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Towards an effective version of Mazur conjecture

Vers une version effective de la conjecture de Mazur

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ABSTRACT

In this Note we present an axiomatic exposition of Cornut–Vatsal's main ingredient in their proof of Mazur conjecture. Our aim is to extract clean ergodic statements and discuss a possible strategy towards effectivity.

RÉSUMÉ

Dans cette Note on présente une vue axiomatique de l'argument principal de la démonstration par Cornut–Vatsal de la conjecture de Mazur. Notre objectif est d'extraire des énoncés ergodiques clairs et de discuter une possible stratégie vers une version effective.

Mazur conjecture [6] has attracted a lot of attention in the recent years. The proof by Cornut and Vatsal [1,3,4,9] combines in a nice way Galois properties of Heegner points and ergodic properties of flows on large products of trees. It is our feeling that the details of the proof are not yet clearly understood. In this Note we present the central part of the argument in a concise way.

We differ slightly from the original proof at several places. Our main observation is that only the topological density of the relevant unipotent orbits is required (versus equidistribution). Statements that follow may be extracted from the above mentioned papers, the most relevant being Sections 2.2–2.7 of [3]. We make no claim of originality.

1. Orbit closure

Let $G = PGL_2(\mathbb{Q}_p)$ with U a one-parameter unipotent subgroup and T a one-dimensional torus, which may or may not be split. Let an integer $r \geqslant 2$ and a sequence of lattices Γ_i for $1 \leqslant i \leqslant r$ be given.

Lemma 1.1. *Assume the following*:

(i) *For all i, the commensurator of Γ_i does not contain a non-trivial unipotent and intersects T non-trivially* (*infinite*).
(ii) *For all pairs $i \neq j$, the lattices Γ_i and Γ_j are not commensurable.*
(iii) *For all pairs $i \neq j$, the lattices Γ_i and Γ_j are either T-commensurable or not G-commensurable.*

Then there is a countable *set $\mathcal{T} \subset T$ such that for all $t \in T - \mathcal{T}$ and all pairs $i \neq j$, the lattices $t^{-1}\Gamma_i t$ and $t^{-1}\Gamma_j t$ are not U-commensurable.*

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Proof. We may consider each pair $i \neq j$ separately. From the assumption (iii) we may assume that Γ_i and Γ_j are T-commensurable. Choose $s \in T$ such that Γ_i and $s^{-1}\Gamma_j s$ are commensurable. Let $t \in T$ such that:

$$\exists u \in U, \quad tut^{-1} \in Comm(\Gamma_i).s. \tag{1}$$

We have to prove that the set of such t is countable. Observe that u is necessarily non-trivial because of assumption (ii). There are two cases:

If T normalizes U, then T is split and TU is a Borel subgroup. For all $r \in Comm(\Gamma_i) \cap T$, the commutator $[tut^{-1}s^{-1}, r]$ is unipotent and belongs to $Comm(\Gamma_i)$. Thus it is trivial by assumption (i), and $tut^{-1}s^{-1}$ commutes with $Comm(\Gamma_i) \cap T$ (infinite). This implies that $tut^{-1}s^{-1}$ is in T which would contradict the earlier fact that u is non-trivial. Therefore in that case there is no $t \in T$ satisfying (1).

If T does not normalize U. We observe that the map $T \times U - \{e\}$, $(t,u) \mapsto tut^{-1}$ is injective. Since $Comm(\Gamma_i).s$ is countable, this concludes the proof of the lemma.

There are several ways to see that the commensurator is countable. Conjugation by an element $g \in Comm(\Gamma_i)$ sends a finite index subgroup Γ' inside Γ_i. There are countably many finite index subgroups in Γ_i. Choose a finite set of generators for Γ'. The centralizer of Γ' is the center of G thus trivial. The element g is uniquely determined by the images of these generators of Γ. These images are in Γ_i which is countable. $\square$

Consider G^r with U and T diagonally imbedded and the product lattice $\Gamma = \prod_{i=1}^r \Gamma_i$. Let $Y = \Gamma \backslash G^r$ and $\pi : G \to G^r \to Y$ be the composition of the diagonal map and the natural projection.

