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Determination of source terms in a degenerate parabolic equation from a locally distributed observation

Détermination d'un terme source dans une équation parabolique dégénérée à partir d'une observation interne localisée

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A R T I C L E   I N F O

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A B S T R A C T

The aim of this Note is to prove a Lipschitz stability and uniqueness result for an inverse source problem relative to a one-dimensional degenerate parabolic equation. We use the method introduced by Imanuvilov and Yamamoto in 1998, with the help of some recent Carleman estimate for degenerate equations obtained by Cannarsa, Martinez and Vancostenoble.

R É S U M É

Le but de cette Note est de montrer un résultat d'unicité et stabilité pour un problème inverse consistant à déterminer un terme source dans une équation parabolique dégénérée en dimension 1. On reprend la méthode introduite par Imanuvilov et Yamamoto en 1998 en précisant une inégalité de Carleman récente obtenue par Cannarsa, Martinez et Vancostenoble.

Version française abrégée

L'étude de problèmes inverses pour des équations paraboliques non dégénérées fait l'objet de nombreux livres et articles. Les premiers travaux sur le sujet étaient basés sur des inégalités de Carleman dites locales et débouchaient sur des résultats d'unicité et stabilité avec des normes de type Hölder. En 1998, dans [8], Imanuvilov et Yamamoto montrent un théorème de stabilité lipschitzienne à l'aide d'une inégalité de Carleman globale. Depuis, leur méthode a été adaptée à divers problèmes inverses, non seulement dans le cas parabolique [6,12] mais aussi pour d'autres types d'équations [1,10]. Nous reprenons dans cet article la démarche de [8] dans le cas d'une équation parabolique dégénérée en utilisant l'inégalité de Carleman globale démontrée dans [9].

On fixe un instant $t_0 \in (0,T)$ et on note $T' = \frac{T+t_0}{2}$. On étudie les questions d'unicité et stabilité lipschitzienne des termes sources g dans le problème (2) sous l'hypothèse que ces termes sources vérifient (3). Pour tout $u_0 \in L^2(0,1)$, et tout $g \in H^1(0,T;L^2(0,1))$ vérifiant (3), le problème (2) est bien posé dans des espaces de Sobolev à poids décrits dans la Section 1.2. Le théorème principal de cet article est le suivant :

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Théorème 0.1. *Soient $\alpha \in [0,2)$ et pour tout $x \in (0,1)$, $a(x) := x^\alpha$. Soit $u_0 \in L^2(0,1)$ et $\omega := (x_0, x_1)$ avec $0 < x_0 < x_1 < 1$. Alors il existe $C = C(T, t_0, C_0, \omega, \alpha) > 0$ telle que pour tout g vérifiant (3), la solution u de (2) vérifie :*

$$\|g\|^2_{L^2(Q_T)} \leqslant C\big(\big\|(au_x)_x\big(T', .\big)\big\|^2_{L^2(0,1)} + \|u_t\|^2_{L^2(\omega_T^{t_0})}\big) \quad \text{avec } Q_T := (0,T) \times (0,1) \text{ et } \omega_T^{t_0} := (t_0, T) \times \omega.$$

La démonstration du Théorème 0.1 repose sur l'inégalité de Carleman globale (7) et sur une inégalité de type Hardy. La question de l'unicité du terme source n'est pas résolue directement par le résultat ci-dessus dans la mesure où l'ensemble des termes sources considéré n'est pas un espace vectoriel. Néanmoins on déduit aisément du théorème ci-dessus l'unicité, en ajoutant une hypothèse supplémentaire sur les termes sources (par exemple en prenant g de la forme rf où r est connue et f fonction de la variable d'espace est à déterminer).

1. Introduction

Numerous articles and books deal with inverse problems for non-degenerate parabolic equations. The first ones used local Carleman estimates in order to get uniqueness and stability in Hölder norms. The founding paper [8] for Lipschitz stability results for parabolic equations was written in 1998. The method is based on global Carleman estimates which were first introduced in [7] to prove observability and thus controllability results. Up to now, there have been various other papers dealing with Lipschitz stability, not only for parabolic equations (see [6,12] for instance) but also for the wave equation [10] and the Schrödinger equation [1]. As for degenerate parabolic equations, where the diffusion coefficient vanishes at one extreme point of the domain, very few results are known in the field of controllability [3,9] and to our knowledge, the question of Lipschitz stability has not been tackled yet. In this Note, we are interested in determining source terms in the following one-dimensional degenerate parabolic equation:

$$u_t - \big(x^\alpha u_x\big)_x = g, \quad \alpha \in [0,2). \tag{1}$$

Our method is based on the global Carleman estimate given in [9]. Let us observe that we only focus on the case of $\alpha \in [0,2)$ because no such Carleman estimates are available for $\alpha \geqslant 2$.

