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Differential Geometry

A harmonic mean bound for the spectral gap of the Laplacian on Riemannian manifolds

Une borne de type moyenne harmonique pour le trou spectral du laplacien sur les variétés riemanniennes

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A B S T R A C T

Most known lower bounds on the spectral gap of the Laplacian using Ricci curvature are based on the infimum of the Ricci curvature, and can be really poor when the Ricci curvature is large everywhere but on a small subset on which it is small. Here we show that the harmonic mean of the Ricci curvature is a lower bound on the spectral gap of the Laplacian, which partially solves the problem (unfortunately, we have to assume that the Ricci curvature is everywhere nonnegative).

R É S U M É

La plupart des minorants connus pour le trou spectral du laplacien faisant intervenir la courbure de Ricci sont basés sur l'infimum de cette courbure, et peuvent être de piètre qualité si la courbure de Ricci est élevée partout sauf sur un petit sous-ensemble sur lequel elle est faible. On montre ici que la moyenne harmonique de la courbure de Ricci est un minorant du trou spectral du laplacien, ce qui résout partiellement le problème (malheureusement, il faut supposer que la courbure de Ricci est partout positive ou nulle).

1. Introduction

This article deals with a lower bound on the spectral gap λ_1 of the Laplacian operator for compact Riemannian manifolds without boundary, using Ricci curvature. Here we will take the convention

$$\Delta f = g^{ij}\nabla^2_{ij} f$$

for the Laplacian operator. One of the numerous interests of estimating this spectral gap is that it gives the speed of convergence to the equilibrium of the Brownian motion on the manifold. In [4], Berger devotes an entire chapter to the study of the spectrum of the Laplacian.

The simplest bound of λ_1 by the Ricci curvature was obtained by Lichnerowicz [7] in the case of positive Ricci curvature:

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Theorem 1.1. *Let $\mathcal{M}$ be a complete connected n-dimensional Riemannian manifold with positive Ricci curvature. Let κ be the infimum of the Ricci curvature, then we have*:

$$\lambda_1 \geqslant \frac{n}{n-1}\kappa.$$

Chen and Wang found better bounds for λ_1, using the infimum of the Ricci curvature and the diameter of the manifold (see [5]):

Theorem 1.2. *Let $\mathcal{M}$ be a compact connected n-dimensional Riemannian manifold, K be the infimum of the Ricci curvature on $\mathcal{M}$, and D be the diameter of $\mathcal{M}$. Then, if $K \geqslant 0$, we have the following bounds*:

$$\lambda_1 \geqslant \frac{\pi^2}{D^2} + \max\left(\frac{\pi}{4n}, 1 - \frac{2}{\pi}\right)$$

and when $n > 1$,

$$\lambda_1 \geqslant \frac{nK}{(n-1)(1-\cos^n(\frac{D\sqrt{K(n-1)}}{2}))}.$$

If $K \leqslant 0$, we have the following bounds:

$$\lambda_1 \geqslant \frac{\pi^2}{D^2} + \left(\frac{\pi}{2} - 1\right)K$$

and when $n > 1$,

$$\lambda_1 \geqslant \frac{\pi^2\sqrt{1 - \frac{2D^2K}{\pi^4}}}{D^2\operatorname{ch}(\frac{D\sqrt{-K(n-1)}}{2})}.$$

In a recent work, [1], Aubry gives a lower bound of λ_1 when the Ricci curvature is approximately constant in L^p norm for p large enough ($> \frac{n}{2}$).

The main theorem of this article is the following one:

Theorem 1.3. *Let $\mathcal{M}$ be a compact connected Riemannian manifold without boundary, with nonnegative Ricci curvature. Then the following inequality holds*:

$$\frac{1}{\lambda_1} \leqslant \int_{\mathcal{M}} \frac{\mathrm{d}\mu(x)}{\kappa(x)}$$

where μ is the renormalized volume measure ($\mathrm{d}\mu = \frac{\mathrm{dvol}}{\mathrm{vol}(\mathcal{M})}$), and $\kappa(x) = \inf_{v\in \mathrm{T}_x\mathcal{M}, \|v\|=1}(\mathrm{Ric}(v,v))$.

The bound on the spectral gap given by this theorem is the harmonic mean of the infimum of the Ricci curvature on every direction. It can be better than the estimates based on the infimum of the Ricci curvature on the manifold, when the Ricci curvature is small on a set of small measure, and large everywhere else. Unfortunately, this does not work with negative curvature.

