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L^2-Alexander invariant for torus knots

Invariant d'Alexander L^2 pour les nœuds toriques

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ABSTRACT

The aim of this Note is to present the explicit computation of the L^2-Alexander invariant (defined by Li and Zhang, 2006 [5,6]) for all torus knots.

RÉSUMÉ

Le but de cette Note est de calculer explicitement l'invariant d'Alexander L^2 (défini par Li et Zhang, 2006 [5,6]) dans le cas des nœuds toriques.

Version française abrégée

La torsion de Milnor–Reidemeister est définie pour les opérateurs à spectre fini et utilise la notion usuelle de déterminant. La torsion analytique de Ray–Singer est quant à elle définie pour les opérateurs à spectre discret (infini). D'après le célèbre théorème de Cheeger–Müller [2,10], elle est égale à la torsion de Reidemeister dans le cas des variétés tridimensionnelles compactes et sans bord. Pour certaines familles d'opérateurs à spectre continu, la notion de torsion L^2 a été introduite il y a un peu plus de 15 ans par Carey–Mathai, Lott, Lück–Rothenberg, Novikov–Shubin. Le lecteur intéressé pourra consulter la très complète monographie de W. Lück [7]. Pour les variétés hyperboliques tridimensionnelles, Lück et Schick a montré que la torsion L^2 de la variété est proportionnelle à son volume hyperbolique.

La torsion L^2 est en général très difficile à calculer explicitement. Le but de cette note est de la calculer, ou plus exactement de calculer l'invariant d'Alexander L^2 introduit par Li et Zhang [5,6], pour une famille particulière de nœuds : les nœuds toriques, qui sont non hyperboliques et fibrés. L'invariant d'Alexander L^2 d'un nœud K de S^3 est défini de la façon suivante. Soit $P = \langle g_1, \dots, g_k \mid r_1, \dots, r_{k-1} \rangle$ une présentation de Wirtinger du groupe Γ de K et considérons l'homomorphisme $\phi : \Gamma \to \mathbb{Z}$ tel que $\phi(g_i) = 1$. Pour $t \in \mathbb{C}^*$, on a un homomorphisme d'anneaux :

$$\psi_t : \mathbb{C}\Gamma \to \mathbb{C}\Gamma, \qquad \sum_{g\in\Gamma} c_g \cdot g \mapsto \sum_{g\in\Gamma} c_g \cdot t^{\phi(g)} \cdot g.$$

Soit $F = (\partial r_i/\partial g_j)$ la matrice de Fox ; notons F_j la matrice obtenue à partir de F en lui otant sa j-ème colonne. On obtient ainsi une matrice $\psi_t(F_j) \in M((k-1) \times (k-1); \mathbb{C}\Gamma)$ en appliquant ψ_t à chaque cœfficient. L'*invariant d'Alexander*

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L^2 de K associé à la présentation P, noté $\Delta^{(2)}_{K,P}(t)$, est le *déterminant de Fuglede–Kadison* (voir Eq. (2) pour la définition) de l'application $r^{(2)}_{\psi_t(F_1)} : l^2(\Gamma)^{k-1} \to l^2(\Gamma)^{k-1}$ donnée par la multiplication à droite par la matrice $\psi_t(F_1)$:

$$\Delta^{(2)}_{K,P}(t) = \det_{\mathcal{N}(\Gamma)}\bigl(r^{(2)}_{\psi_t(F_1)} : l^2(\Gamma)^{k-1} \to l^2(\Gamma)^{k-1}\bigr) \in [0,\infty).$$

Sous une hypothèse technique supplémentaire (injectivité de l'application $\psi_t(F_j)$ pour tout $t \in \mathbb{C}^*$, voir Proposition 3.1), Li et Zhang ont montré que $\Delta^{(2)}_{K,P}(t)$ est indépendant de la présentation de Wirtinger P choisie à un facteur $|t|^p$, $p \in \mathbb{Z}$, près. Ainsi $\Delta^{(2)}_{K,P}(t)$ définit, à une puissance de $|t|$ près, un invariant du nœud K (voir Définition 3.1).