1.1. The issue

Let us describe our problem more precisely. Let $\alpha \in [0,2)$ be given and let us introduce the notation $\forall x \in [0,1]$, $a(x) := x^\alpha$. We consider the following initial–boundary value problem:

$$\begin{cases} u_t - (au_x)_x = g & (t,x) \in (0,T) \times (0,1), \\ u(t,1) = 0 & t \in (0,T), \\ \text{and } \begin{cases} u(t,0) = 0 & \text{for } 0 \leqslant \alpha < 1, \\ (au_x)(t,0) = 0 & \text{for } 1 \leqslant \alpha < 2, \end{cases} & t \in (0,T), \\ u(0,x) = u_0(x) & x \in (0,1). \end{cases} \tag{2}$$

We set $Q_T := (0,T) \times (0,1)$. Let $t_0 \in (0,T)$ and $T' := \frac{T+t_0}{2}$. The topic of this paper can be stated as follows: is it possible to recover source terms g from the knowledge of $(au_x)_x(T', .)$ and $(u_t)|_{(t_0,T)\times\omega}$ where ω is some nonempty open subinterval of $(0,1)$?

1.2. Well-posedness of problem (2)

The classical theory of uniformly parabolic equations does not apply here because $a(0) = 0$. Nevertheless the above problem is well-posed in appropriate weighted spaces which replace the classical Sobolev spaces. Problem (2) can be formulated in terms of an evolution equation in $L^2(0,1)$ with an unbounded operator. For that purpose, let us introduce, for $0 \leqslant \alpha < 2$,

$$H^1_a(0,1) := \big\{u \in L^2(0,1) \bigm| \sqrt{a}u_x \in L^2(0,1)\big\}.$$

In the $\alpha \in [0,1)$ case, one can show that the elements of $H^1_a(0,1)$ have a boundary value both at $x = 0$ and $x = 1$. Therefore, let us define $H^1_{a,0}(0,1) := \{u \in H^1_a(0,1) \mid u(0) = u(1) = 0\}$. In the other case ($1 \leqslant \alpha < 2$), the boundary value at $x = 0$ for an element of H^1_a does not exist anymore. Then, in this case, we define $H^1_{a,0}(0,1) := \{u \in H^1_a(0,1) \mid u(1) = 0\}$.

In both cases, define the unbounded operator $A : D(A) \subset L^2(0,1) \to L^2(0,1)$ by

$$D(A) := \big\{u \in H^1_{a,0}(0,1) \bigm| au_x \in H^1(0,1)\big\} \quad \text{and} \quad \forall u \in D(A), \quad Au := (au_x)_x.$$

We refer to [4] for more details about the spaces $H^1_a(0,1)$ and $D(A)$. One has the following lemma:

Lemma 1.1. *$(A, D(A))$ is the infinitesimal generator of a strongly continuous semigroup of contractions on $L^2(0,1)$. Moreover this semigroup is analytic.*

As a consequence, we have the following theorem:

Theorem 1.2. *For all $u_0 \in L^2(0,1)$, for all $g \in L^2(0,T;L^2(0,1))$, problem (2) has a unique solution u satisfying, for all $\epsilon \in (0,T)$, $u \in L^2(\epsilon,T;D(A)) \cap H^1(\epsilon,T;L^2(0,1))$.*

If moreover $g \in H^1(0,T;L^2(0,1))$, then, for all $\epsilon \in (0,T)$, $u \in C([\epsilon,T];D(A)) \cap C^1([\epsilon,T];L^2(0,1))$.

1.3. Statement of the main result

Let $C_0 > 0$. We introduce the following condition on source terms:

$$\left|\frac{\partial g}{\partial t}(t,x)\right| \leqslant C_0\big|g\big(T',x\big)\big| \quad \text{for a.e. } (t,x) \in Q_T. \tag{3}$$

Let $\mathcal{G}(C_0) := \{g \in H^1(0,T;L^2(0,1)) \mid g \text{ satisfies (3)}\}$. The main result of this Note is the following one:

