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# A nonparametric test for conditional symmetry in nonstationary and absolutely regular dynamical models

## *Un test non paramétrique de la symétrie conditionnelle des modèles dynamiques non stationnaires et absolument réguliers*

Joseph Ngatchou-Wandji [a], Michel Harel [b,d], Echarif Elharfaoui [c,d]

[a] *EHESP, Rennes and Université Henri-Poincaré, 54506 Vandoeuvre-lès-Nancy cedex, France*
[b] *IUFM du Limousin, 209, boulevard de Vanteaux, 87036 Limoges cedex, France*
[c] *EMMID, Département de mathématiques et informatique, université Chouaîb-Doukkali, faculté des sciences El-Jadida, route Ben-Maachou, B.P. 20, 24000, Morocco*
[d] *IMT (UMR CNRS 5219), Université Paul-Sabatier, 31062 Toulouse cedex, France*

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ABSTRACT

In this Note, we reconsider the test for symmetry of the errors distribution in a class of heteroscedastic models proposed by Ngatchou-Wandji (2009). In the new study, the observations, as well as the errors, are not necessarily stationary but are required to be absolutely regular.



RÉSUMÉ

Dans cette Note, nous reconsidérons le test de symétrie de la loi du bruit dans une classe de modèles hétéroscédastiques proposé par Ngatchou-Wandji (2009). Le cadre de notre étude est celui où, aussi bien les observations que les erreurs, ne sont plus nécessairement stationnaires, mais absolument régulières.



## Version française abrégée

Soit $U_i = \{(Y_i, X_i),\ i \in \mathbb{N}\}$ un processus absolument régulier, non nécessairement stationnaire. La moyenne et la variance conditionnelles de $Y_i$ sachant $Z_i = (Y_{i-1}, Y_{i-2}, \ldots, Y_{i-s}, X_i, X_{i-1}, \ldots, X_{i-q})$ sont notées $m(Z_i;\theta)$ et $\sigma^2(Z_i;\rho)$, où $m$ et $\sigma$ sont des fonctions spécifiées, $\rho$ et $\theta$ des paramètres inconnus. Nous considérons la forme conditionnellement centrée et réduite

$$\varepsilon_i(\psi) = \frac{Y_i - m(Z_i;\rho)}{\sigma(Z_i;\theta)}, \qquad \psi = \left(\rho^\top, \theta^\top\right)^\top \in \Theta \times \widetilde{\Theta} \subset \mathbb{R}^l \times \mathbb{R}^p \tag{1}$$

de $Y_i$ dont la fonction de répartition est notée $F_i$.

Soient $\mathcal{S} = \{G$ fonction de répartition continue: $G(x) = 1 - G(-x),\ x \in \mathbb{R}\}$ et $F$ la limite (en un sens précisé dans la version anglaise ci-dessous) de la suite des fonctions $F_i$. Notre but est de généraliser la procédure proposée dans Ngatchou-Wandji [4] au test de l'hypothèse nulle $\mathcal{H}_0 : F \in \mathcal{S}$ contre l'alternative $\mathcal{H}_1 : F \notin \mathcal{S}$.

*E-mail address:* michel.harel@limousin.iufm.fr (M. Harel).

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On note $U^\top$ la transposée d'un vecteur ou d'une matrice $U$. On définit les fonctions aléatoires suivantes :

$$\widehat{F}_n(t)=\frac{1}{n}\sum_{i=1}^{n}\mathbb{1}_{(\varepsilon_i(\psi_n)\leqslant t)},\quad t\in\mathbb{R},$$

$$S_n(t)=n^{1/2}\int_{\mathbf{Supp}(\mathbf{F})}\sin(tx)\omega(t)\,\mathrm{d}\widehat{F}_n(x)=\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\sin\bigl[t\varepsilon_i(\psi_n)\bigr]\omega(t),\quad t\in\mathbb{R},\tag{2}$$

où **Supp**(**F**) est le support de la loi associée à $F$, $\psi_n=(\rho_n^\top,\theta_n^\top)^\top$ un estimateur consistant du vrai paramètre $\psi_0=(\rho_0^\top,\theta_0^\top)^\top$, $\omega(t)$ une fonction réelle positive continue telle que $\sup_{t\in\mathbf{Supp}(\mathbf{F})}|t\omega(t)|<\infty$.