Notre résultat principal s'énonce ainsi :

Théorème 0.1. *L'invariant d'Alexander L^2 du nœud torique $T(p,q)$ — (p,q) étant une paire d'entiers premiers entre eux — satisfait l'équation suivante* :

$$\Delta^{(2)}_{T(p,q)}(t) = \max\bigl\{|t|, 1\bigr\}^{(p-1)(q-1)}.$$

Idées de la démonstration. — Le groupe du nœud torique $T(p,q)$ de type (p,q) admet une présentation très simple $P' = \langle x, y \mid x^p = y^q\rangle$ qui malheureusement n'est pas une présentation de Wirtinger. On ne peut donc pas utiliser directement la définition pour calculer le déterminant de Fuglede–Kadison. Le calcul passe par l'intermédiaire d'un résultat énoncé dans cette note (Théorème 3.2) mais dont la démonstration sera donnée dans un article en préparation [3] et passe par l'étude des invariants L^2 à poids (voir les commentaires qui suivent le Théorème 3.2).

1. Introduction

The Milnor–Reidemeister torsion is defined using matrices, i.e. operators with finite spectrum, and using the usual notion of determinant. The analytic Ray–Singer torsion is defined for operators with (infinite) discrete spectrum, and it is well known that for closed three-dimensional manifolds analytic and Reidemeister torsions are equal by the celebrated theorem of Cheeger–Müller (see [2,10]). For certain operators whose spectrum is no more discrete but continuous, the notion of L^2-torsion has been introduced around 15 years ago by Carey–Mathai, Lott, Lück–Rothenberg, Novikov–Shubin (see in particular Lück's monograph [7] for a complete history of the story). One of the most significant results in this field is that the L^2-torsion of a hyperbolic three-dimensional manifold is proportional to the hyperbolic volume of the manifold (in fact equal up to a factor $-\frac{1}{6\pi}$). This fundamental result is due to W. Lück and T. Schick (Ref. [8]).

In the sixties, Milnor [9] gave a spectacular interpretation of the (usual) Alexander polynomial as a kind of (abelian) Reidemeister torsion. As a generalization of the Milnor–Reidemeister torsion, the notion of twisted Alexander polynomial has been introduced in the nineties by X.-S. Lin, and next generalized and studied by many authors: one could refer to the excellent survey by Friedl and Vidussi [4] for a complete bibliography. In 2006, Li and Zhang [5,6] introduced the notion of L^2-Alexander invariants for knots, which is a sort of (twisted) Alexander type invariant but using in its definition the Fuglede–Kadison determinant instead of the usual determinant. Further observe that the L^2-Alexander invariant of a knot evaluated at $t = 1$ is precisely the L^2-torsion of the knot complement.

In general, the Fuglede–Kadison determinant and hence also the L^2-Alexander invariant are difficult to compute explicitly. The aim of this note is to present the computation of the L^2-Alexander invariant for a particular family of knots, the so-called torus knots, which are known to be non-hyperbolic. More precisely, our main result is the following:

Theorem 1.1 *(Main Theorem). The L^2-Alexander invariant of the torus knot $T(p,q)$—where (p,q) denotes a pair of coprime integers—is given by*:

$$\Delta^{(2)}_{T(p,q)}(t) = \max\bigl\{|t|, 1\bigr\}^{(p-1)(q-1)}. \tag{1}$$

2. Some background on L^2-invariants

The Hilbert space $l^2(G)$ is defined as the completion of the complex group ring $\mathbb{C}G$ with respect to the inner product

$$\Bigl\langle \sum_{g\in G} c_g \cdot g, \sum_{g\in G} d_g \cdot g\Bigr\rangle = \sum_{g\in G} c_g \cdot \bar{d}_g.$$