Theorem 1.3. *Let $\alpha \in [0,2)$, $u_0 \in L^2(0,1)$ and let $\omega := (x_0,x_1)$ with $0 < x_0 < x_1 < 1$. Then, for all $C_0 > 0$ there exists $C = C(T,t_0,C_0,\omega,\alpha) > 0$ such that, for all $g \in \mathcal{G}(C_0)$, the solution u of (2) satisfies:*

$$\|g\|^2_{L^2(Q_T)} \leqslant C\big(\big\|(au_x)_x\big(T',.\big)\big\|^2_{L^2(0,1)} + \|u_t\|^2_{L^2(\omega_T^{t_0})}\big) \quad \textit{where } \omega_T^{t_0} := (t_0,T)\times\omega. \tag{4}$$

2. Proof of Theorem 1.3

2.1. Fundamental tool: a global Carleman estimate

As in [9], we introduce the following functions. Define, for all $t \in (t_0,T)$

$$\theta(t) := \frac{1}{[(t-t_0)(T-t)]^4} \quad \text{and} \quad \beta(t) := T + t_0 - 2t.$$

Then, for all $(t,x) \in (t_0,T)\times(0,1)$, we set $\sigma(t,x) := \theta(t)p(x)$ where p is a positive bounded function satisfying for a.e. $x \in (0,1)$,

$$\big|p_x(x)\big| \leqslant C\frac{x(1-x)}{x^\alpha} \tag{5}$$

for some constant $C > 0$ (see [5,9] for a precise definition of p). Let us consider the following problem:

$$\begin{cases} z_t - (az_x)_x = h & (t,x) \in Q_T^{t_0}, \\ z(t,1) = 0 & t \in (t_0,T), \\ \text{and} \begin{cases} z(t,0) = 0 & \text{for } 0 \leqslant \alpha < 1, \\ (az_x)(t,0) = 0 & \text{for } 1 \leqslant \alpha < 2, \end{cases} & t \in (t_0,T) \end{cases} \tag{6}$$

where $h \in L^2(t_0,T;L^2(0,1))$, and $Q_T^{t_0} := (t_0,T)\times(0,1)$.

Proposition 2.1. *Let $\alpha \in [0,2)$. Then there exist two constants $C_1 = C_1(T,t_0,\omega,\alpha) > 0$ and $R_0 = R_0(T,t_0,\omega,\alpha) > 0$ such that: $\forall R \geqslant R_0$,*

$$I_0(z) := \iint_{Q_T^{t_0}} \left(R^3\theta^3x^{2-\alpha}(1-x)^2z^2 + R\theta^{\frac{3}{2}}|\beta|pz^2 + R\theta x^\alpha z_x^2 + \frac{1}{R\theta}z_t^2\right)e^{-2R\sigma}$$

$$\leqslant C_1\underbrace{\left(\iint_{Q_T^{t_0}} h^2e^{-2R\sigma} + \iint_{\omega_T^{t_0}} R^3\theta^3z^2e^{-2R\sigma}\right)}_{:=I_1(h,z)} \tag{7}$$

for all solutions $z \in L^2(t_0,T;D(A)) \cap H^1(t_0,T;L^2(0,1))$ of (6).

The above proposition can be proved developing the computations made in [9], in which just the first and third terms of $I_0(z)$ are estimated by $I_1(h,z)$ (see the proof in the forthcoming paper [5]).

2.2. A sketch of the proof of Theorem 1.3

We apply Proposition 2.1 to $z := u_t$ and to $h := g_t$ where u is the solution of problem (2). For all $R \geqslant R_0$, we get $I_0(u_t) \leqslant C_1 I_1(g_t, u_t)$.

Steps of the proof of Theorem 1.3.

(a) *There exists a constant* $C = C(T, t_0, C_0, \omega) > 0$ *such that*:

$$I_1(g_t, u_t) \leqslant C\left[\frac{1}{\sqrt{R}}\int_0^1 g^2(T', x)e^{-2R\sigma(T',x)}\,\mathrm{d}x + \|u_t\|^2_{L^2(\omega_T^{t_0})}\right].$$

(b) *There exists a constant* $C = C(T, t_0, C_0, \omega, \alpha) > 0$ *such that*:

$$\int_0^1 z^2(T', x)e^{-2R\sigma(T',x)}\,\mathrm{d}x \leqslant C I_0(u_t).$$

(c) *Conclusion.*

Proof of (a). The second term of $I_1(g_t, u_t)$ is estimated thanks to the definition of the weight functions. The term $\iint_{Q_T^{t_0}} g_t^2 e^{-2R\sigma}$ is estimated using the fact that $g \in \mathcal{G}(C_0)$ and then a standard argument based on Taylor's expansion applied to σ (which is essential to make the factor $\frac{1}{\sqrt{R}}$ appear).