De l'étude des propriétés asymptotiques du processus $S_n(t)$, plusieurs statistiques de test sont envisageables. Nous nous limitons ici à celle du type Cramér–von Mises définie par

$$\mathcal{T}_n=\int_{\mathbf{Supp}(\mathbf{F})}S_n^2(t)\,\mathrm{d}\widehat{F}_n(t)=\frac{1}{n}\sum_{i=1}^{n}S_n^2\bigl[\varepsilon_i(\psi_n)\bigr].$$

D'après la proposition 1 et les théorèmes 1 et 2 ci-dessous, pour tout $i\in\mathbb{N}$, $\varepsilon_i(\psi_n)$ et $\varepsilon_i(\psi_0)$ sont asymptotiquement égaux en probabilité, le processus empirique $\widehat{F}_n(t)$ des $\varepsilon_i(\psi_n)$ converge uniformément en probabilité vers $F(t)$, et sous $\mathcal{H}_0$ le processus $S_n(t)$ converge en loi vers un processus gaussien centré $S(t)$ de noyau de covariance $\Sigma_0$ définie par (10) avec $u(t)\equiv 0$. Il en résulte que sous $\mathcal{H}_0$, $\mathcal{T}_n$ converge en loi vers une somme pondérée de khi-deux indépendantes à un degré de liberté, les poids intervenant dans cette somme étant les valeurs propres de l'opérateur intégral $\nabla_{\Sigma_0}$ défini pour toute fonction $g$ telle que $\int_{\mathbb{R}}g^2(s)\,\mathrm{d}F(s)<\infty$ par :

$$\nabla_{\Sigma_0}g(t)=\int_{\mathbb{R}}\Sigma_0(s,t)g(s)\,\mathrm{d}F(s).\tag{3}$$

En procédant comme dans Ngatchou-Wandji (2009), ces valeurs propres peuvent être estimées et la convergence des estimateurs établie. Les $p$-valeurs du test peuvent ensuite être approximées en utilisant par exemple les résultats de Imhof [3].

## 1. Introduction

Let $U_i=\{(Y_i,X_i);\ i\in\mathbb{N}\}$ be a sequence of absolutely regular nonnecessarily stationary random vectors. Denote the conditional mean and variance of $Y_i$ given $Z_i=(Y_{i-1},Y_{i-2},\dots,Y_{i-s},X_i,X_{i-1},\dots,X_{i-q})$ by $m(Z_i;\rho)$ and $\sigma(Z_i;\theta)$ where the functions $m(z;\rho)$ and $\sigma(z;\theta)$ have known forms and $\rho$ and $\theta$ are unknown parameters.

Consider the random variables $\varepsilon_i(\psi)$'s defined by (1), and denote by the $F_i$'s their cumulative distribution functions. Assume that the sequence $\{F_i\}$ converges in a sense to be made precise to a limiting distribution function $F$. Let $\mathcal{S}=\{G$ continuous distribution function: $G(x)=1-G(-x),\ x\in\mathbb{R}\}$. Our main objective in this paper is to test $\mathcal{H}_0$: $F\in\mathcal{S}$ against $\mathcal{H}_1$: $F\notin\mathcal{S}$. This testing problem generalizes the one in Bai and Ng [1], Pérez-Alonso [5] or Ngatchou-Wandji [4], in the sense that if the $\varepsilon_i$'s are iid, then all the $F_i$'s equal the distribution function $F$. Another relevent paper is Delgado and Escanciano [2] where testing conditional symmetry in dynamical models has been considered for stationary observations and nonnecessarily independent errors. However, many time series encountered in practice are not stationary. This can justify the need of reconsidering the above testing problem in a nonstationary context.