There are several equivalent definitions of the von Neumann algebra $\mathcal{N}(G)$. Let us take the following one: $\mathcal{N}(G)$ is the algebra of all bounded linear endomorphisms of $l^2(G)$ that commute with the left $\mathbb{C}G$-action. The *trace* of an element $\phi \in \mathcal{N}(G)$ is defined by $\mathrm{tr}_{\mathcal{N}(G)}(\phi) := \langle \phi(e), e\rangle$ where $e \in \mathbb{C}G \subset l^2(G)$ denotes the unit element. We can extend this trace to $n \times n$-matrices over $\mathcal{N}(G)$ by considering the sum of the traces of the entries on the diagonal.

A finitely generated Hilbert $\mathcal{N}(G)$-module V is a Hilbert space V with a linear left G-action such that there exists a $\mathbb{C}G$-linear embedding of V into an orthogonal direct sum of a finite number of copies of $l^2(G)$. The von Neumann dimension of a finitely generated Hilbert $\mathcal{N}(G)$-module V is defined by $\dim_{\mathcal{N}(G)}(V) := \mathrm{tr}_{\mathcal{N}(G)}(\mathrm{pr}_V) \in \mathbb{R}^{>0}$. Here

$$\mathrm{pr}_V : \bigoplus_{i=1}^{k} l^2(G) \to \bigoplus_{i=1}^{k} l^2(G)$$

denotes the orthogonal projection onto V. The von Neumann dimension does not depend on the choice of the embedding of V into a finite number of copies of $l^2(G)$. We list three fundamental properties of the von Neumann dimension:

(i) $\dim_{\mathcal{N}(G)}(V) = 0$ if and only if $V = 0$.
(ii) $\dim_{\mathcal{N}(G)}(l^2(G)) = 1$.
(iii) If G is finite then $\dim_{\mathcal{N}(G)}(V) = \dim_{\mathbb{C}}(V)/|G|$.

Let $f : U \to V$ be a map of finitely generated Hilbert $\mathcal{N}(G)$-modules. Its spectral density function is the function $F(f)(\lambda)$ defined as follows:

$$F(f)(\lambda) = \dim_{\mathcal{N}(G)}\left(\mathrm{im}\left(E_{\lambda^2}^{f^*f}\right)\right).$$

Here $\{E_\lambda^{f^*f} : U \to U \mid \lambda \in \mathbb{R}\}$ is the family of spectral projections of the positive endomorphism $f^*f : U \to U$. Observe that $F(t)(\lambda)$ is monotonous and right-continuous. It defines a measure on the Borel σ-algebra on $\mathbb{R}$ which is uniquely determined by $dF(f)((a,b]) = F(f)(b) - F(f)(a)$ for $a < b$.

Let $f : U \to V$ be a map of finitely generated Hilbert $\mathcal{N}(G)$-modules. We define the *Fuglede–Kadison determinant* of f by

$$\det_{\mathcal{N}(G)}(f) = \exp\left(\int_{0^+}^{\infty} \ln(\lambda)\,\mathrm{d}F(f)(\lambda)\right) \tag{2}$$

if $\int_{0^+}^{\infty} \ln(\lambda)\,\mathrm{d}F(f)(\lambda) > -\infty$ and by $\det_{\mathcal{N}(G)}(f) = 0$ otherwise.

3. The L^2-Alexander invariants for knots

3.1. Preliminaries

We first recall the definition of the L^2-Alexander invariants for knots (see [5,6]). Let $K \subset S^3$ be a knot and consider a Wirtinger presentation

$$P = \langle g_1, \dots, g_k \mid r_1, \dots r_{k-1} \rangle$$

of the knot group $\Gamma = \pi_1(M_K)$, where $M_K = S^3 \setminus V(K)$ denotes the knot exterior (here $V(K)$ is a tubular neigborhood of K).