2. Proof of the main theorem

Let f be an eigenfunction of Δ for the eigenvalue $-\lambda_1$. The existence of f is obtained by minimizing $\int_{\mathcal{M}} \|\nabla u\|^2 \,\mathrm{dvol}$ on the set $\{u \in H_2^1(\mathcal{M}), \int_{\mathcal{M}} u^2 \,\mathrm{dvol} = 1, \int_{\mathcal{M}} u \,\mathrm{dvol} = 0\}$, the Rellich–Kondrakov theorem proves that the infimum is reached. We refer to [6] for the regularity issues.

We define $h(x) = \|\nabla f(x)\|$, and $u(x) = \frac{\nabla f(x)}{h(x)}$ if $\nabla f(x) \neq 0$.

We have $\langle \nabla f, \nabla(\Delta f)\rangle = -\lambda_1 h^2$.

The Böchner–Weitzenböck formula can be found in [4]. It states that:

$$\frac{1}{2}\Delta\left(\|\nabla f\|^2\right) = \mathrm{Ric}(\nabla f, \nabla f) + \left\langle \nabla(\Delta f), \nabla f\right\rangle + \left\|\nabla^2 f\right\|^2.$$

At each point where h does not vanish, we have:

$$\mathrm{Ric}(\nabla f, \nabla f) = h^2 \mathrm{Ric}(u, u) \geqslant \kappa h^2$$

and

$$\left\|\nabla^2 f\right\|^2 = \|h\nabla u + \nabla h.u\|^2 = h^2\|\nabla u\|^2 + \|\nabla h\|^2 + 2g^{ij}g^{kl}h\nabla_i u_k + \nabla_j h u_l.$$

Differentiating $g^{kl}u_k u_l = 1$, we get $g^{kl}\nabla_j(u_k)u_l = 0$. So we have:

$$\left\|\nabla^2 f\right\|^2 = \|\nabla h\|^2 + h^2\|\nabla u\|^2 \geqslant \|\nabla h\|^2.$$

By integration over $\mathcal{M}$ of the Böchner–Weitzenböck formula, we have:

$$0 = \int_{\mathcal{M}} \frac{1}{2}\Delta\left(h^2\right)\mathrm{d}\mu \geqslant \int_{\mathcal{M}} \kappa h^2 - \lambda_1 h^2 + \|\nabla h\|^2\,\mathrm{d}\mu$$

(the integration on the set where h vanishes is 0). By the Poincaré inequality, we get:

$$0 \geqslant \int_{\mathcal{M}} (\kappa - \lambda_1)h^2\,\mathrm{d}\mu + \lambda_1\left[\int_{\mathcal{M}} h^2\,\mathrm{d}\mu - \left(\int_{\mathcal{M}} h\,\mathrm{d}\mu\right)^2\right] \geqslant \int_{\mathcal{M}} \kappa h^2\,\mathrm{d}\mu - \lambda_1\int_{\mathcal{M}} \kappa h^2\,\mathrm{d}\mu \int_{\mathcal{M}} \frac{\mathrm{d}\mu}{\kappa}$$

where the second inequality follows from the Cauchy–Schwarz inequality applied to $\int_{\mathcal{M}} h\sqrt{\kappa}.\frac{1}{\sqrt{\kappa}}\,\mathrm{d}\mu$. We can assume that $\kappa > 0$, μ-almost surely, otherwise there is nothing to prove. Since f is smooth and non-constant, h is not 0 μ-almost surely, and then $\int_{\mathcal{M}} \kappa(x)h^2(x)\,\mathrm{d}\mu(x) > 0$, so we get:

$$0 \geqslant 1 - \lambda_1\int_{\mathcal{M}} \frac{\mathrm{d}\mu(x)}{\kappa(x)}$$

which was to be proved.

3. A generalization of the main theorem

A similar estimate for the spectral gap can be proved in a similar way for reversible diffusions, whose generator has the form:

$$L(f) = \Delta f + \langle\nabla f, \nabla\varphi\rangle.$$

Here we use the generalized notion of curvature developed by Bakry and Emery [2,3], that we recall in the following definitions:

Definition 3.1. Let L be the generator of a Markovian process. The carré du champ operator is defined by the formula:

$$\Gamma(f,g) = \frac{1}{2}\left[L(fg) - fL(g) - L(f)g\right].$$

The iterated carré du champ operator is defined in the same way, except that we put Γ instead of the product of two functions:

$$\Gamma_2(f,g) = \frac{1}{2}\left[L\left(\Gamma(f,g)\right) - \Gamma\left(f, L(g)\right) - \Gamma\left(L(f), g\right)\right].$$

Remark 1. In our case, we have $\Gamma(f,f) = \|\nabla f\|^2$, which is why it is called "carré du champ". We also have $\Gamma_2(f,f) = (\mathrm{Ric} - \nabla^2\varphi)(\nabla f, \nabla f) + \|\nabla^2 f\|^2$.