## 2. Notations and general assumptions

In the sequel, we assume that the sequence $\{U_i\}_{i\in\mathbb{N}}$ is absolutely regular with a geometrical rate, that is

$$\beta(n)=\mathcal{O}\bigl(\tau^n\bigr),\quad 0<\tau<1,\tag{4}$$

where

$$\beta(k)=\sup_{n\in\mathbb{N}}\max_{0\leqslant j\leqslant n-k}E\Bigl\{\sup_{A\in\mathcal{A}_{j+k}^{\infty}}\bigl|P\bigl(A\mid\mathcal{A}_0^j\bigr)-P(A)\bigr|\Bigr\}$$

with $\mathcal{A}_i^j$ standing for the $\sigma$-algebra generated by $U_i,\dots,U_j$, $i,j\in\mathbb{N}\cup\{\infty\}$. We also assume that the distribution function of $U_i$ is continuous, and converges for the norm of total variation $\|.\|_{TV}$, to a distribution function with positive density marginals. Denoting $H_{i,j}$ the distribution function of $(U_i,U_j)$, we furthermore assume that for any $l>1$, there exists a continuous distribution function $\widetilde{H}_l$ on $\mathbb{R}^{2+2d}$ admitting a strictly positive density such that

$$\|H_{i,j}-\widetilde{H}_{j-i}\|_{TV}=\mathcal{O}\bigl(\varrho_0^i\bigr),\quad i<j,\ i,j\in\mathbb{N},\ 0<\varrho_0<1\tag{5}$$

and there exists a sequence $\{\widetilde{U}_i=(\widetilde{Y}_i,\widetilde{X}_i),\ i\geqslant 1\}$ of stationary absolutely regular random vectors of rate (4) with $(\widetilde{U}_i,\widetilde{U}_j)$ having $\widetilde{H}_{j-i}$ as distribution function $(i<j+1)$.

For $z \in \mathbb{R}^{s+dq}$, $\partial Q(x;z) = \partial Q(x;z)/\partial x$ and $\partial^2 Q(x;z)/\partial x^2$ denote respectively, the gradient and the Hessian of a function $Q(x;z)$ differentiated with respect to $x \in \mathbb{R}^l$ or $x \in \mathbb{R}^p$. The notation $\partial^\top Q(x;z)$ stands for the transpose of this function. Let $\kappa$ and $\kappa_i$, $i \geqslant 1$ be positive integers. For given $b \in \mathbb{R}$ and vectors $U = (U_1, \dots, U_\kappa)^\top$ and $D = (D_1, \dots, D_\kappa)^\top$ where the $D_i$'s are $1 \times \kappa_i$ matrices, we denote $bU = Ub = (bU_1, \dots, bU_\kappa)^\top$ and $D \odot U = U \odot D = (D_1 U_1, \dots, D_\kappa U_\kappa)^\top$. We also denote by $\|\mathcal{V}\|_E$ the Euclidean norm of a vector $\mathcal{V}$. The interior set of a set $\Omega$ is denoted by $\mathrm{int}(\Omega)$ and its closure by $\bar{\Omega}$. Henceforth, we restrict to models (1) for which:

(A1) For all $\theta \in \widetilde{\Theta}$ and $z \in \mathbb{R}^{s+dq}$, $|\sigma(z;\theta)| > \tau$, for some positive real number $\tau$.
(A2) There exists $\delta > 2$ such that $\sup_{i \geqslant 0} \int_{\mathbb{R}} |x|^{2+2\delta}\, \mathrm{d}F_i(x) < \infty$.
(A3) The functions $m(z;\rho)$ and $\sigma(z;\theta)$ are each continuously differentiable with respect to $\rho \in \mathrm{int}(\Theta)$ and $\theta \in \mathrm{int}(\widetilde{\Theta})$ respectively, and there exist finite positive numbers $r_1$, $r_2$ such that $\bar{B}(\rho_0, r_1) \subset \mathrm{int}(\Theta)$, $\bar{B}(\theta_0, r_2) \subset \mathrm{int}(\widetilde{\Theta})$ and a positive function $\gamma(z)$ with $\sup_{i \geqslant 0} E[\gamma^{2+2\delta}(Z_i)] < \infty$, such that

$$\max\Big\{ \sup_{\rho \in \bar{B}(\rho_0, r_1)} \big\|\partial m(z;\rho)\big\|_E,\ \sup_{\theta \in \bar{B}(\theta_0, r_2)} \big\|\partial \sigma(z;\theta)\big\|_E \Big\} \leqslant \gamma(z).$$