Let $\phi : \Gamma \to \mathbb{Z}$ be the abelianizing homomorphism given by $g_i \mapsto 1$, for all $i = 1, \dots, k$. For $t \in \mathbb{C}^*$ we obtain a ring homomorphism

$$\psi_t : \mathbb{C}\Gamma \to \mathbb{C}\Gamma, \qquad \sum_{g\in\Gamma} c_g \cdot g \mapsto \sum_{g\in\Gamma} c_g \cdot t^{\phi(g)} \cdot g.$$

Let F_j $(1 \leqslant j \leqslant k)$ be the matrix obtained from the Fox matrix $F = (\partial r_i/\partial g_j)$ by removing its jth column. We obtain a matrix $\psi_t(F_j) \in M((k-1)\times(k-1); \mathbb{C}\Gamma)$ by applying ψ_t entry-wise to the matrix F_j.

The *L^2-Alexander invariant $\Delta_{K,P}^{(2)}(t)$ of the knot K with respect to the Wirtinger presentation P* is defined as the Fuglede–Kadison determinant of the map $r_{\psi_t(F_1)}^{(2)} : l^2(\Gamma)^{k-1} \to l^2(\Gamma)^{k-1}$ given by right multiplication with the matrix $\psi_t(F_1)$:

$$\Delta_{K,P}^{(2)}(t) = \det_{\mathcal{N}(\Gamma)}\left(r_{\psi_t(F_1)}^{(2)} : l^2(\Gamma)^{k-1} \to l^2(\Gamma)^{k-1}\right) \in [0,\infty).$$

For the special values $|t| = 1$, the L^2-Alexander invariant is defined and studied in [5,6], whereas [5, Section 7] deals with the general case.

By [5, Lemma 3.1] (which also holds for $|t| \neq 1$) one knows that if $r_{\psi_t(F_j)}^{(2)}$ is injective for some j then it is injective for all j. In this case $\det_{\mathcal{N}(\Gamma)}(r_{\psi_t(F_j)}^{(2)})$ does not depend on j. Moreover, one has $\Delta_{K,P}^{(2)}(t) = \Delta_{K,P}^{(2)}(|t|)$ for all $t \in \mathbb{C}^*$ (this equality has been proved for $|t| = 1$ in [5, Theorem 6.1]).

The next result—due to Li and Zhang [5,6]—ensures that the Fuglede–Kadison determinant does not depend on the Wirtinger presentation.

Proposition 3.1. *Let P and P' be Wirtinger presentations of the knot group of K. We respectively denote the associated Fox matrices by F and F'. Suppose that $r^{(2)}_{\psi_t(F_1)}$ is injective. Then $r^{(2)}_{\psi_t(F'_1)}$ is injective and there exists $p \in \mathbb{Z}$ such that*

$$\Delta^{(2)}_{K,P}(t) = \Delta^{(2)}_{K,P'}(t) \cdot |t|^p.$$

The proof follows by examining the proof of [5, Proposition 3.4].

The proposition above allows us to define the L^2-Alexander invariant $\Delta^{(2)}_K$ of the knot K.

Definition 3.1. Let K be a knot. Suppose that one (and hence all) Wirtinger presentation P of the knot group of K has the property that for the associated Fox matrix F and all $t \in \mathbb{C}^*$ the map $r^{(2)}_{\psi_t(F_1)}$ is injective with $\det_{\mathcal{N}(\Gamma)}(r^{(2)}_{\psi_t(F_1)}) > 0$. Notice that $\{t \mapsto |t|^p \mid p \in \mathbb{Z}\}$ is a subgroup of the multiplicative group $\operatorname{map}(\mathbb{C}^*, \mathbb{R}^{>0})$. We define the L^2-Alexander invariant

$$\Delta^{(2)}_K \in \operatorname{map}\big(\mathbb{C}^*, \mathbb{R}^{>0}\big) \big/ \big\{t \mapsto |t|^p \bigm| p \in \mathbb{Z}\big\}$$

by $t \mapsto \Delta^{(2)}_{K,P}(t)$.