Proof of (b). Using that for a.e. $x \in (0, 1)$, $z^2(t_0, x)e^{-2R\sigma(t_0,x)} = 0$, we easily get

$$\int_0^1 z^2(T', x)e^{-2R\sigma(T',x)}\,\mathrm{d}x = \underbrace{\int_{t_0}^{T'}\int_0^1 \left[2zz_t - 2R\sigma_t z^2\right]e^{-2R\sigma(t,x)}\,\mathrm{d}x\,\mathrm{d}t}_{J}.$$

After a standard Young's inequality to estimate the first term of J, we get

$$J \leqslant \int_{t_0}^{T}\int_0^1 \left(R\theta z^2 + \frac{z_t^2}{R\theta} - 2R\sigma_t z^2\right)e^{-2R\sigma(t,x)}\,\mathrm{d}x\,\mathrm{d}t. \tag{8}$$

In the right-hand side of (8), the second term appears in $I_0(u_t)$ and the third term is easily estimated by $\iint_{Q_T^{t_0}} R\theta^{\frac{3}{2}}|\beta|pz^2e^{-2R\sigma}$ thanks to the definition of σ. In order to estimate the first term, we use the following lemma (see [9] for a proof of the lemma):

Lemma 2.2 *(Hardy-type inequality). Let* $1 < \alpha^\star < 2$. *Then there exists* $C = C(\alpha^\star) > 0$ *such that*

$$\int_0^1 \frac{x^{\alpha^\star}}{x^2(1-x)^2} f^2 \leqslant C(\alpha^\star)\int_0^1 x^{\alpha^\star} f_x^2 \quad \forall f \in H^1_{a^\star,0}(0,1),$$

where, for all $x \in [0, 1]$, $a^\star(x) = x^{\alpha^\star}$.

Recalling that $\alpha \in [0, 2)$, consider $\alpha^\star \in (max(1, \alpha), 2)$ and apply Lemma 2.2 to $x \longmapsto z(t, x)e^{-R\sigma(t,x)}$ for a.e. $t \in (t_0, T)$. Then

$$\int_{t_0}^{T}\int_0^1 R\theta z^2 e^{-2R\sigma} \leqslant \frac{8}{(1-\alpha^\star)^2}\left(\int_{t_0}^{T}\int_0^1 R\theta x^{\alpha^\star} z_x^2 e^{-2R\sigma} + \int_{t_0}^{T}\int_0^1 R^3\theta^3 p_x^2 x^{\alpha^\star} z^2 e^{-2R\sigma}\right).$$

Using that $x^{\alpha^\star} \leqslant x^\alpha$, the conclusion follows from the choice of p (use (5)).

Proof of (c). Using steps (a) and (b) and noting that $z(T',x) = u_t(T',x) = (au_x)_x(T',x) + g(T',x)$, we get

$$\int_0^1 g^2(T',x)e^{-2R\sigma(T',x)}\,\mathrm{d}x \leqslant C\big(\|u_t\|^2_{L^2(\omega_T^{t_0})} + \big\|(au_x)_x(T',.)\big\|^2_{L^2(0,1)}\big),$$

for $R = R(T, t_0, \alpha)$ chosen large enough. The theorem follows from the fact that $g \in \mathcal{G}(C_0)$ and $x \mapsto e^{-2R\sigma(T',x)}$ is a positive function.

3. Further results and comments

3.1. Uniqueness for the inverse source problem

Theorem 1.3 provides a useful stability result but it does not ensure that the inverse problem has a unique solution as $\mathcal{G}(C_0)$ is not a vector space. A possible way to get uniqueness from Theorem 1.3 is to find a vector space included in $\mathcal{G}(C_0)$. Let $r : [0,T] \times [0,1] \to \mathbb{R}$ be a given function of class $C^1([0,T] \times [0,1])$ and $d > 0$ be a positive constant such that $r(T',.) > d$. Then the space $\mathcal{E} := \{rf \mid f \in L^2(0,1)\}$ is included in $\mathcal{G}(C_0)$ for $C_0 := C_0(r) > 0$. Then, when g takes the form $g = rf$, it is easy to deduce a uniqueness result for the coefficients f from Theorem 1.3.

3.2. The case of a boundary observation

Replacing the distributed observation $(u_t)|_{(t_0,T)\times\omega}$ by a boundary observation $(u_t)_x(.,1)|_{(t_0,T)}$., one gets another result similar to Theorem 1.3. This is the subject of the forthcoming paper [5]. Let us also mention two other recent Lipschitz stability results for non-classical parabolic equations: see [2] for the case of a discontinuous diffusion coefficient and see [11] for the case of a heat equation with a singular potential.

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