(A4) The true parameter $\psi_0 = (\rho_0^\top, \theta_0^\top)^\top \in \Theta \times \widetilde{\Theta}$ has an estimator $\psi_n = (\rho_n^\top, \theta_n^\top)^\top$ satisfying the Bahadur representation

$$n^{1/2}(\psi_n - \psi_0) = n^{-1/2} \sum_{i=1}^{n} \Pi(\psi_0; Z_i) \odot \Gamma\big[\varepsilon_i(\psi_0)\big] + o_p(1)$$

where for $z \in \mathbb{R}^{s+dp}$, $\Pi(\psi_0;z) = (\Pi_1^\top(\psi_0;z); \Pi_2^\top(\psi_0;z))^\top$, $\Pi_1(\psi_0;z) = (\Pi_{11}(\psi_0;z), \dots, \Pi_{1l}(\psi_0;z))^\top \in \mathbb{R}^l$, $\Pi_2(\psi_0;z) = (\Pi_{21}(\psi_0;z), \dots, \Pi_{2p}(\psi_0;z))^\top \in \mathbb{R}^p$ such that $\sup_{i \geqslant 0} E[\|\Pi(\psi_0;Z_i)\|_E^{2+2\delta}] < \infty$, and for $x \in \mathbb{R}$, $\Gamma(x) = (\Gamma_1(x); \Gamma_2(x))^\top \in \mathbb{R}$ such that $\sup_{i \geqslant 0} \int_{\mathbb{R}} \|\Gamma(x)\|_E^{2+2\delta}\, \mathrm{d}F_i(x) < \infty$ and $\int_{\mathbb{R}} \Gamma(x)\, \mathrm{d}F_i(x) = 0 \in \mathbb{R}^2$, $\forall i \in \mathbb{N}$.

**Remark 1.** For all $i \in \mathbb{N}$, let $\widetilde{\varepsilon}_i(\psi_0)$ be the random variable obtained by substituting $\widetilde{Y}_i$ and $\widetilde{Z}_i = (\widetilde{Y}_{i-1}, \widetilde{Y}_{i-2}, \dots, \widetilde{Y}_{i-s}, \widetilde{X}_i, \widetilde{X}_{i-1}, \dots, \widetilde{X}_{i-q})$ for $Y_i, Z_i$ on the right-hand side of (1). Then the sequence $\{\widetilde{\varepsilon}_i(\psi_0)\}_{i \in \mathbb{N}}$ is stationary with stationary cumulative distribution function $F$. Denoting $F_{i,j}$ the distribution function of $(\varepsilon_i, \varepsilon_j)$, one can show that for all $k > 1$, there exists a continuous distribution function $\widetilde{F}_k$ defined on $\mathbb{R}^2$, with marginals $F$ such that

$$\|F_{i,j} - \widetilde{F}_{j-i}\|_{TV} = \mathcal{O}\big(\varrho^i\big), \quad i < j,\ i, j \in \mathbb{N}, \tag{6}$$

for some $\varrho$, $0 < \varrho_0 < \varrho < 1$, and $\widetilde{F}_{j-i}$ is the distribution function of $(\widetilde{\varepsilon}_i(\psi_0), \widetilde{\varepsilon}_j(\psi_0))$.

**Remark 2.** Assumptions (A1) and (A3) are verified by a large class of common models. Assumption (A2) is a weak assumption on the distribution of the error. It is satisfied by most of the common distributions. Assumption (A4) is satisfied by least-squares and likelihood estimators. Under this assumption, one can show that $n^{1/2}(\psi_n - \psi_0)$ converges in distribution to a Gaussian random vector with mean $0 \in \mathbb{R}^{l+p}$ and covariance matrix

$$\Bigg( E\big[\Pi_k(\psi_0; \widetilde{Z}_0)\Pi_j^\top(\psi_0; \widetilde{Z}_0)\big] \int_{\mathbb{R}} \Gamma_k(x)\Gamma_j(x)\, \mathrm{d}F(x) + 2\sum_{i=1}^{\infty} E\big\{\Pi_k(\psi_0; \widetilde{Z}_0)\Pi_j^\top(\psi_0; \widetilde{Z}_i)\Gamma_k\big[\widetilde{\varepsilon}_0(\psi_0)\big]\Gamma_j\big[\widetilde{\varepsilon}_i(\psi_0)\big]\big\}\colon 1 \leqslant k, j \leqslant 2 \Bigg).$$

## 3. The test statistic

**Proposition 1.** *Assume that* (A1)–(A2) *hold. Then for all* $i \in \mathbb{N}$,

$$\varepsilon_i(\psi_n) = \varepsilon_i(\psi_0) + r_{i,n}$$

*where* $\max_{1 \leqslant i \leqslant n} |r_{i,n}| = o_P(1)$.