Remark 1. Observe that $\Delta^{(2)}_K(t) = \Delta^{(2)}_K(|t|)$. Furthermore, $\Delta^{(2)}_K(1)$ is equal to the L^2-torsion of the knot exterior.

3.2. L^2-Alexander invariant for knots

For some knots (e.g. the trefoil knot) there exist such simple Wirtinger presentations that one can directly calculate the L^2-Alexander invariant from the definition. But in general it is difficult to determine the L^2-Alexander invariant. We are mostly interested in torus knots. The knot group of the torus knot of type (p,q) admits the very simple well-known presentation with two generators and a single relation: $P' = \langle x, y \mid x^p = y^q \rangle$ (see [1]). Unfortunately, this is not a Wirtinger presentation. By a result of Li and Zhang, the L^2-Alexander invariant can be defined using any presentation which is strongly Tietze-equivalent to a Wirtinger one (see [5, Remark 3.6]). But P' is not strongly Tietze-equivalent to a Wirtinger presentation. Nevertheless, the following result gives us a method to compute the Fuglede–Kadison determinant:

Theorem 3.2. *Let K be a knot in S^3 and let P be a Wirtinger presentation of its knot group Γ. Let $\phi : \Gamma \to \mathbb{Z}$ and $\psi_t : \mathbb{C}\Gamma \to \mathbb{C}\Gamma$ be as described in the preliminaries. Consider a further presentation (not necessarily a Wirtinger presentation) P' of Γ with k generators and $k-1$ relations for some $k > 0$. We denote the associated Fox matrix by F'. Suppose there exists j such that $r^{(2)}_{\psi_t(F'_j)}$ is injective and $\det_{\mathcal{N}(\Gamma)}(r^{(2)}_{\psi_t(F'_j)}) > 0$ for all $t \in \mathbb{C}^*$. Then the Wirtinger presentation P satisfies the assumption of Definition* 3.1 *and*

$$\Delta^{(2)}_K(t) = \det_{\mathcal{N}(\Gamma)}\big(r^{(2)}_{\psi_{|t|}(F'_j)}\big) \cdot \max\big\{|t|, 1\big\}^{1-\phi(g'_j)}.$$

The proof of the theorem above will appear in a forthcoming paper [3]. It is based on a systematical study of *weighted L^2-invariants*, i.e. we consider the weighted Hilbert spaces $l^2(G, \varrho)$ instead of $l^2(G)$. Here, $l^2(G, \varrho)$ denotes the completion of the complex group ring $\mathbb{C}G$ with respect to the inner product

$$\left\langle \sum_{g\in G} c_g \cdot g, \sum_{g\in G} d_g \cdot g \right\rangle_{\varrho} = \sum_{g\in G} c_g \cdot \bar{d}_g \cdot \varrho(g)$$

where $\varrho : G \to \mathbb{R}^{>0}$ is a group homomorphism.

3.3. Computation for torus knots

Let (p,q) be a pair of coprime integers. We let $T(p,q)$ denote the torus knot of type (p,q) and adopt the following notations: $M(p,q) = S^3 \setminus V(T(p,q))$ denotes the knot exterior of $T(p,q)$, and $\Gamma(p,q) = \pi_1(M(p,q))$ its group. Using Theorem 3.2 we can calculate the L^2-Alexander invariant for torus knots, and obtain our Main Theorem: the L^2-Alexander invariant of the torus knot $T(p,q)$ is given by:

$$\Delta^{(2)}_{T(p,q)}(t) = \max\big\{|t|, 1\big\}^{(p-1)(q-1)}. \tag{3}$$

The proof of Eq. (3) is as follows. The knot group of $T(p,q)$ admits the following well-known presentation $P' = \{x, y \mid x^p = y^q\}$ (see [1]). Since the natural image of x (resp. y) in $H_1(M(p,q)) \simeq \mathbb{Z} = \langle h \rangle$ is h^q (resp. h^p), we obtain $\phi(x) = q$ (resp. $\phi(y) = p$). The entry of the matrix F'_2 is

