. Fr. 64 (1996).

[5] M. Kashiwara, P. Schapira, Ind-sheaves, Astérisque 271 (2001).
[6] M. Kashiwara, P. Schapira, Microlocal study of ind-sheaves I: Microsupport and regularity, Astérisque 284 (2003) 143–164.
[7] A.R. Martins, Functorial properties of the microsupport and regularity for ind-sheaves, Math. Z. 260 (3) (2008) 541–556.
[8] L. Prelli, Sheaves on subanalytic sites, Rend. Semin. Mat. Univ. Padova 120 (2008) 167–216.
[9] L. Prelli, Microlocalization of subanalytic sheaves, C. R. Acad. Sci. Paris, Ser. I 345 (2007) 127–132.
[10] P. Schapira, Microdifferential Systems in the Complex Domain, Grundlehren Math., vol. 269, Springer-Verlag, Berlin, 1985.