Recall that $F(t)$ is the common distribution function of the $\widetilde{\varepsilon}_i(\psi_0)$'s and define the following function:

$$u(t) = E\big(\sin\big[t\widetilde{\varepsilon}_1(\psi_0)\big]\big) = \int_{\mathbb{R}} \sin(tx)\, \mathrm{d}F(x), \quad t \in \mathbb{R}.$$

**Remark 3.** The distribution of $\widetilde{\varepsilon}_0(\psi_0)$ is symmetric around 0 if and only if the function $u(t)$ is identically nil on $\mathbb{R}$.

Denote by $\mathbf{Supp}(\mathbf{F}) \subset \mathbb{R}$ the support of the distribution associated with $F(t)$. Let $\omega(t)$ be a real-valued nonnegative continuous function such that $\sup_{t\in\mathbf{Supp}(\mathbf{F})} |t\omega(t)| < \infty$. For all $t \in \mathbb{R}$, define the following random functions:

$$\widehat{F}_n(t) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}_{(\varepsilon_i(\psi_n)\leqslant t)}, \tag{7}$$

$$S_n(t) = n^{1/2} \int\limits_{\mathbf{Supp}(\mathbf{F})} \sin(tx)\omega(t)\,\mathrm{d}\widehat{F}_n(x) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n} \sin\bigl[t\varepsilon_i(\psi_n)\bigr]\omega(t) \tag{8}$$

where $\mathbb{1}_A$ denotes the indicator function of $A$.

**Remark 4.** One can prove easily that the random function $u_n(t) = (1/n)\sum_{i=1}^n \sin[t\varepsilon_i(\psi_n)]$ defined on $\mathbb{R}$ converges uniformly in probability to $u(t)$.

From Remark 3, a test for symmetry can be based on suitable normalized sample versions of the function $u(t)$, as for example, $n^{-1/2}\sum_{i=1}^n \sin[t\varepsilon_i(\psi_0)]$. But since the $\varepsilon_i(\psi_0)$'s are not observable, Remark 4 indicates that a more suitable sample version may be $n^{1/2}u_n(t)$. However, since the weight function $w(t)$ allows for some flexibility, we instead consider the random function $S_n(t)$ given by (8), and study the following Cramér–von Mises statistic:

$$\mathcal{T}_n = \int\limits_{\mathbf{Supp}(\mathbf{F})} S_n^2(t)\,\mathrm{d}\widehat{F}_n(t) = \frac{1}{n}\sum_{i=1}^{n} S_n^2\bigl[\varepsilon_i(\psi_n)\bigr].$$

## 4. Some theoretical results

Denote by $\mathcal{C}(\mathbf{Supp}(\mathbf{F}))$ the set of all real-valued continuous functions on $\mathbf{Supp}(\mathbf{F})$ endowed with the usual sup-norm $\|f\|_\infty = \sup_{t\in\mathbf{Supp}(\mathbf{F})} |f(t)|$, $f \in \mathcal{C}(\mathbf{Supp}(\mathbf{F}))$.

Make the following assumptions:

(B1) The set $\mathbf{Supp}(\mathbf{F})$ is bounded and the function $\omega(t)$ is constant on it.
(B2) There exist a positive constant $C$ and a positive real-valued function $\phi(x)$ defined on $\mathbb{R}$ with $\sup_{i\geqslant 0}\int_{\mathbb{R}} \phi^2(x)\,\mathrm{d}F_i(x) < \infty$ such that for all $s, t \in \mathbf{Supp}(\mathbf{F})$,
  (i) $\|t\Lambda(t)\omega(t) - s\Lambda(s)\omega(s)\|_E \leqslant C|t-s|$,
  (ii) $|\sin(tx)\omega(t) - \sin(sx)\omega(s)| \leqslant \phi(x)|t-s|$, $x \in \mathbb{R}$.

**Remark 5.** One can check that the assumption (B2) will generally hold for the functions $\omega(t)$ for which the derivative of the functions $t \mapsto \sin(tz)\omega(t)$ and $t \mapsto t\Lambda(t)\omega(t)$ are bounded on $\mathbf{Supp}(\mathbf{F})$.