$$\frac{\partial r}{\partial x} = x^{p-1} + x^{p-2} + \cdots + x + 1.$$

Thus, we conclude

$$\det_{\mathcal{N}(\Gamma)}\big(r^{(2)}_{\psi_{|t|}(F_2')}\big)\cdot\det_{\mathcal{N}(\Gamma)}\big(r^{(2)}_{\psi_{|t|}(x-1)}\big)=\det_{\mathcal{N}(\Gamma)}\big(r^{(2)}_{\psi_{|t|}(x^p-1)}\big)$$

where

$$\det_{\mathcal{N}(\Gamma)}\big(r^{(2)}_{\psi_{|t|}(x-1)}\big)=\det_{\mathcal{N}(\Gamma)}\big(r^{(2)}_{|t|^p\cdot x-1}\big)=\max\big\{|t|^p,1\big\}=\max\big\{|t|,1\big\}^p,$$

and

$$\det_{\mathcal{N}(\Gamma)}\big(r^{(2)}_{\psi_{|t|}(x^p-1)}\big)=\det_{\mathcal{N}(\Gamma)}\big(r^{(2)}_{|t|^{pq}\cdot x^p-1}\big)=\max\big\{|t|^{pq},1\big\}=\max\big\{|t|,1\big\}^{pq}.$$

In the calculation above we used the fact that $\det_{\mathcal{N}(\Gamma)}(r^{(2)}_{cg-1})=\max\{|c|,1\}$ for any $c\in\mathbb{C}$ and any $g\in\Gamma$ of infinite order (see [7, Example 3.22 and Theorem 3.14(6)] or [5, Remark 3.3]).

This shows $\det_{\mathcal{N}(\Gamma)}(r^{(2)}_{\psi_{|t|}(F_2')})=\max\{|t|,1\}^{pq-q}$. Using Theorem 3.2, we obtain

$$\Delta^{(2)}_K(t)=\det_{\mathcal{N}(\Gamma)}\big(r^{(2)}_{\psi_{|t|}(F_2')}\big)\cdot\max\big\{|t|,1\big\}^{1-\phi(y)}=\max\big\{|t|,1\big\}^{pq-q}\cdot\max\big\{|t|,1\big\}^{1-p},$$

thus

$$\Delta^{(2)}_K(t)=\max\big\{|t|,1\big\}^{(p-1)(q-1)}.$$

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References

[1] G. Burde, H. Zieschang, Knots, de Gruyter Studies in Mathematics, vol. 5, Walter de Gruyter, 2003.
[2] J. Cheeger, Analytic torsion and the heat equation, Ann. Math. 109 (1979) 259–322.
[3] J. Dubois, C. Wegner, L^2-Alexander invariant for knots, in preparation.
[4] S. Friedl, S. Vidussi, A survey of twisted Alexander polynomials, preprint arXiv:GT/0905.0591v2, 2009.
[5] W. Li, W. Zhang, An L^2-Alexander invariant for knots, Commun. Contemp. Math. 8 (2) (2006) 167–187.
[6] W. Li, W. Zhang, An L^2-Alexander–Conway invariant for knots and the volume conjecture, in: Differential Geometry and Physics, in: Nankai Tracts Math., vol. 10, World Sci. Publ., Hackensack, NJ, 2006, pp. 303–312.
[7] W. Lück, L^2-Invariants: Theory and Applications to Geometry and K-Theory, Ergebnisse der Mathematik und Ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, vol. 44, Springer-Verlag, Berlin, 2002.
[8] W. Lück, T. Schick, L^2-torsion of hyperbolic manifolds of finite volume, Geometric and Functional Analysis 9 (1999) 518–567.
[9] J. Milnor, A duality theorem for Reidemeister torsion, Ann. of Math. 76 (1962) 134–147.
[10] W. Müller, Analytic torsion and R-torsion of Riemannian manifolds, Adv. in Math. 28 (1978) 233–305.