C. R. Acad. Sci. Paris, Ser. I 348 (2010) 959–962

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

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Lie Algebras/Harmonic Analysis

On a branching law of unitary representations and a conjecture of Kobayashi

Sur une loi de branchement des représentations unitaires et une conjecture de Kobayashi

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A R T I C L E   I N F O

Article history:
Received 3 March 2009
Accepted after revision 3 September 2010

Presented by Michel Duflo

A B S T R A C T

In this Note we consider the equivalence of different properties of the restriction of an irreducible unitary representation of a real reductive group to a closed reductive subgroup. As a corollary, we prove a weak form of a conjecture of Kobayashi.

R É S U M É

Dans cette Note, nous considérons l'équivalence de différentes propriétés de la restriction d'une représentation unitaire irréductible d'un groupe de Lie réel réductif á un sous-groupe réductif fermé. Comme corollaire, nous prouvons une forme faible d'une conjecture de Kobayashi.

1. Introduction

Let G be a real reductive Lie group with the Lie algebra $\mathfrak{g}$ and K the maximal compact subgroup of G corresponding to a Cartan involution θ. Let π be an irreducible unitary representation of G. A **branching law** is the irreducible decomposition of π when restricted to a subgroup G' [7]:

$$\pi|_{G'} \cong \int_{\widehat{G'}}^{\oplus} m_\pi(\tau)\tau \, \mathrm{d}\mu(\tau) \quad \text{(direct integral)},$$

where $\mathrm{d}\mu$ is a Borel measure on the unitary dual $\widehat{G}'$ of G' with Fell topology and

$$m_\pi(\cdot) : \widehat{G}' \to \mathbb{N} \cup \{\infty\}$$

is the multiplicity defined almost everywhere with respect to $\mathrm{d}\mu$.

The restriction $\pi|_{G'}$ is called G'-admissible [2,3] if the restriction $\pi|_{G'}$ splits into a discrete sum of irreducible unitary representations:

$$\pi|_{G'} \cong \sum_{\tau \in \widehat{G}'}^{\oplus} m_\pi(\tau)\tau \quad \text{(discrete Hilbert sum)},$$

with multiplicity $m_\pi(\tau) \in \mathbb{N} = \{0, 1, 2, \ldots\}$.

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1631-073X/$ – see front matter
doi:10.1016/j.crma.2010.09.006

In this Note, we will always assume that G' is a θ-stable closed reductive subgroup of G with maximal compact subgroup $K' = G' \cap K$. Usually, the branching laws involve continuous spectrum and/or infinite multiplicity. The idea of looking at K'-structure was proposed in [3] for the study of G'-admissible restrictions. Indeed, it was proved that K'-admissibility implies G'-admissibility (see [3, Theorem 1.2]) and T. Kobayashi conjectured the converse statement:

Conjecture 1.1. *(See [6, Conjecture D].) If $\pi|_{G'}$ is G'-admissible, then $\pi|_{K'}$ is K'-admissible.*

As was explained in [6], one important example is Harish–Chandra's celebrated theorem, which says that $\pi|_K$ is K-admissible. This is the case for $G' = G$. Recently, Duflo and Vargas [1] proved the above conjecture when π is a discrete series representation of G, which we will discuss in the last section using our method. For the general case, Conjecture 1.1 still remains open. We just direct the readers' attention to the extensive reviews of [7,8] and references therein.

There is a one–one correspondence between the unitary dual of G and the set of irreducible unitarizable $(\mathfrak{g}_{\mathbb{C}}, K)$-modules. Here and in the following, $\mathfrak{g}_{\mathbb{C}}$ denotes the complexification of $\mathfrak{g}$. We can consider the restriction of the corresponding Conjecture Harish–Chandra module π_K of π.

Definition 1.2. (See [4, Definition 1.1].) Let $\mathfrak{g}'$ be the Lie algebra of G'. We say that the restriction $\pi|_{G'}$ is $\mathfrak{g}'$-discretely decomposable, if we have an isomorphism as $(\mathfrak{g}'_{\mathbb{C}}, K')$-modules:

$$\pi_K \cong \bigoplus_{\tau \in \widehat{G'}} n_\pi(\tau)\tau_{K'}.$$

This is equivalent to the condition that there is an irreducible $(\mathfrak{g}'_{\mathbb{C}}, K')$-submodule occurring in π_K (see [4, Lemma 1.3]).

Kobayashi [4, Theorem 4.2] proved that if $\pi = \overline{A_{\mathfrak{q}}(\lambda)}$ for λ in the weakly fair range and (G, G') is a reductive symmetric pair, then $\mathfrak{g}'$-discretely decomposability is equivalent to K'-admissibility. For general cases, we say $\pi|_{G'}$ is $\mathfrak{g}'$-admissible if $n_\pi(\tau) < \infty$, for each $\tau \in \widehat{G'}$. Now we can state our main result.