For all $i \in \mathbb{N}$, define the following matrices:

$$\begin{aligned}
\dot{\Omega}_{i,11} &= E\bigl\{\Pi_1(\psi_0;\widetilde{Z}_0)\Pi_1^\top(\psi_0;\widetilde{Z}_i)\Gamma_1\bigl[\widetilde{\varepsilon}_0(\psi_0)\bigr]\Gamma_1\bigl[\widetilde{\varepsilon}_i(\psi_0)\bigr]\bigr\},\\
\dot{\Omega}_{i,12} &= E\bigl\{\Pi_1(\psi_0;\widetilde{Z}_0)\Pi_2^\top(\psi_0;\widetilde{Z}_i)\Gamma_1\bigl[\widetilde{\varepsilon}_0(\psi_0)\bigr]\Gamma_2\bigl[\widetilde{\varepsilon}_i(\psi_0)\bigr]\bigr\},\\
\dot{\Omega}_{i,22} &= E\bigl\{\Pi_2(\psi_0;\widetilde{Z}_0)\Pi_2^\top(\psi_0;\widetilde{Z}_i)\Gamma_2\bigl[\widetilde{\varepsilon}_0(\psi_0)\bigr]\Gamma_2\bigl[\widetilde{\varepsilon}_i(\psi_0)\bigr]\bigr\},\\
\Omega_{11}(\psi_0) &= E\bigl[\Pi_1(\psi_0;\widetilde{Z}_0)\Pi_1^\top(\psi_0;\widetilde{Z}_0)\bigr]\int\limits_{\mathbb{R}} \Gamma_1^2(x)\,\mathrm{d}F(x) + 2\sum_{i=1}^{\infty}\dot{\Omega}_{i,11},\\
\Omega_{12}(\psi_0) &= E\bigl[\Pi_1(\psi_0;\widetilde{Z}_0)\Pi_2^\top(\psi_0;\widetilde{Z}_0)\bigr]\int\limits_{\mathbb{R}} \Gamma_1(x)\Gamma_2(x)\,\mathrm{d}F(x) + 2\sum_{i=1}^{\infty}\dot{\Omega}_{i,12},\\
\Omega_{22}(\psi_0) &= E\bigl[\Pi_2(\psi_0;\widetilde{Z}_0)\Pi_2^\top(\psi_0;\widetilde{Z}_0)\bigr]\int\limits_{\mathbb{R}} \Gamma_2^2(x)\,\mathrm{d}F(x) + 2\sum_{i=1}^{\infty}\dot{\Omega}_{i,22},\\
\Omega(\psi_0) &= \begin{pmatrix} \Omega_{11}(\psi_0) & \Omega_{12}(\psi_0)\\ \Omega_{12}^\top(\psi_0) & \Omega_{22}(\psi_0)\end{pmatrix}.
\end{aligned} \tag{9}$$

Also, for all $s, t \in \mathbb{R}$, define the following functions:

$$\varphi(t)=\int_{\mathbb{R}}\cos(tx)\,\mathrm{d}F(x),$$

$$\Lambda(t)=\left(\varphi(t)E\left[\frac{\partial^\top m(\rho_0;\widetilde{Z}_0)}{\sigma(\theta_0;\widetilde{Z}_0)}\right];E\left[\frac{\partial^\top\sigma(\theta_0;\widetilde{Z}_0)}{\sigma(\theta_0;\widetilde{Z}_0)}\right]\int_{\mathbb{R}}x\cos(tx)\,\mathrm{d}F(x)\right)^\top,$$

$$\varphi_1(s,t)=\varphi(s-t)+2\sum_{i=1}^{\infty}\int_{\mathbb{R}^2}\cos(sx-ty)\,\mathrm{d}\widetilde{F}_i(x,y),$$

$$\varphi_2(s,t)=\varphi(s+t)+2\sum_{i=1}^{\infty}\int_{\mathbb{R}^2}\cos(sx+ty)\,\mathrm{d}\widetilde{F}_i(x,y),$$