We fix an isomorphism $u : \mathbb{Q}_p \to U$ and a compact open subgroup κ of $\mathbb{Z}_p^\times$. Let H be a compact open subgroup of G. We consider the finite quotient $X = \Gamma \backslash G^r / H^r$ and let $\pi_H : G \to Y \to X$ be the composition of π and the natural projection.

Theorem 1.2. *Assumptions are as in Lemma* 1.1. *For all $t \in T - \mathcal{T}$, the orbit $\pi(tU)$ is dense in Y. More precisely, for all $t \in T - \mathcal{T}$ we have $\pi_H(tu(p^{-m}\kappa)) = X$ for all integer m large enough.*

The above theorem is consequence of Ratner orbit closure theorem [7]. Indeed the orbit closure theorem says that the closure of $\pi(tU)$ is equal to $\pi(tv^{-1}Mv)$ for some $v \in U^r$ and a sugbroup M of G^r which is the product of a certain number of copies of G. The fact that for all $i \neq j$ the lattices $t^{-1}\Gamma_i t$ and $t^{-1}\Gamma_j t$ are not U-commensurable implies that $M = G^r$ and therefore the closure of $\pi(tU)$ is Y. The orbit closure theorem is established as well in Margulis–Tomanov [5]. An exposition of their proof in the present setting is given by Shah [8].

2. Translations of torus orbits

Theorem 2.1. *Assumptions are as in Lemma* 1.1. *For all but finitely many $g \in T\backslash G/H$, $\pi_H(Tg) = X$.*

Theorem 2.1 follows from Theorem 1.2 and the decomposition (2) below. This implication is traditionally pictured by saying that a far translate of a geodesic is approximated by a long horocycle.

2.1. Levels

The quotient $T\backslash G/H$ is countable. It is possible to define a level function that takes values in $\mathbb{N}$ and roughly measures the distance to the origin. This is related to the notion of level for Heegner points in the arithmetic context. Details on this and the next claim may be found in [3, Section 2.6] and are generalized in [2].

Let U be a one-parameter unipotent subgroup of G and $u : \mathbb{Q}_p \to U$ an isomorphism. A way to understand the approximation by horocycles is to study the locally constant map $u \mapsto TugH$ for various elements g. There exists a finite index subgroup κ of $\mathbb{Z}_p^\times$ and a finite sequence of group elements $g_k \in G$ such that (disjoint union):

$$G = \bigsqcup_k \bigsqcup_{m=0}^{\infty} Tu\left(p^{-m}\kappa\right) g_k H, \tag{2}$$

where each $Tu(p^{-m}\kappa)g_k H$ is actually a single (T,H)-coset. When T is split and normalizes U it is a rephrasing of the Iwasawa decomposition (at least for H an adapted maximal compact subgroup).

2.2. Proof of Theorem 2.1

We may fix one of the finitely many indices k. We have to prove that for m large enough:

$$\pi_H\left(Tu\left(p^{-m}\right)g_k\right) = X. \tag{3}$$

This is equivalent to $\pi_H(Tu(p^{-m}\kappa)g_k) = X$. Changing H by its conjugate $H' = g_k H g_k^{-1}$, it is the same as saying that

$$\pi_{H'}\big(Tu\big(p^{-m}\kappa\big)\big) = X' \tag{4}$$

for m large enough, where $X' = \Gamma\backslash G/H'^{r}$. This fact is consequence of Theorem 1.2.

2.3. Effective version

An effective version of Theorem 2.1 would mean the following. One ought to find a constant n_0 depending *effectively* on T, Γ_i and H inside G, such that for all $g \in T\backslash G/H$ of level greater than n_0, we have $\pi_H(Tg) = X$.

2.4. Possible strategy

From the above observations a possible approach for obtaining an effective version of Mazur conjecture emerges. It may be divided into two distinct steps.

1) One ought to establish an effective version of the orbit closure theorem for products of $PGL_2(Q_p)$ or the slightly weaker form given in Theorem 1.2. The arguments by Shah [8, Sections 2–9] are mostly self-contained, and although intricated can probably be made effective. The main difficulty is in bypassing the use of minimal sets.

2) Then one ought to establish an effective version of the Diophantine assumptions (i)–(iii). The proof of these assumptions in [4] consists mostly in algebraic identifications, which is an indication that it can probably be made effective.

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