Theorem 1.3. *If $\pi|_{G'}$ is $\mathfrak{g}'$-admissible, then $\pi|_{K'}$ is K'-admissible.*

Remark 1.4. Theorem 1.3 is the opposite implication of Kobayashi's result in [4]. Thanks to [6, Theorem 2.7], [4, Proposition 1.6], and Theorem 1.3, one sees Kobayashi's Conjecture 1.1 is equivalent to "$\mathfrak{g}'$-admissibility $\Leftrightarrow$ G'-admissibility". Our result provides an evidence to Kobayashi's Conjecture 1.1.

Following the idea of [4, Section 1], we shall use Kostant's Theorem as below:

Theorem 1.5. *(See [9, Theorem 1.2].) Let $Z(\mathfrak{g}')$ be the center of the universal enveloping algebra of $\mathfrak{g}'$. Assume that M is a $Z(\mathfrak{g}')$-finite $\mathfrak{g}'$-module. Then M is strongly $Z(\mathfrak{g}')$-finite: that is, if F is any finite-dimensional $\mathfrak{g}'$-module, then $M \otimes F$ is $Z(\mathfrak{g}')$-finite. More precisely, if M has generalized infinitesimal character $\lambda \in \mathfrak{h}^*$, where $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}'$, then every infinitesimal character occurring in $M \otimes F$ is of the form $\lambda + \gamma$, with γ a weight of $\mathfrak{h}$ in F.*

2. The proof of the main result

Assume that $\pi|_{G'}$ is $\mathfrak{g}'$-discretely decomposable, i.e.,

$$\pi_K \cong \bigoplus_{\tau \in \widehat{G'}} n_\pi(\tau)\tau_{K'}. \tag{2.1}$$

Assume that $\tau_{K'}$ with infinitesimal character λ occurs in π_K. Let $\mathfrak{q}$ be the orthogonal complement of $\mathfrak{g}'$ in $\mathfrak{g}$ with respect to the Killing form, i.e., $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{q}$. Following the idea of [4, Lemma 1.5], we consider the linear subspace $\mathfrak{q}\pi_{K'}$ of π_K generated by $\{X \cdot v \mid X \in \mathfrak{q}, v \in \pi_{K'}\}$, which is a $(\mathfrak{g}'_{\mathbb{C}}, K')$-submodule of π_K and is a quotient module of $\tau_{K'} \otimes \mathfrak{q}$. One can easily see that $\pi_K = \sum_n \mathfrak{q}^n \tau_{K'}$, where $\mathfrak{q}^n \tau_{K'} = \{X_1 \dots X_n \cdot v \mid X_i \in \mathfrak{q},\ v \in \tau_{K'}\}$. By Kostant's Theorem, we have the following result:

Lemma 2.1. *For any irreducible $(\mathfrak{g}'_{\mathbb{C}}, K')$-module $\tau_{K'}$, which occurs in π_K, the infinitesimal character of $\tau_{K'}$ is of the form*

$$\lambda + \gamma$$

for some $\gamma \in \Delta(\mathfrak{q})$, where $\Delta(\mathfrak{q})$ is the lattice generated by all the weight of $\mathfrak{h}$ in $\mathfrak{q}$. Here $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}'$.

Definition 2.2. Let λ_1 and λ_2 be the infinitesimal characters of two irreducible G'-representations. We say that they are $\mathfrak{q}$-related, if $\lambda_1 - \lambda_2 \in \Delta(\mathfrak{q})$.

Now assume furthermore that $\pi|_{G'}$ is $\mathfrak{g}'$-admissible, we want to show that any irreducible K'-module V_μ just occurs in τ_K with finite multiplicity. We just need to prove that there are finitely many irreducible $(\mathfrak{g}'_{\mathbb{C}}, K')$-submodules $\tau_{K'}$ (see Eq. (2.1)) of τ_K which contain V_μ as a K'-type. Now we need the following inequality:

Lemma 2.3. *Let V be an irreducible unitary $(\mathfrak{g}'_{\mathbb{C}}, K')$-module and infinitesimal character λ. Assume $V_\mu \in \widehat{K'}$ occurs in V with highest weight μ. Then*

$$(\lambda, \lambda) \leqslant (\mu + \rho_c, \mu + \rho_c) - (\rho_c, \rho_c) + (\rho, \rho). \tag{2.2}$$

Here $(\,,)$ is the Killing form, ρ and ρ_c are the half sum of the positive roots of $\mathfrak{g}'$ and $\mathfrak{k}' = \mathrm{Lie}\, K'$ respectively.