$$\begin{aligned}\Xi(s,t)=&-E\left[\Pi^\top(\psi_0;\widetilde{Z}_0)\right]\left\{t\left(\int_{\mathbb{R}}\left[\sin(sx)-u(s)\right]\Gamma(x)\,\mathrm{d}F(x)\right)\odot\Lambda(t)\right.\\&\left.+s\left(\int_{\mathbb{R}}\left[\sin(tx)-u(t)\right]\Gamma(x)\,\mathrm{d}F(x)\right)\odot\Lambda(s)\right\}\\&-\sum_{j=1}^{\infty}\left\{E\left[\Pi^\top(\psi_0;\widetilde{Z}_j)\left(\sin\left[s\widetilde{\varepsilon}_0(\psi_0)\right]\Gamma\left[\widetilde{\varepsilon}_j(\psi_0)\right]\right)\odot\Lambda(t)\right]t\right.\\&\left.+E\left[\Pi^\top(\psi_0;\widetilde{Z}_0)\left(\sin\left[t\widetilde{\varepsilon}_j(\psi_0)\right]\Gamma\left[\widetilde{\varepsilon}_0(\psi_0)\right]\right)\odot\Lambda(s)\right]s\right\},\end{aligned}$$

$$\Sigma_u(s,t)=\left[\frac{1}{2}\left\{\varphi_1(s,t)-\varphi_2(s,t)-2u(t)u(s)\right\}+\Xi(s,t)+st\Lambda^\top(t)\Omega(\psi_0)\Lambda(s)\right]\omega(s)\omega(t). \tag{10}$$

**Theorem 1.** *Assume that* (A1)–(A4), *the conditions* (4), (5) *and* (6) *hold and* $F(t)$ *is continuous. Then for* $\widehat{F}_n(t)$ *defined by* (7), *in probability,*

$$\sup_{t\in\mathbb{R}}\left|\widehat{F}_n(t)-F(t)\right|\to 0,\quad \text{as } n\to\infty.$$

**Theorem 2.** *Assume that the conditions of Theorem* 1 *hold.*

(i) *Under* $\mathcal{H}_0$, *if either* (B1) *or* (B2) *holds, then* $S_n(t)$ *converges in distribution to a zero-mean Gaussian process* $S\in\mathcal{C}(\mathbf{Supp}(\mathbf{F}))$ *with covariance kernel* $\Sigma_0(s,t)$ *defined by* (10) *for* $u(t)\equiv 0$.
(ii) *Under* $\mathcal{H}_1$, *if either* (B1) *or* (B2) *holds with* $\sin[t\varepsilon_1(\psi_0)]$ *substituted for* $\sin[t\varepsilon_1(\psi_0)]-u(t)$, *then* $S_n(t)-n^{1/2}u(t)\omega(t)$ *converges in distribution to a zero-mean Gaussian process* $S_1\in\mathcal{C}(\mathbf{Supp}(\mathbf{F}))$ *with covariance kernel* $\Sigma_u(s,t)$ *given by* (10).

**Sketch of the proof.** For Part (i), it suffices to study the tightness and the finite-dimensional distributions of $S_n(t)$. Part (ii) can be handled in the same way. □

**Theorem 3.** *Assume that the assumptions* (A1)–(A4) *hold.*

(i) *If the assumptions in Part* (i) *of Theorem* 2 *hold and if under* $\mathcal{H}_0$ *the eigenvalues* $\zeta_k$*'s of the integral operator* $\nabla_{\Sigma_0}$ *given by* (3) *are positive, then under* $\mathcal{H}_0$, *for all* $x\in\mathbb{R}$,

$$P(\mathcal{T}_n\leqslant x)\to P\left(\sum_{k\geqslant 1}\zeta_k\chi_k^2\leqslant x\right), \tag{11}$$

*where* $\chi_1^2,\chi_2^2,\ldots$ *are iid chi-squared random variables with one degree of freedom.*
(ii) *If the assumptions in Part* (ii) *of Theorem* 2 *hold and if* $\int_{\mathbb{R}}\omega^2(t)\,\mathrm{d}F(t)<\infty$, *then, under* $\mathcal{H}_1$, *for all* $\chi>0$, $P(\mathcal{T}_n>\chi)\to 1$, *as n tends to infinity.*

**Sketch of the proof.** Part (i) hinges on the Karhumen–Loève expansion of $S(t)$, the application of Theorem 1 and the continuous mapping theorem. Part (ii) can be established in the same spirit. □

**Remark 6.** Proceeding as in Ngatchou-Wandji [4], it is possible to obtain estimates of the $\zeta_k$'s and establish their consistency. The $p$-values of our test can be approximated by making use of Imhof's results [3].

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