Definition 3.2. We say that a Markovian operator L satisfies a curvature-dimension inequality $CD(\rho, n')$ if and only if

$$\Gamma_2(f,f)(x) \geqslant \rho(x)\Gamma(f,f)(x) + \frac{1}{n'(x)}\left(L(f)(x)\right)^2$$

holds for every regular enough f.

Remark 2. In our settings, for having the $CD(\rho, n')$ inequality, we must have $\frac{1}{n'} \leqslant \frac{1}{n}$, and equality may occur only at points where $\nabla\varphi = 0$. For a given n', the optimal ρ is $\rho(x) = \inf_{v\in \mathrm{T}_x\mathcal{M}, \|v\|=1}[\mathrm{Ric}_{ij} - \nabla^2_{ij}\varphi - \frac{1}{n'-n}(\nabla_i\varphi\nabla_j\varphi)]v^i v^j$.

Theorem 3.3. *Let $\mathcal{M}$ be a compact connected n-dimensional Riemannian manifold, and φ a smooth function from $\mathcal{M}$ to $\mathbb{R}$. Let L be the diffusion operator defined by*:

$$Lf = \Delta f + \langle\nabla\varphi, \nabla f\rangle.$$

Let $\nu = \frac{e^{\varphi}\mathrm{vol}}{\int_{\mathcal{M}} e^{\varphi}\,\mathrm{dvol}(x)}$ be the reversible probability measure associated to the diffusion. Let λ_1 be the spectral gap of L. Assume that L satisfies a curvature-dimension inequality $CD(\rho, n')$, with n' constant and positive (so greater than n, and possibly $+\infty$) whereas ρ may vary as a function of $\mathcal{M}$. If $\inf_{\mathcal{M}} \rho \geqslant 0$, then we have:

$$\forall 0 \leqslant c \leqslant \inf_{\mathcal{M}} \kappa, \qquad \lambda_1 \geqslant \frac{n'}{n'-1} c + \frac{1}{\int_{\mathcal{M}} \frac{\mathrm{d}\nu}{\rho - c}}.$$

The proof of this theorem in the special case $n' = \infty$ is very similar of the one of Theorem 1.3. We just have to replace Δ with L, and μ with ν. For the general case, we make the suitable convex combination of the inequality $\int_{\mathcal{M}} \rho h^2\,\mathrm{d}\nu - \lambda_1 (\int_{\mathcal{M}} h\,\mathrm{d}\nu)^2 \leqslant 0$, obtained in the case $n' = \infty$ with a better ρ, and of the more classical one $\int_{\mathcal{M}} (\rho - \lambda_1 \frac{n'-1}{n'}) h^2\,\mathrm{d}\nu \leqslant 0$, and thereafter use the Cauchy–Schwarz inequality to get the result.

Remark 3. The function $c \mapsto \frac{n'}{n'-1} c + \frac{1}{\int_{\mathcal{M}} \frac{\mathrm{d}\nu}{\rho(x) - c}}$ is concave on the interval $[0, \inf \rho]$, so the optimal c is

- 0 when $\mathrm{Var}_{\nu}(\frac{1}{\rho}) \geqslant \frac{1}{n-1} (\mathbb{E}_{\nu}[\frac{1}{\rho}])^2$;
- $\inf \rho$ when $\mathrm{Var}_{\nu}(\frac{1}{\rho - \inf \rho}) \leqslant \frac{1}{n-1} (\mathbb{E}_{\nu}[\frac{1}{\rho - \inf \rho}])^2$ or $\nu(\rho = \inf \rho) \geqslant \frac{n-1}{n}$;
- somewhere in the interior of the interval $[0, \inf \rho]$ in the other cases.

So all the values of c in the interval are useful in our theorem, and not only the extremal ones.

References

[1] E. Aubry, Finiteness of π_1 and geometric inequalities in almost positive Ricci curvature, Ann. Sci. Ec. Norm. Super. 40 (July–August 2007).
[2] D. Bakry, M. Émery, Hypercontractivité de semi-groupes de diffusion, C. R. Acad. Sci. Paris Sér. I Math. 299 (15) (1984).
[3] D. Bakry, M. Émery, Diffusions hypercontractives, in: Séminaire de probabilités, XIX, 1983/84, in: Lecture Notes in Math., vol. 1123, Springer, Berlin, 1985.
[4] M. Berger, A Panoramic View of Riemannian Geometry, Springer-Verlag, 2003.
[5] M.F. Chen, F.Y. Wang, General formula for lower bound of the first eigenvalue on Riemannian manifolds, Sci. Sin. (1997).
[6] D. Gilbarg, N.S. Trudinger, Elliptic Partial Differential Equations of Second Order, Springer-Verlag, 1977.
[7] A. Lichnerowicz, Géométrie des groupes de transformations, Dunod, 1958.