Proof. Let $\mathfrak{g}' = \mathfrak{k}' + \mathfrak{p}'$ be a Cartan decomposition of $\mathfrak{g}'$ and let $\{x_i\}$ be an orthonormal basis of $\mathfrak{p}'$ with respect to the Killing form. Denote by $\langle\,,\rangle$ the invariant Hermitian form on V, then for any $v \in V_\mu$, we have

$$\langle x_i v, x_i v\rangle \geqslant 0 \quad \Rightarrow \quad \langle x_i^2 v, v\rangle \leqslant 0 \quad \Rightarrow \quad \langle (c - c_{\mathfrak{k}'})v, v\rangle \leqslant 0,$$

where c and $c_{\mathfrak{k}'}$ are the Casimir elements of $\mathfrak{g}'$ and $\mathfrak{k}'$ respectively, hence $c = c_{\mathfrak{k}'} + \sum x_i^2$. Since c acts on V by the scalar $(\lambda, \lambda) - (\rho, \rho)$ and $c_{\mathfrak{k}'}$ acts on V_μ by the scalar $(\mu + \rho_c, \mu + \rho_c) - (\rho_c, \rho_c)$, the assertion follows. □

Remark 2.4. The well-known Dirac inequality will also work for our proof.

Let V_μ be a K'-type of π_K. Denote by Λ_μ the set of infinitesimal characters of irreducible $(\mathfrak{g}', K')$-submodules of π_K which contain V_μ as a K'-type. Then we have the following lemma:

Lemma 2.5. *The set Λ_μ is finite.*

Proof. By Lemma 2.1, all $\lambda \in \Lambda_\mu$ share the same imaginary part, hence the real parts of $\lambda \in \Lambda_\mu$ are bounded by Lemma 2.3. Furthermore, all the infinitesimal characters of irreducible $(\mathfrak{g}', K')$-submodules of π_K are contained in $\lambda + \Delta(\mathfrak{q})$, for some $\lambda \in \Lambda_\mu$, by Lemma 2.1 again. So the set Λ_μ is discrete and bounded, hence finite. □

Now we come to the proof to our main result. By a theorem of Harish and Chandra, there are finitely many irreducible unitary representations of G' with a given infinitesimal character. So there are finitely many irreducible $(\mathfrak{g}', K')$-submodules of π_K with infinitesimal character $\lambda \in \Lambda_\mu$. Thus there are finitely many irreducible $(\mathfrak{g}', K')$-submodules of π_K which contain V_μ as a K-type. The multiplicity of V_μ in any irreducible $(\mathfrak{g}', K')$-module is finite and $\tau|_{G'}$ is $\mathfrak{g}'$-admissible, so V_μ is of finite multiplicity in π_K, which completes the proof of our main result.

3. A weak form of Conjecture 1.1

Assume that $\pi \in \widehat{G}$ is G'-admissible, i.e., $\pi|_{G'} = \sum^{\oplus}_{\tau \in \widehat{G'}} m_\pi(\tau)\tau$, $m_\pi(\tau) < \infty$. Let V_μ be an irreducible K'-module and let $\Pi = \{\tau \in \widehat{G'} \mid m_\pi(\tau) > 0$ and μ occurs in $\tau\}$. Conjecture 1.1 is equivalent to that Π is a finite set. Then we have the following weak form of the conjecture:

Proposition 3.1. *Assume that $\pi|_{G'}$ is G'-admissible. If all the infinitesimal characters occurring in $\pi|_{G'}$ are $\mathfrak{q}$-related, then $\pi|_{K'}$ is K'-admissible.*

Proof. Let $V_\mu \in \widehat{K'}$. By the same argument in the above section, there are finitely many irreducible subrepresentations τ of π, which contain V_μ as a K'-type. So one can see that Π is finite. Thus the multiplicity of V_μ in π is finite. □

Remark 3.2. If Conjecture 1.1 holds, then one can easily see that all the infinitesimal characters occurring in $\pi|_{G'}$ are $\mathfrak{q}$-related.

Example 3.3. Let π be a discrete series for G. Assume that $\pi|_{G'} = \sum^{\oplus}_{\tau \in \widehat{G'}} m_\pi(\tau)\tau$ with $m_\pi(\tau) < \infty$. It is well known that any τ which occurs in $\pi|_{G'}$ is a discrete series for G' (see [5, Corollary 8.7]). Duflo and Vargas [1] proved that in this case $\pi|_{K'}$ is admissible, using a theorem of Harish and Chandra: given an irreducible representation τ of K', the number of discrete series τ of G' in which τ occurs is finite.

By our method, once we know that all the τ occurring in $\pi|_{G'}$ are discrete series representations, we can see that all the infinitesimal characters of such τ are analytically integral, so they are real and form a discrete set. By the same argument in the proof of Proposition 3.1, we can easily see that $\pi|_{K'}$ is K'-admissible.

Actually, we have the following more general result:

Proposition 3.4. *Assume that $\pi|_{G'}$ is G'-admissible. If all the infinitesimal characters occurring in $\pi|_{G'}$ are real and form a discrete set, then $\pi|_{K'}$ is K'-admissible.*

Acknowledgements

Both the authors are partially supported by the National Science Foundation of China (No. 10971103). The authors thank Professors Duflo, Kobayashi and the referee for their helpful suggestions.

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